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PARAMETER AUGMENTATION FOR SOME BASIC HYPERGEOMETRIC SERIES IDENTITIES

ADARSH KUMAR[1,*], MOHD.SADIQ KHAN[1] AND KESHAV PRASAD YADAV[2]

[1]Department of Mathematics, Shibli National P.G. College, Azamgarh, Uttar Pradesh, India

[2]Department of Mathematics, Rama Institute of Engineering & Technology, Kanpur, Uttar Pradesh, India

Abstract: In present paper, an attempt has been made to establish some interesting q-series theorems which verify the special case of companion identity by making use of augmentation operator introduced by Chen & Liu and q-difference operator. Also, this technique of parameter augmentation for basic hypergeometric series can be helpful for providing q-summation and integral formulae.

Keywords: q-exponential operator, q-difference operator, parameter augmentation and basic hypergeometric series.

2000 AMS Subject Classification: 33D15, 33D60

1. Introduction:

Zhang and Yang [1] extended the results of Chen and Liu [2, 3], then make use of them several q- series identities are obtained involving a q- series identity in Ramanujan's Lost Note book. We motivated to [1]and establish certain q- series identities on the same pattern of [1] by making use of special case of companion identity which is established in the chapter (2) of dissertation [4] and q-exponential operator technique due to Chen & Liu [2,3].

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*Corresponding author

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For any integer n the q- shifted factorial $(a: q)_n$ is defined as

$$(a;q)_n = \frac{(a\,;\,q)_\infty}{(aq^n\,;\,q)_\infty} \tag{1.1}$$

$$(a;q)_{-n} = \frac{1}{(aq^{-n};\,q)_n} = \frac{(-q/a)^n}{(\frac{q}{a}\,;\,q)_n} q^{\binom{n}{2}} \tag{1.2}$$

The q- binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}} = \frac{\left(q^{-n};q\right)_k}{\left(q;q\right)_k} \left(-q^n\right)^k q^{-\binom{k}{2}} \qquad \text{(cf[9], p-20, 1.2)} \tag{1.3}$$

Multiple q- shifted factorials is defined as:

$$(a_1, a_2, a_3, ..., a_m; q)_n = (a_1;q)_n (a_2;q)_n (a_3;q)_n ... (a_m;q)_n \tag{1.4}$$

The q- shifted operator is defined as

$$\eta\{f(a)\} = f(aq) \tag{1.5}$$

$$\eta^{-1}\{f(a)\} = f(aq^{-1}) \tag{1.6}$$

These can be found in the paper of Rogers [5, 6, 7]. Due to Roman [8], θ is defined as

$$\theta = \eta^{-1} D_q \tag{1.7}$$

The operator introduced by Chen and Liu [2, 3] built from θ is as under

$$E(d\theta) = \sum_{n=0}^{\infty} \frac{(d\theta)^n q^{\binom{n}{2}}}{(q;q)_n} \tag{1.8}$$

The q- Saalschütz formula [9, (II.12)]

$$ {}_3\varphi_2[a,b,q^{-n};c,abc^{-1}q^{1-n};q,q]=\frac{(c/a,c/b;q)_n}{(c,c/ab;q)_n} \tag{1.9}$$

turns out to be self-dual. The companion identity is

$$\frac{c}{ab}\frac{(a,b;q)_{k+1}}{(c,q;q)_k}\sum_{n=k}^{\infty}\frac{(c,c/ab;q)_n(q^{-n};q)_k}{(c/a,c/b;q)_{n+1}(abq^{1-n}/c;q)_k}q^n=1-\frac{(c,c/ab;q)_\infty}{(c/a,c/b;q)_\infty}\sum_{j=0}^{k}\frac{c^j(a,b;q)_j}{a^jb^j(c,q;q)_j},\quad (k\geq 0) \tag{1.10}$$

Set $k=0$, replace c by cab and then put $b=0$ respectively in (1.10) we get the special case of companion identity as

$$(1-a)\sum_{n=0}^{\infty}\frac{(caq^{n+1};q)_\infty}{(cq^n;q)_\infty}q^n=\frac{(ca;q)_\infty}{c(c;q)_\infty}-\frac{1}{c} \tag{1.11}$$

Zhang & Wang [10] provided the Leibnitz rule for θ^n, for $n\geq 0$

$$\theta^n\{f(a)g(a)\}=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}\theta^n\{f(a)\}\theta^{n-k}\{g(aq^{-k})\}. \tag{1.12}$$

Lemma: *For non negative integer n, Zhang & Wang [10] established the following*

$$\theta^n\left\{\frac{(at\,;q)_\infty}{(av\,;q)_\infty}\right\}=v^nq^{-\binom{n}{2}}(t/v\,;q)_n\frac{(at\,;q)_\infty}{(avq^{-n}\,;q)_\infty} \tag{1.13}$$

$$\theta^n\left(c^{-k}\right)=\begin{cases}0, & if\ \ n>k,\\ (-1)^nq^n\left(q^k;q\right)_nc^{-k-n}, & if\ \ n\leq k\end{cases} \tag{1.14}$$

2. Main results

Theorem 2.1: *Let* $E(d\theta)$ *is operator defined by the equation (1.9). Then*

$$E(d\theta)\left\{c^{-k}\right\}=c^{-k}\sum_{j=0}^{\infty}\frac{(q^k;q)_j}{(q;q)_j}q^{\binom{j}{2}}\left(-\frac{dq}{c}\right)^j \tag{2.1.1}$$

Proof: We have

$$E(d\theta)=\sum_{j=0}^{\infty}\frac{(d\theta)^j}{(q;q)_j}q^{\binom{j}{2}} \tag{2.1.2}$$

Multiply by $\{c^{-k}\}$ on both sides of (2.1.2),then simplify by using (1.14), we get (2.1.1).

Theorem 2.2: *Let* $E(d\theta)$ *is operator defined by the equation (1.9). Then*

$$E(d\theta)\left\{\frac{(ca;q)_\infty}{(c;q)_\infty}c^{-k}\right\}=\frac{(ca;q)_\infty}{(c;q)_\infty}c^{-k}\sum_{m=0}^{\infty}\frac{(a;q)_m q^{\binom{m+1}{2}}}{(q;q)_m(q/c;q)_m}\left(-\frac{dq^k}{c}\right)^m\sum_{j=0}^{\infty}\frac{\left(q^k;q\right)_j q^{\binom{j+1}{2}}}{\left(q;q\right)_j}\left(-\frac{dq^m}{c}\right)^j \tag{2.2.1}$$

where c *is treating as variable.*

Proof: We have

$$E(d\theta)\left\{\frac{(ca;q)_\infty}{(c;q)_\infty}c^{-k}\right\}=\sum_{j=0}^{\infty}\frac{d^j q^{\binom{j}{2}}}{(q;q)_j}\theta^j\left\{\frac{(ca;q)_\infty}{(c;q)_\infty}c^{-k}\right\} \tag{2.2.2}$$

Apply Leibnitz rule given by (1.12) on right hand side (R.H.S) of (2.2.2), we get

$$E(d\theta)\left\{\frac{(ca;q)_\infty}{(c;q)_\infty}c^{-k}\right\}=\sum_{j=0}^{\infty}\frac{d^j}{(q;q)_j}q^{\binom{j}{2}}\sum_{m=0}^{j}\begin{bmatrix} j\\ m\end{bmatrix}\theta^m\left\{\frac{(ca;q)_\infty}{(c;q)_\infty}\right\}\theta^{j-m}\left\{\left(cq^{-m}\right)\right\}^{-k} \tag{2.2.3}$$

Changing the order of summation, replace j by $j+m$ and then simplify R.H.S by applying the two operators defined by (1.13) and (1.14), we get the required result (2.2.1).

Theorem 2.3: *Let* $E(d\theta)$ *is operator defined by the equation (1.8). Then*

$$E\left(d\theta\right)\left\{\sum_{n=0}^{\infty}\frac{(caq^{n+1};q)_\infty}{(cq^n;q)_\infty}q^n c^{-k+1}\right\}=c^{-k+1}\sum_{n=0}^{\infty}\frac{(caq^{n+1};q)_\infty}{(cq^n;q)_\infty}q^n\sum_{m=0}^{\infty}\frac{\left(aq;q\right)_m}{\left(q;q\right)_m\left(q^{1-n}/c;q\right)_m}q^{\binom{m}{2}}\left(\frac{-dq^k}{c}\right)^m\sum_{j=0}^{\infty}\frac{(q^{k-1};q)_j}{\left(q;q\right)_j}q^{\binom{j+1}{2}}\left(-\frac{d}{c}\right)^j \tag{2.3.1}$$

where c *is treating as variable.*

Proof: *We have*

$$E\left(d\theta\right)\left\{\sum_{n=0}^{\infty}\frac{(caq^{n+1};q)_{\infty}}{(cq^{n};q)_{\infty}}q^{n}c^{-k+1}\right\}=\sum_{n=0}^{\infty}q^{n}E\left(d\theta\right)\left\{\frac{(caq^{n+1};q)_{\infty}}{(cq^{n};q)_{\infty}}c^{-k+1}\right\} \tag{2.3.2}$$

The R.H.S. of (2.3.2) can be written as

$$\sum_{n=0}^{\infty}q^{n}E\left(d\theta\right)\left\{\frac{(caq^{n+1};q)_{\infty}}{(cq^{n};q)_{\infty}}c^{-k+1}\right\}=\sum_{n=0}^{\infty}q^{n}\sum_{j=0}^{\infty}\frac{d^{j}}{(q;q)_{j}}q^{\binom{j}{2}}\theta^{j}\left\{\frac{(caq^{n+1};q)_{\infty}}{(cq^{n};q)_{\infty}}c^{-k+1}\right\} \tag{2.3.3}$$

Apply Leibnitz rule given by equation (1.12) on R.H.S. of (2.3.3), we get

$$\sum_{n=0}^{\infty}q^{n}E\left(d\theta\right)\left\{\frac{(caq^{n+1};q)_{\infty}}{(cq^{n};q)_{\infty}}c^{-k+1}\right\}=\sum_{n=0}^{\infty}q^{n}\sum_{j=0}^{\infty}\frac{d^{j}}{(q;q)_{j}}q^{\binom{j}{2}}\sum_{m=0}^{j}\begin{bmatrix}j\\m\end{bmatrix}\theta^{m}\left\{\frac{(caq^{n+1};q)_{\infty}}{(cq^{n};q)_{\infty}}\right\}\theta^{j-m}\left\{cq^{-m}\right\}^{-k+1} \tag{2.3.4}$$

Changing the order of summation, replace j by $j+m$ and then simplify R.H.S. of (2.3.4) by applying the two operators defined by (1.13) and (1.14), we get the required result (2.3.1).

Theorem 2.4: *We have*

$$c\sum_{n=0}^{\infty}\frac{(caq^{n+1};q)_{\infty}}{(cq^{n};q)_{\infty}}q^{n}\sum_{m=0}^{\infty}\frac{\left(aq;q\right)_{m}}{\left(q;q\right)_{m}\left(q^{1-n}/c;q\right)_{m}}q^{\binom{m}{2}}\left(\frac{-dq^{k}}{c}\right)^{m}\sum_{j=0}^{\infty}\frac{(q^{k-1};q)_{j}}{\left(q;q\right)_{j}}q^{\binom{j+1}{2}}\left(-\frac{d}{c}\right)^{j}$$

$$=\frac{(ca;q)_{\infty}}{(c;q)_{\infty}}\sum_{m=0}^{\infty}\frac{(a;q)_{m}q^{\binom{m+1}{2}}}{(q;q)_{m}(q/c;q)_{m}}\left(-\frac{dq^{k}}{c}\right)^{m}\sum_{j=0}^{\infty}\frac{\left(q^{k};q\right)_{j}q^{\binom{j+1}{2}}}{\left(q;q\right)_{j}}\left(-\frac{dq^{m}}{c}\right)^{j}-\sum_{j=0}^{\infty}\frac{(q^{k};q)_{j}}{(q;q)_{j}}q^{\binom{j}{2}}\left(-\frac{dq}{c}\right)^{j} \tag{2.4.1}$$

Proof: To prove theorem (2.4) recall the equation (1.11) as

$$(1-a)\sum_{n=0}^{\infty}\frac{(caq^{n+1};q)_{\infty}}{(cq^{n};q)_{\infty}}q^{n}=\frac{(ca;q)_{\infty}}{c(c;q)_{\infty}}-\frac{1}{c} \tag{2.4.2}$$

Multiply by c^{-k} on both sides of (2.4.2), we get

$$(1-a)\sum_{n=0}^{\infty}q^{n}\frac{(caq^{n+1};q)_{\infty}}{(cq^{n};q)_{\infty}}c^{-k+1}=\frac{(ca;q)_{\infty}}{(c;q)_{\infty}}c^{-k}-c^{-k}. \tag{2.4.3}$$

Apply the operator defined by equation (1.8) on both sides of (2.4.3) with respect to variable c, we get

$$(1-a)E\left(d\theta\right)\left\{\sum_{n=0}^{\infty}q^{n}\frac{(caq^{n+1};q)_{\infty}}{(cq^{n};q)_{\infty}}c^{-k+1}\right\}=E\left(d\theta\right)\left\{\frac{(ca;q)_{\infty}}{(c;q)_{\infty}}c^{-k}\right\}-E\left(d\theta\right)\left\{c^{-k}\right\} \tag{2.4.4}$$

Using theorem 2.1, theorem 2.2 and theorem 2.3 on equation (2.4.4), we get the complete proof of the theorem (3.1.1).

Now, we find the special case of the theorem 2.4.

Special case 2.5: If we set $j=0$ in equation (2.4.1), we get

$$\sum_{n=0}^{\infty}\frac{(caq^{n+1};q)_{\infty}}{(cq^{n};q)_{\infty}}q^{n}\sum_{m=0}^{\infty}\frac{\left(aq;q\right)_{m}}{\left(q;q\right)_{m}\left(q^{1-n}/c;q\right)_{m}}q^{\binom{m}{2}}\left(\frac{-dq^{k}}{c}\right)^{m}=\frac{1}{c}\frac{(ca;q)_{\infty}}{(c;q)_{\infty}}\sum_{m=0}^{\infty}\frac{(a;q)_{m}q^{\binom{m+1}{2}}}{(q;q)_{m}(q/c;q)_{m}}\left(-\frac{dq^{k}}{c}\right)^{m}-\frac{1}{c} \tag{2.5.1}$$

Also, if we set $m=0$ in (2.5.1), we get (1.11).

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