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SYNCHRONIZATION BETWEEN TWO NON IDENTICAL FRACTIONAL ORDER HYPERCHAOTIC SYSTEMS

AYUB KHAN[1], DINESH KHATTAR[2], NEHA AGRAWAL[3,*]

[1]Department of Mathematics, Jamia Millia Islamia, Delhi 110025, India

[2]Department of Mathematics,Kirori Mal College, University of Delhi, Delhi 110007, India

[3]Department of Mathematics, University of Delhi, Delhi 110007, India

Abstract. In this paper we have investigated chaos synchronization between the two non- identical fractional order hyperchaotic systems using feedback control technique.The hyperchaotic system introduced by Xin and Ling has been synchronized with the Lü like hyperchaotic system.The analytical conditions for the synchronization of this pair of different fractional order hyperchaotic systems are derived by using Laplace transform. Numerical simulations are carried out using Matlab to show the effectiveness of the method.

Keywords: synchronization; hyperchaotic system; Lü like system.

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1. Introduction

Chaotic systems are characterized by their sensitive dependence on initial conditions.Chaos synchronization has attracted a great deal of attention since the seminal work by Pecora and

*Corresponding author

E-mail address: neha.maths10@gmail.com

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Carroll [1] in which they established a chaos synchronization scheme for two identical chaotic systems with different initial conditions.

As fractional order chaotic behaviour has a wide range of applications in image processing, secure communication, information encrypion, etc.[2-5]. Therefore, synchronization of fractional order chaotic systems gained a lot of attention due to its importance in applications in biology [6], economics [7], physics [8] etc.Some of the approaches which have been presented to achieve chaos synchronization in fractional order chaotic systems are adaptive control method [9-11], sliding mode control method [12-13], linear and non-linear feedback control method [14-16], active control method [17], PC control [18], backstepping control method [19], etc.

Recently, the study of dynamics of fractional-order chaotic systems has received interest of many researchers. Yu and Li in [20] used Laplace transformation theory and variational iteration method to study Rössler system, Wu, Lu and Shen discussed the synchronization of a new fractional- order hyperchaotic system via active control [21], Wang, Yu and Diao in [22] studied the hybrid projective synchronization between fractional-order chaotic systems of different dimensions, Sahab and Ziabari [23] analyzed the chaos between two different hyperchaotic systems by generalized backstepping method, S.T. Mohammad and H. Mohammad in [24] proposed a controller based on active sliding mode theory to synchronize the chaotic fractional order systems, Zhang and Lu introduced a new type of hybrid synchronization called full state hybrid lag projective synchronization and applied it to the Rössler system and the hyperchaotic Lorenz system to verify their results numerically [25]. A. Ouannas in [26] studied the Q-S synchronization of chaotic dynamical systems in continuous-time, Boutefnouchet, Taghvafard and Erjaee in their paper [27] discussed the phase synchronization in coupled chaotic systems.

This paper is organized as follows: in section 2, the fractional order derivative and its approximation is given. In section 3, the synchronization between the two non-identical fractional-order hyperchaotic systems using feedback control method is discussed. Section 4 presentS the numerical results to verify the effectiveness of the method. Finally, the conclusion is given in section 5.

2. Fractional order Derivative and its Approximation

Fractional calculus is a generalization of integration and differentiation to a non-integer-order integro-differential operator ${}_aD_t^q$ defined by

$$
{}_aD_t^q = \begin{cases} \frac{d^q}{dt^q} & \text{if } \mathbb{R}(q) > 0 \\ 1 & \text{if } \mathbb{R}(q) = 0 \\ \int_a^t (d\tau)^{-q} & \text{if } \mathbb{R}(q) < 0 \end{cases}
$$

where q is the fractional order which can be a complex number,$\mathbb{R}(q)$ denotes the real part of q and $a < t$, where a is the fixed lower terminal and t is the moving upper terminal.

There are two commonly used definitions for fractional derivatives [28], they are Grunward -Letnikov definition and Riemann- Liouville definition. The Riemann- Liouville definition is given by

$$
D^q x(t) = \frac{d^\eta}{dt^\eta} J^{\eta - q} x(t), \qquad q > 0
$$

where η is the first integer that is not less than q , J^β is the β - order Riemann- Liouville integral operator defined as follows:

$$
J^\beta f(t) = \frac{1}{\Gamma(\beta)} \int_0^t f(t)(t-\tau)^{\beta-1} d\tau
$$

where $\Gamma(.)$ is the Gamma function, $0 < \beta \leq 1$.

The Laplace transform of the Riemann- Liouville fractional derivative is given by

$$
L\left\{\frac{d^q f(t)}{dt^q}\right\} = s^q L\{f(t)\} - \sum_{k=0}^{n-1} s^k \left[\frac{d^{q-1-k} f(t)}{dt^{q-1-k}}\right]_{t=0}
$$

where L means the Laplace transform and s is a complex variable.Assuming the initial conditions to be zero, the above equation reduces to

$$
L\left\{\frac{d^q f(t)}{dt^q}\right\} = s^q L\{f(t)\}
$$

Thus the fractional integral operator of order ”q” can be represented by the transfer function $F(s) = \frac{1}{s^q}$ in the frequency domain.

The standard definitions of fractional order calculus do not allow direct implemetation of the fractional operators in time- domain simulations. An efficient method to circumvent this problem is to approximate fractional operators by using standard integer- order operators.In [29], an effective algorithm is developed to approximate fractional order transfer functions, which has been adopted in [30-32] and has sufficient accuracy for time domain implementations. In table 1 of [30], approximations for $\frac{1}{s^q}$ with q from 0.1 to 0.9 in steps 0.1 were given with errors of approximately $2dB$.We will use these approximations in the simulations.

3. Synchronization between the new fractional order hyperchaotic system and Lü like fractional order hyperchaotic system.

In this section, our goal is to achieve synchronization between the new fractional order hyperchaotic system and Lü like fractional order hyperchaotic system. The drive and response systems are given as follows:

As a drive system, consider a fractional order hyperchaotic system proposed by Xin and Ling in [33]

$$\left.\begin{aligned}
\frac{d^q x_1}{dt^q} &= a_1(x_2 - x_1) + x_4,\\
\frac{d^q x_2}{dt^q} &= b_1 x_1 + x_1 x_3 - x_4,\\
\frac{d^q x_3}{dt^q} &= -c_1 x_3 - d_1 x_1^2,\\
\frac{d^q x_4}{dt^q} &= c_1 x_1
\end{aligned}\right\} \tag{1}$$

where $(x_1,x_2,x_3,x_4) \in \mathbb{R}^4$ and a_1,b_1,c_1,d_1 are real parameters. When $a_1 = 10, b_1 = 40, c_1 = 2.5, d_1 = 4$ the system exhibits a hyperchaotic nature.The trajectories of the drive system are shown in figure 1. As a response system, consider the Lü like fractional order hyperchaotic system [14] given by

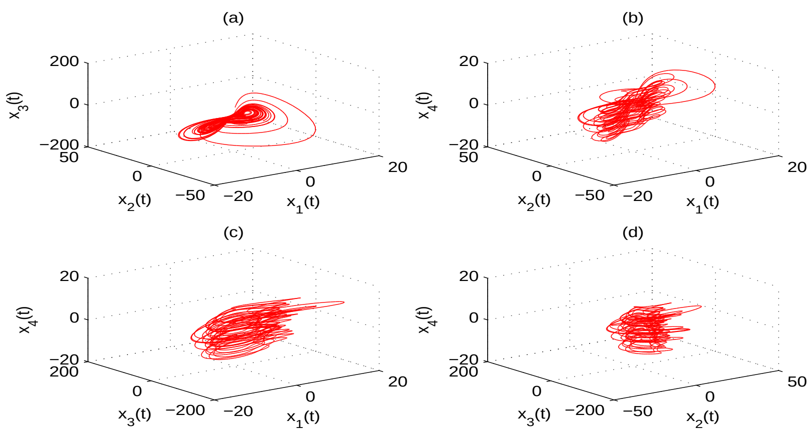

FIGURE 1. Phase Potraits of New fractional order hyperchaotic Dynmical system in (a) the $x_1-x_2-x_3$ space , (b) the $x_1-x_2-x_4$ space , (c) the $x_1-x_3-x_4$ space and (d) the $x_2-x_3-x_4$ space.

$$\left.\begin{aligned}
\frac{d^q y_1}{dt^q} &= a_2(y_2-y_1)+y_4+u_1,\\
\frac{d^q y_2}{dt^q} &= c_2y_1-y_1y_3+u_2,\\
\frac{d^q y_3}{dt^q} &= y_1y_2-b_2y_3+u_3,\\
\frac{d^q y_4}{dt^q} &= -y_1y_3+d_2y_4+u_4
\end{aligned}\right\} \tag{2}$$

where $(y_1,y_2,y_3,y_4)\in\mathbb{R}^4$, u_1,u_2,u_3 and u_4 are the linear or nonlinear control functions to be determined and a_2,b_2,c_2,d_2 are real parameters of the system. When $a_2=10, b_2=\frac{8}{3}, c_2=28, d_2=1.3$, the four Lyapunov exponents calculated with the help of Wolf Algorthm of the system are $L_1=0.7340, L_2=0.2492, L_3=0, L_4=-11.3437$. Since the system possesses two positive lyapunov exponents, it exhibits a hyperchaotic nature. The phase potraits of the response system is shown in fig.2.

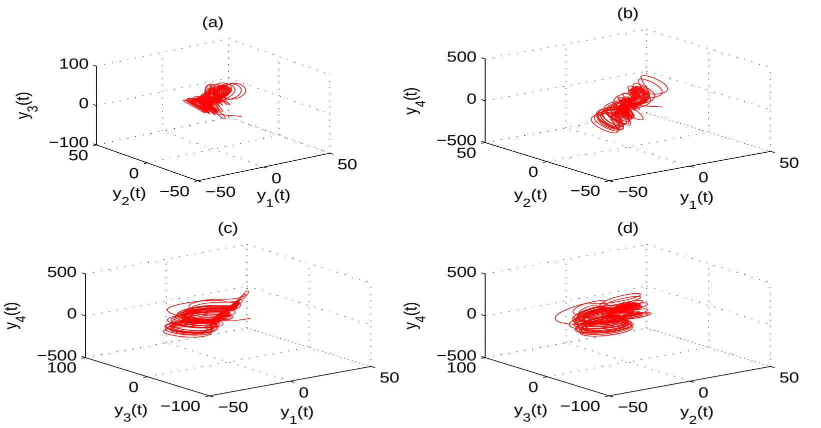

FIGURE 2. Phase Potraits of Lü like fractional order hyperchaotic Dynamical system in (a) the $y_1 - y_2 - y_3$ space , (b) the $y_1 - y_2 - y_4$ space , (c) the $y_1 - y_3 - y_4$ space and (d) the $y_2 - y_3 - y_4$ space.

Define the error functions as

$$\left.\begin{aligned} e_1 &= y_1 - x_1, \\ e_2 &= y_2 - x_2, \\ e_3 &= y_3 - x_3, \\ e_4 &= y_4 - x_4 \end{aligned}\right\} \tag{3}$$

Subtracting (1) from (2) and using (3), we get

$$\left.\begin{aligned} \frac{d^q e_1}{dt^q} &= a_2(e_2 - e_1) + e_4 + (a_2 - a_1)(x_2 - x_1) + u_1, \\ \frac{d^q e_2}{dt^q} &= (b_2 + c_2)e_1 - y_1 e_3 - y_1(2x_3 + b_1) + x_3 e_1 + c_2 x_1 + x_4 + u_2, \\ \frac{d^q e_3}{dt^q} &= y_1 y_2 - b_2 e_3 + (c_1 - b_2)x_3 + d_1 x_1^2 + u_3, \\ \frac{d^q e_4}{dt^q} &= -y_1 y_3 + d_2 e_4 + d_2 x_4 - c_1 x_1 + u_4 \end{aligned}\right\} \tag{4}$$

Choosing the control functions $u_i,\ \ i = 1,2,3,4$ as

$$\left.\begin{aligned} u_1 &= -a_2e_2 - e_4 + (a_1 - a_2)(x_2 - x_1) - k_1e_1, \\ u_2 &= -(b_2 + c_2)e_1 + y_1(2x_3 + b_1) + y_1e_3 - x_3e_1 - c_2x_1 - x_4 - k_2e_2, \\ u_3 &= -y_1y_2 + (b_2 - c_1)x_3 - d_1x_1^2, \\ u_4 &= y_1y_3 - d_2x_4 + c_1x_1 - k_4e_4 \end{aligned}\right\} \tag{5}$$

the error system (4) reduces to

$$\left.\begin{aligned} \tfrac{d^qe_1}{dt^q} &= -(a_2 + k_1)e_1, \\ \tfrac{d^qe_2}{dt^q} &= -k_2e_2, \\ \tfrac{d^qe_3}{dt^q} &= -b_2e_3, \\ \tfrac{d^qe_4}{dt^q} &= (d_2 - k_4)e_4 \end{aligned}\right\} \tag{6}$$

where $k_1, k_2, k_4 \geq 0$ are real parameters.

Taking Laplace transform [34] on both sides of equation (6) and letting $E_i(s) = L\{e_i(t)\}, i = 1,2,3,4$ and using

$$L\left\{\frac{d^qe_i}{dt^q}\right\} = s^qE_i(s) - s^{q-1}e_i(0), i = 1,2,3,4$$

we get

$$\left.\begin{aligned} s^qE_1(s) - s^{q-1}e_1(0) &= -(a_2 + k_1)E_1(s), \\ s^qE_2(s) - s^{q-1}e_2(0) &= -k_2E_2(s), \\ s^qE_3(s) - s^{q-1}e_3(0) &= -b_2E_3(s), \\ s^qE_4(s) - s^{q-1}e_4(0) &= (d_2 - k_4)E_4(s) \end{aligned}\right\} \tag{7}$$

If $E_1(s), E_2(s), E_3(s)$ and $E_4(s)$ are bounded and $d_2 - k_4 \neq 0$ then the drive and response systems will be synchronized with suitable choice of k_1, k_2 and k_4. Rewriting (7), we have

$$\left.\begin{aligned} E_1(s) &= \tfrac{s^{q-1}e_1(0)}{s^q+a_2+k_1}, \\ E_2(s) &= \tfrac{s^{q-1}e_2(0)}{s^q+k_2}, \\ E_3(s) &= \tfrac{s^{q-1}e_3(0)}{s^q+b_2}, \\ E_4(s) &= \tfrac{s^{q-1}e_4(0)}{s^q-d_2+k_4} \end{aligned}\right\} \tag{8}$$

We know that the *Final value theorem* of the Laplace transform is described as [34]

$$e_{i,ss} = \lim_{t\to\infty} e_i(t) = \lim_{s\to 0} sE_i(s)$$

Applying the above method to $E_i(s)$, we get

$$\begin{aligned} \lim_{t\to\infty} e_1(t) &= \lim_{s\to 0} sE_1(s) \\ &= \lim_{s\to 0} \frac{e_1(0)}{1+\frac{a_2+k_1}{s^q}} \\ &= 0 \end{aligned}$$

$$\begin{aligned} \lim_{t\to\infty} e_2(t) &= \lim_{s\to 0} sE_2(s) \\ &= \lim_{s\to 0} \frac{e_2(0)}{1+\frac{k_2}{s^q}} \\ &= 0 \end{aligned}$$

$$\begin{aligned} \lim_{t\to\infty} e_3(t) &= \lim_{s\to 0} sE_3(s) \\ &= \lim_{s\to 0} \frac{e_3(0)}{1+\frac{b_2}{s^q}} \\ &= 0 \end{aligned}$$

$$\begin{aligned}\lim_{t\to\infty} e_4(t) &= \lim_{s\to 0} sE_4(s)\\ &= \lim_{s\to 0}\frac{e_4(0)}{1+\frac{k_4-d_2}{s^q}}\\ &= 0\end{aligned}$$

Since $E_1(s), E_2(s), E_3(s), E_4(s)$ are bounded and $d_2 - k_4 \neq 0$, there exists a $\eta > 0$(owing to attractiveness of the attractor (1) and (2)) such that $|x_i(t)| \leq \eta < \infty$ and $|y_i(t)| \leq \eta < \infty$, $i = 1,2,3,4$.

Therefore, $lim_{t\to\infty} e_i(t) = 0 \quad \forall\ i = 1,2,3,4$, which implies that the synchronization between the drive and the response systems (1) and (2) is achieved.

4. Numerical simulations

Based on the scheme discussed above, the systems (1) and (2) are integrated numerically with fractional order $q = 0.95$ and the initial conditions $[x_1(0), x_2(0), x_3(0), x_4(0)] = [1,2,3,4]$ and $[y_1(0), y_2(0), y_3(0), y_4(0)] = [10,-15,-10,11]$ respectively. Therefore the initial condition-s for the error system are $[e_1(0), e_2(0), e_3(0), e_4(0)] = [9,-17,-13,7]$.The chaotic tracjectories of the drive and response systems before and after the controllers are applied are shown in figures (3) and (4) respectively. It can be seen from figure 4 that the trajectories of the drive and response systems aymptotically synchronize and that from figure 5 that the error system converges to zero which shows that the systems (1) and (2) are synchronized.

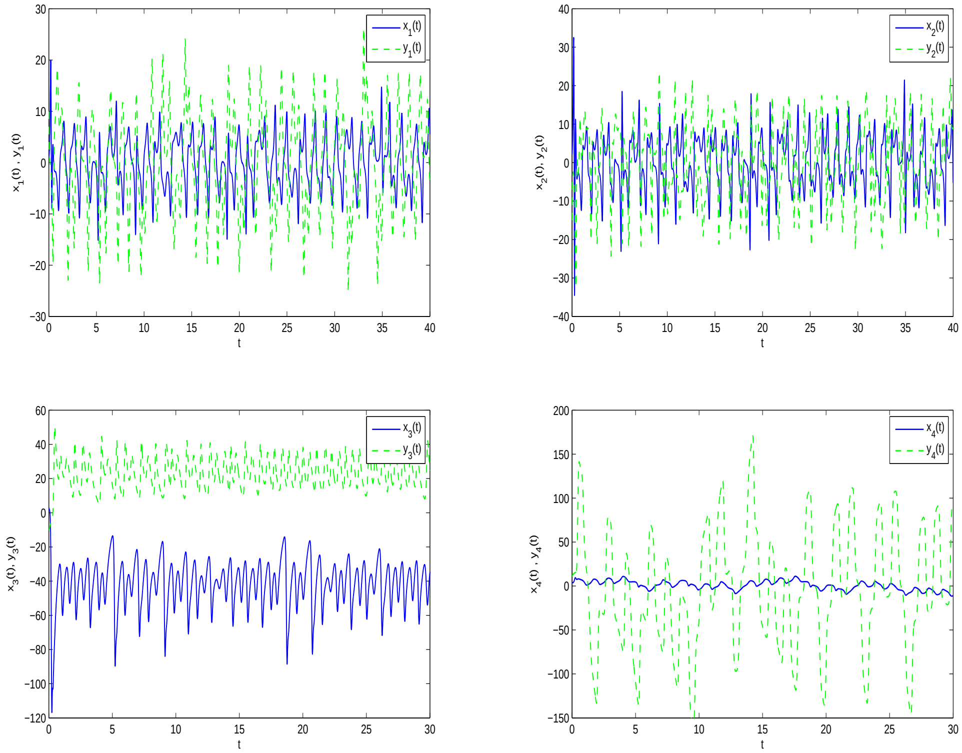

FIGURE 3. The chaotic tracjectories of the drive and response system before the controllers are applied.

5. Conclusion

Chaos synchronization between two fractional order hyperchaotic systems has been studied using feedback control technique for a short interval of time. The hyperchaotic system introduced by Xin and Ling has been used to drive the Lü like hyperchaotic system. Numerical simulations are carried out to show the effectiveness of the method.

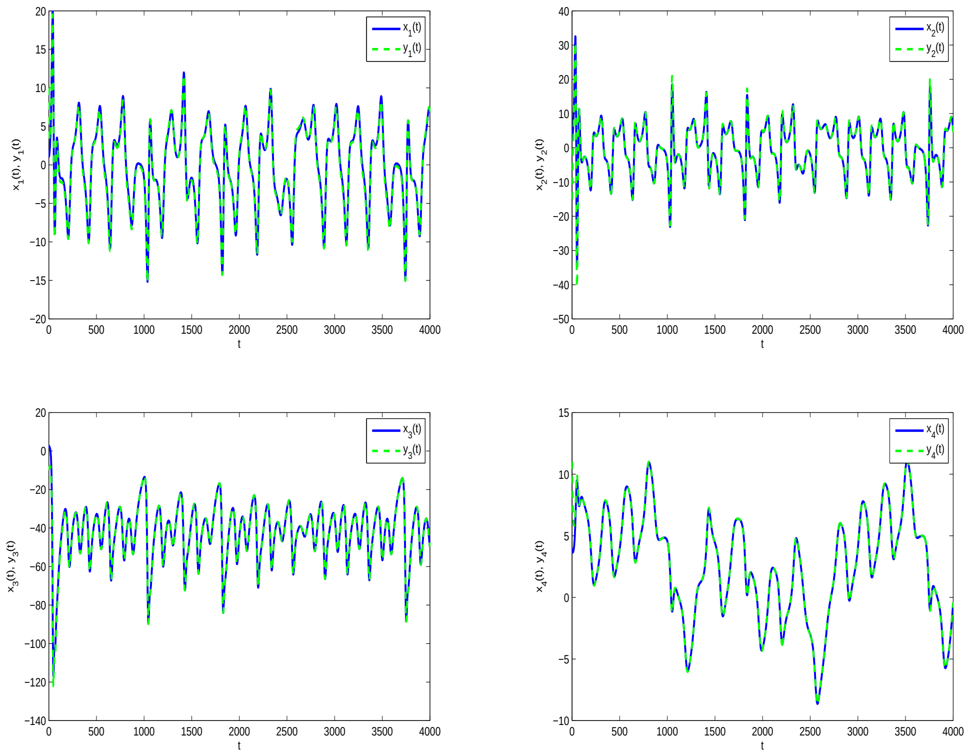

FIGURE 4. The chaotic tracjectories of the drive and response system after the controllers are applied.

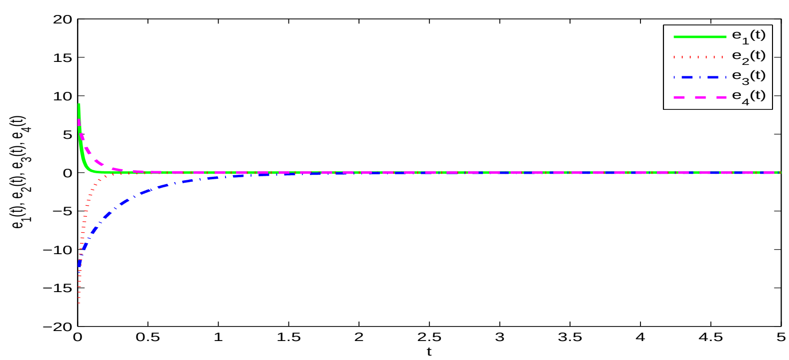

FIGURE 5. Error functions comparison of four state variables versus the time t.

Conflict of Interests

The authors declare that there is no conflict of interests.

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