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PROPERTIES OF CHARACTERISTIC POLYNOMIAL OF MARKER SET DISTANCE AND ITS LAPLACIAN

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Abstract. In our previous papers, we had introduced the marker set distance matrix and its eigenvalues and the marker set Laplacian eigenvalues. Also, expressions for the characteristic polynomials of the marker set distance matrix and its Laplacian had been found. In this paper, we discuss the properties of the characteristic polynomials of M-set distance matrix and its Laplacian.

Keywords: marker set of a graph; M-set distance matrix; M-set distance Laplacian; characteristic polynomial.

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1. Introduction

The study of matrices associated with graphs involves the study of their characteristic polynomials. More often than not, the characteristic polynomial of a matrix reveals a lot of information about the underlying graph. The coefficients and roots of the characteristic polynomial are another set of parameters whose relation to the underlying graph is an interesting premise which we are looking in to in this paper. We refer the reader to our earlier papers [12], [13], [14] for

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a detailed study of marker set distance and matrix and its characteristic polynomial as well as that of marker set Laplacian. For easy understanding, we list out a few of the definitions and results here again.

2. Preliminaries

Definition 2.1. [12] Let $G=(V,E)$ be a simple connected graph of order p. Let M be a subset of vertices of G, referred to as a marker set or M-set (in short). M-set distance between two vertices v_i and v_j is defined as $d_M(v_i,v_j)=d_{ij}=|d(v_i,M)-d(v_j,M)|$. Here $d(v_i,M)=\min\{d(v_i,w): w\in M\}$. The $p\times p$ matrix $D_M(G)=[d_{ij}]$ is called the *M-set distance matrix* of the marker set M in the graph G. The characteristic polynomial can be written as $\Phi(G:M,\mu)=\Delta(D_M(G)=\mu^p-S_1\mu^{p-1}+S_2\mu^{p-2}-\ldots+(-1)^pS_p$. It is clear from [15] that $(-1)^iS_i=\Sigma M_{D_i}$ where M_{D_i} are the principal minors of $D_M(G)$ with order i. (Minors whose diagonal elements belong to the main diagonal of $D_M(G)$). $S_0=1$ and $S_1=traceD_M(G)=0$.

Definition 2.2. [12] The M-set eccentricity of a vertex v of G, denoted by $e_M(v)$ is defined as the maximum of all the M-set distances of v.

Definition 2.3. [12] The M-set diameter of a graph G with respect to a marker set M is denoted by $diam_M(G)$ and is defined as the maximum of all the M-set eccentricities of the vertices of G.

Definition 2.4. [13] Given a simple connected graph G and a marker set M of G, the distance degree sequence denoted by $DDS_G(M)$ can be defined as

$DDS_G(M)=(k_0,k_1,k_2,\ldots,k_n)$ written in a non-decreasing order where k_i is the number of vertices of G at distance i from M where, $0\le i\le m$ and $m=diam_M(G)$.

Definition 2.5. [12],[13] Let G be a simple connected graph of order p and M be a marker set with $|M|=k$ and $diam_M(G)=m$. Let k_i be the number of vertices of G at M-distance i $(1\le i\le m)$ so that $k+\Sigma_{i=1}^m k_i=p$. Permuting the vertices in such a way that the first k vertices are at M-distance 0 from the set M, the next k_1 vertices are at distance 1 from the set M, the next k_2 vertices at distance 2 from the set M and so on. The M-distance matrix is given by

$$D_M(G) = \begin{bmatrix} 0_{k\times k} & 1_{k\times k_1} & 2_{k\times k_2} & \ldots & m_{k\times k_m} \\ 1_{k_1\times k} & 0_{k_1\times k_1} & 1_{k_1\times k_2} & \ldots & (m-1)_{k_1\times k_m} \\ 2_{k_2\times k} & 1_{k_2\times k_1} & 0_{k_2\times k_2} & \ldots & (m-2)_{k_2\times k_m} \\ . & . & . & \ldots & . \\ . & . & . & \ldots & . \\ . & . & . & \ldots & . \\ m_{k_m\times k} & (m-1)_{k_m\times k_1} & (m-2)_{k_m\times k_2} & \ldots & 0_{k_m\times k_m} \end{bmatrix}$$

where $j_{t_1\times t_2}$ is a matrix of order $t_1 \times t_2$ with all entries equal to $j, 0 \leq j \leq m$ and $t_1, t_2 \in \{k, k_1, k_2, ...k_m\}$. This is called the *Standard form of marker set distance matrix.*

Definition 2.6. [13] Let $G = (V,E)$ be a simple connected graph of order p. Let $M \subseteq V(G)$ be a non empty marker set of G. The M-set distance Laplacian is defined as $L_M(G) = D_M(G) - diag[DDS_G(M)]$,
where $D_M(G)$ is the marker set distance matrix in the standard form.

We recall a theorem.

Theorem 2.1. [13] *Let G be a simple connected graph on p vertices and M be a dominating marker set of G with $|M| = k$. Then the characteristic polynomial of the M-set Laplacian is $\lambda^p - S_1\lambda^{p-1} + S_2\lambda^{p-2} - S_3\lambda^{p-3} + S_4\lambda^{p-4}$ where $S_1 = -p$, $S_2 = 0$,*

$$S_3 = \begin{cases} (p-1)(p-2), & \text{when} \quad k = 1 \\ (k-1)p(p-k), & \text{when} \quad k \geq 2 \end{cases}$$

$$S_4 = \begin{cases} 0, & \text{when} \quad k \leq 2 \\ -k(k-1)(p-k)^2, & \text{when} \quad k \geq 3 \end{cases}$$

3. Main results

Theorem 3.1. *A simple connected graph G of order p with a dominating set as a marker set M has integral M- set distance eigenvalues if and only if p is even and $|M| = p/2$.*

Proof. *Let G be a simple connected graph of order p with a dominating marker set M of cardinality k. Then the marker set distance matrix is given by* $D_M(G) = \begin{bmatrix} 0_{k\times k} & 1_{k\times(p-k)} \\ 1_{(p-k)\times k} & 0_{(p-k)\times(p-k)} \end{bmatrix}$

The characteristic polynomial of $D_M(G)$ *is hence given by* $\mu^p - k(p-k)\mu^{(p-2)} = 0$. *which implies* $\mu^{(p-2)}(\mu^2 - k(p-k)) = 0$ *and the the eigenvalues are* 0 *of multiplicity* $(p-k)$, $\pm\sqrt{k(p-k)}$ *of multiplicity* 1. $\sqrt{k(p-k)}$ *is an integer if and only if* $k = (p-k)$.*That is, if and only if* $k = p/2$ *and p is even.*

Theorem 3.2. *A simple connected graph G has two nonzero real skew M- set distance eigenvalues if and only if the M-set is a dominating set.*

Proof. *Let G be a simple connected graph of order p and M be a marker set of cardinality k. Let G have two nonzero real skew M- set distance eigenvalues say* $\pm l$. *Then the corresponding M-set distance characteristic polynomial is*
$x^{p-2}(x^2 - l^2) = 0$. *This corresponds to the M-set distance matrix*
$D_M(G) = \begin{bmatrix} 0_{l\times l} & 1_{l\times l} \\ 1_{l\times l} & 0_{l\times l} \end{bmatrix}$. *This corresponds to a simple connected graph G of order 2l and a dominating marker set M of cardinality l.*
The converse part can be proved by reversing the arguments in the proof above.

Now, we define marker set distance isomorphic graphs.

Definition 3.1. *Two simple connected graphs* G_1 *and* G_2 *are said to be marker set distance isomorphic if there exist marker sets* M_1 *of* G_1 *and* M_2 *of* G_2 *such that* $DDS_{M_1}(G) = DDS_{M_2}(G)$.

For any two marker set distance isomorphic graphs G_1 *and* G_2 *with marker sets* M_1 *of* M_2 *respectively, the graph* $G_1 - M_1$ *is isomorphic to the graph* $G_2 - M_2$.
A graph G is said to be determined by its spectrum if there exists no cospectral nonisomorphic pair. Such graphs are called DS graphs [19].

We now define DS graphs with respect to their marker set distance spectrum as follows.
Definition 3.2. *A graph G with a marker set M is said to be marker set distance DS graph if there exists no marker set cospectral marker set isomorphic graph.*

We have characterised the marker set distance spectrum as follows.

Theorem 3.3. *A simple connected graph G with a dominating set as a marker set is determined by its spectrum.*

Proof. *The proof is obvious from Theorem 3.2.*

Lemma 3.4. *A simple connected graph G with a dominating marker set has atmost four nonzero M-set distance Laplacian eigenvalues.*

Proof. *Let G be a simple connected graph on p vertices and M be a dominating marker set of G with $|M| = k$.*
Case 1: *Let $k = 1$. Then the characteristic polynomial of the M-set Laplacian is $\lambda^p + p\lambda^{p-1} - (p-1)(p-2)\lambda^{p-3}$ and hence has 3 nonzero roots. Therefore, G has 3 nonzero eigenvalues.*
Case 2: *Let $k = 2$. Then the characteristic polynomial of the M-set Laplacian is $\lambda^p + p\lambda^{p-1} - p(p-2)\lambda^{p-3}$ and hence has 3 nonzero roots. Therefore, G has 3 nonzero eigenvalues.*
Case 3: *Let $k \geq 3$. Then the characteristic polynomial of the M-set Laplacian is $\lambda^p + p\lambda^{p-1} - (k-1)p(p-k)\lambda^{p-3} + k(k-1)(p-k)^2\lambda^{p-4}$ and hence has 4 nonzero roots. Therefore, G with marker set M has 4 nonzero marker set eigenvalues.*

Theorem 3.5. *Let G be a graph on p vertices and M be a marker set of cardinality k_0. Let $diam_M(G) = r$ and $dds_M(G) = (k_0, k_1, k_2, k_3, \ldots, k_r)$. Then the characteristic polynomial of the marker set distance matrix is given by $\mu^p - S_1\mu^{p-1} + S_2\mu^{p-2} - \ldots + S_r\mu^{p-r}$. $S_0 = 1$ and $S_1 = traceD_M(G) = 0$,*

$$S_i = \sum_{0 \leq u_1 < u_2 < u_3 < \ldots < u_i \leq r} \prod_{m=1}^{i} k_{u_m} \begin{vmatrix} 0 & u_2 - u_1 & u_3 - u_1 & \ldots & u_i - u_1 \\ u_2 - u_1 & 0 & u_3 - u_2 & \ldots & u_i - u_2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ u_i - u_1 & u_i - u_2 & \ldots & \ldots & 0 \end{vmatrix}$$

where $2 \leq i \leq (r-1)$ and

$$S_r = \prod_{i=0}^{r} k_{u_i} \begin{vmatrix} 0 & u_2 - u_1 & u_3 - u_1 & \ldots & u_r - u_1 \\ u_2 - u_1 & 0 & u_3 - u_2 & \ldots & u_r - u_2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ u_r - u_1 & u_r - u_2 & \ldots & \ldots & 0 \end{vmatrix}$$

Proof. *The M-set distance matrix is given by*

$$D_M(G) = \begin{bmatrix} 0_{k_0\times k_0} & 1_{k_0\times k_1} & 2_{k_0\times k_2} & \dots & m_{k_0\times k_r} \\ 1_{k_1\times k_0} & 0_{k_1\times k_1} & 1_{k_1\times k_2} & \dots & (r-1)_{k_1\times k_r} \\ 2_{k_2\times k_0} & 1_{k_2\times k_1} & 0_{k_2\times k_2} & \dots & (r-2)_{k_2\times k_r} \\ . & . & . & \dots & . \\ . & . & . & \dots & . \\ . & . & . & \dots & . \\ r_{k_r\times k_0} & (r-1)_{k_{r\times k_1}} & (r-2)_{k_r\times k_2} & \dots & 0_{k_r\times k_r} \end{bmatrix}$$

The characteristic polynomial can be written as $\mu^p - S_1\mu^{p-1} + S_2\mu^{p-2} - \ldots + (-1)^p S_p$. *It is clear from* [12] *that* $(-1)^i S_i = \Sigma M_{D_i}$ *where* M_{D_i} *are the principal minors of* $D_M(G)$ *with order* i. *(Minors whose diagonal elements belong to the main diagonal of* $D_M(G)$*).* $S_0 = 1$ *and* $S_1 = trace D_M(G) = 0$. $D_M(G)$ *has r distinct rows and each row occurs* k_i *times in the matrix. The* $i\times i$ *minors can be chosen from these rows in* $\binom{r}{i}$ *ways. Each row of the minor can be chosen in* k_{u_m} *ways where* $0 \le m \le r$. *Hence*

$$S_i = \sum_{0\le u_1<u_2<u_3<\ldots<u_i\le r}\prod_{m=1}^{i} k_{u_m} \begin{vmatrix} 0 & u_2-u_1 & u_3-u_1 & \dots & u_i-u_1 \\ u_2-u_1 & 0 & u_3-u_2 & \dots & u_i-u_2 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ u_i-u_1 & u_i-u_2 & \dots & \dots & 0 \end{vmatrix}$$

where $2 \le i \le (r-1)$

Obviously,

$$S_r = \prod_{i=0}^{r} k_{u_i} \begin{vmatrix} 0 & u_2-u_1 & u_3-u_1 & \dots & u_r-u_1 \\ u_2-u_1 & 0 & u_3-u_2 & \dots & u_r-u_2 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ u_r-u_1 & u_r-u_2 & \dots & \dots & 0 \end{vmatrix}.$$

Corollary 3.6. *Let G be a graph on p vertices and M be a marker set of cardinality* k_0. *Let* $diam_M(G) = r$ *and* $dds_M(G) = (k_0, k_1, k_2, k_3, \ldots, k_r)$. *Then the characteristic polynomial of the marker set distance matrix is given by* $\mu^p - S_1\mu^{p-1} + S_2\mu^{p-2} - \ldots + S_r\mu^{p-r}$ *where* $S_2 = \sum_{0\le i<j\le r}\{-(j-i)^2 k_i k_j\}$.

Example 3.1. *For $M = \{u_4, u_5\}$ a marker set of the graph G (Figure 1), $DDS_G(M) = (2,1,1,1)$.*

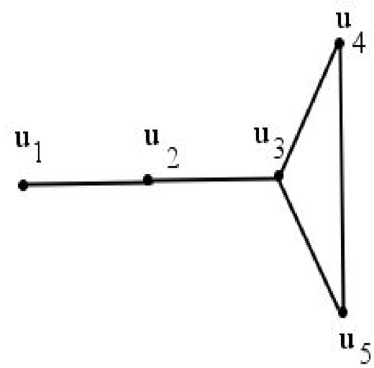

FIGURE 1

$$D_M(G) = \begin{bmatrix} 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 & 3 \\ 1 & 1 & 0 & 1 & 2 \\ 2 & 2 & 1 & 0 & 1 \\ 3 & 3 & 2 & 1 & 0 \end{bmatrix}$$

The characteristic polynomial of the M-set distance matrix is $\mu^5 - 34\mu^3 - 60\mu^2 - 24\mu$. We shall verify the previous theorem now.
According to the theorem,

$$S_i = \sum_{0 \le u_1 < u_2 < u_3 < \ldots < u_i \le r} \prod_{m=1}^{i} k_{u_m} \begin{vmatrix} 0 & u_2 - u_1 & u_3 - u_1 & \ldots & u_i - u_1 \\ u_2 - u_1 & 0 & u_3 - u_2 & \ldots & u_i - u_2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ u_i - u_1 & u_i - u_2 & \ldots & \ldots & 0 \end{vmatrix}$$

where $2 \le i \le (r-1)$

$$S_2 = \sum_{0\le u_1<u_2\le 3}\prod_{m=1}^{2} k_{u_m}\begin{vmatrix} 0 & u_2-u_1 \\ u_2-u_1 & 0 \end{vmatrix}$$

$$= \sum_{0\le u_1<u_2\le 3} k_{u_1}k_{u_2}\begin{vmatrix} 0 & u_2-u_1 \\ u_2-u_1 & 0 \end{vmatrix}$$

$$= k_0k_1\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + k_0k_2\begin{vmatrix} 0 & 2 \\ 2 & 0 \end{vmatrix} + k_0k_3\begin{vmatrix} 0 & 3 \\ 3 & 0 \end{vmatrix} + k_1k_2\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + k_1k_3\begin{vmatrix} 0 & 2 \\ 2 & 0 \end{vmatrix} + k_2k_3\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$$

$$= -2-8-18-1-4-1$$

$$= -34.$$

$$S_3 = \sum_{0\le u_1<u_2<u_3\le 3}\prod_{m=1}^{3} k_{u_m}\begin{vmatrix} 0 & u_2-u_1 & u_3-u_1 \\ u_2-u_1 & 0 & u_3-u_2 \\ u_3-u_1 & u_3-u_2 & 0 \end{vmatrix}$$

$$= \sum_{0\le u_1<u_2<u_3\le 3} k_{u_1}k_{u_2}k_{u_3}\begin{vmatrix} 0 & u_2-u_1 & u_3-u_1 \\ u_2-u_1 & 0 & u_3-u_2 \\ u_3-u_1 & u_3-u_2 & 0 \end{vmatrix}$$

$$= k_0k_1k_2\begin{vmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{vmatrix} + k_0k_1k_3\begin{vmatrix} 0 & 1 & 3 \\ 1 & 0 & 2 \\ 3 & 2 & 0 \end{vmatrix} + k_0k_2k_3\begin{vmatrix} 0 & 2 & 3 \\ 2 & 0 & 1 \\ 3 & 1 & 0 \end{vmatrix} + k_1k_2k_3\begin{vmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{vmatrix}$$

$$= 8+24+24+4$$

$$= 60$$

$$S_4 = \sum_{0\le u_1<u_2<u_3<u_4\le 3}\prod_{m=1}^{4} k_{u_m}\begin{vmatrix} 0 & u_2-u_1 & u_3-u_1 & u_4-u_1 \\ u_2-u_1 & 0 & u_3-u_2 & u_4-u_2 \\ u_3-u_1 & u_3-u_2 & 0 & u_4-u_3 \\ u_4-u_1 & u_4-u_2 & u_4-u_3 & 0 \end{vmatrix}$$

$$= \sum_{0\leq u_1<u_2<u_3<u_4\leq 3} k_{u_1}k_{u_2}k_{u_3}k_{u_4} \begin{vmatrix} 0 & u_2-u_1 & u_3-u_1 & u_4-u_1 \\ u_2-u_1 & 0 & u_3-u_2 & u_4-u_2 \\ u_3-u_1 & u_3-u_2 & 0 & u_4-u_3 \\ u_4-u_1 & u_4-u_2 & u_4-u_3 & 0 \end{vmatrix}$$

$$= k_0k_1k_2k_3k_4 \begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{vmatrix} .$$

$$= -24.$$

Hence, the theorem is verified.

Theorem 3.7 *Let G be a simple connected graph on p vertices and M be a marker set of cardinality k_0. Let $diam_M(G) = r$ and $DDS_M(G) = (k_0, k_1, k_2, k_3, \ldots, k_r)$. The marker set distance Laplacian matrix of G has atleast r nonzero eigenvalues.*

Proof. *The marker set distance Laplacian matrix of G is given by $L_M(G) = D_M(G) - diag[DDS_G(M)]$. Since $(k_0, k_1, k_2, k_3, \ldots, k_r)$ are the only nonzero diagonal entries of the Laplacian, there are atleast r distinct rows and hence atleast r nonzero eigenvalues.*

Lemma 3.8. *Let G be a simple connected graph on p vertices and M be a marker set of cardinality k_0. Let $diam_M(G) = r$ and $DDS_M(G) = (k_0, k_1, k_2, k_3, \ldots, k_r)$. Then the characteristic polynomial of the marker set Laplacian matrix $L_M(G)$ of is $\lambda^p - S_1\lambda^{p-1} + S_2\lambda^{p-2} - \ldots + (-1)^p S_p$. $S_0 = 1$ and $S_1 = traceL_M(G) = -p$ and $S_2 = \sum_{i=0}^{r-2}\sum_{j=i+2}^{r}[(j-i)^2 - 1]k_ik_j$ with $k_0 \geq (r+1)$.*

Proof. *The marker set Laplacian matrix of graph $L_M(G) = D_M(G) - diag[DDS_G(M)]$, where $D_M(G)$ is the marker set distance matrix in the standard form given by*

$$D_M(G) = \begin{bmatrix} 0_{k_0\times k_0} & 1_{k_0\times k_1} & 2_{k_0\times k_2} & \dots & r_{k_0\times k_r} \\ 1_{k_1\times k_0} & 0_{k_1\times k_1} & 1_{k_1\times k_2} & \dots & (r-1)_{k_1\times k_r} \\ 2_{k_2\times k_0} & 1_{k_2\times k_1} & 0_{k_2\times k_2} & \dots & (r-2)_{k_2\times k_r} \\ . & . & . & \dots & . \\ . & . & . & \dots & . \\ . & . & . & \dots & . \\ r_{k_r\times k_0} & (r-1)_{k_0\times k_1} & (r-2)_{k_r\times k_2} & \dots & 0_{k_r\times k_r} \end{bmatrix}$$

It is clear from [15] *that* $(-1)^i S_i = \Sigma M_{L_{M(i)}}$, *where* $M_{L_{M(i)}}$ *are the principal minors of* $L_M(G)$ *with order i. (Minors whose diagonal elements belong to the main diagonal of* $L_M(G)$*).* $S_0 = 1$ *and* $S_1 = traceL_M(G) = -p$.

It can be seen that, the only nonzero entries of the principal diagonal of $L_M(G)$ *are the first* $r+1$ *entries* $k_0, k_1, k_2, \dots, k_r$. *All these nonzero principal diagonal entries occur only in the first block* $0_{k_0\times k_0}$ *as* $k_0 \geq (r+1)$. *Every subset of the principal diagonal gives a principal minor. The nonzero* 2×2 *principal minors and their sums are as follows.*

1. All minors of the form $\begin{vmatrix} -k_i & 0 \\ 0 & -k_j \end{vmatrix}$ *where* $0 \leq i \leq (r-1)$ *and* $1 \leq j \leq r$ *with* $i < j$. *These minors sum to*

$$\sum_{0\leq i<j\leq r} k_i k_j = \sum_{i=0}^{r-1} k_i k_{i+1} + \sum_{i=0}^{r-2} \sum_{j=i+2}^{r} k_i k_j.$$

.

2. All minors of the form $\begin{vmatrix} -k_j & i \\ i & 0 \end{vmatrix}$ *with* $1 \leq i \leq r$ *and* $0 \leq j \leq r$ *each of them occurring* k_i *times. These minors sum to*

$$(r+1) \sum_{1\leq i\leq r} (-i^2) k_i.$$

3. All minors of the form $\begin{vmatrix} 0 & i \\ i & 0 \end{vmatrix}$ *where* $1 \leq i \leq r$ *with each occurring* $(k_0 - r - 1)k_i$ *times. These sum to*

$$\sum_{1\leq i\leq r} (-i^2)(k_0 - r - 1)k_i = \sum_{i=1}^{r} (-i^2) k_0 k_i + (r+1) \sum_{i=1}^{r} i^2 k_i.$$

4. All minors of the form $\begin{vmatrix} 0 & (j-i) \\ (j-i) & 0 \end{vmatrix}$ *each occurring* $k_i k_j$ *times where* $1 \leq i \leq (r-1)$ *and*

$2 \leq j \leq r$. *These sum to*

$$\sum_{1\leq i<j\leq r}[-(j-i)^2]k_ik_j = \sum_{i=1}^{r-1}(-k_ik_{i+1}) + \sum_{i=1}^{r-2}\sum_{j=i+2}^{r}[-(j-i)^2]k_ik_j.$$

Hence, the sum of all the 2×2 *minors is*

$$\sum_{i=0}^{r-1}k_ik_{i+1} + \sum_{i=0}^{r-2}\sum_{j=i+2}^{r}k_ik_j + (r+1)\sum_{1\leq i\leq r}(-i^2)k_i + \sum_{i=1}^{r}(-i^2)k_0k_i$$

$$+(r+1)\sum_{i=1}^{r}i^2k_i + \sum_{i=1}^{r-1}(-k_ik_{i+1}) + \sum_{i=1}^{r-2}\sum_{j=i+2}^{r}[-(j-i)^2k_ik_j$$

which in turn implies that

$$S_2 = \sum_{i=0}^{r-2}\sum_{j=i+2}^{r}[(j-i)^2-1]k_ik_j.$$

Example 3.2. *For* $M = \{u_4, u_5\}$ *a marker set of the graph G (Figure 1),* $DDS_G(M) = (2,1,1,1)$.

$$L_M(G) = \begin{bmatrix} -2 & 0 & 1 & 2 & 3 \\ 0 & -1 & 1 & 2 & 3 \\ 1 & 1 & -1 & 1 & 2 \\ 2 & 2 & 1 & -1 & 1 \\ 3 & 3 & 2 & 1 & 0 \end{bmatrix}$$

The characteristic polynomial of the M-set distance Laplacian matrix is

$\lambda^5 + 5\lambda^4 - 25\lambda^3 - 164\lambda^2 - 280\lambda + 153$.

Using the above result,

$$\begin{aligned} S_2 &= \sum_{i=0}^{1}\sum_{j=i+2}^{3}[(j-i)^2-1]k_ik_j \\ &= [(2-0)^2-1](2)(1) + [(3-0)^2-1](2)(1) + [(3-1)^2-1](1)(1) \\ &= 6+16+3 \\ &= 25. \end{aligned}$$

Conflict of Interests

The authors declare that there is no conflict of interests.

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