Available online at http://scik.org
J. Math. Comput. Sci. 8 (2018), No. 3, 331-344
https://doi.org/10.28919/jmcs/3616
ISSN: 1927-5307

BIVARIATE JACOBSTHAL AND BIVARIATE JACOBSTHAL-LUCAS MATRIX POLYNOMIAL SEQUENCES

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Abstract. In this study, we consider sequences named bivariate Jacobsthal, bivariate Jacobsthal Lucas polynomial sequences. After that, by using these sequences, we define bivariate Jacobsthal and bivariate Jacobsthal-Lucas matrix polynomial sequences. Finally we investigate some properties of these sequences, present some important relationship between bivariate Jacobsthal matrix polynomial sequence and bivariate Jacobsthal Lucas matrix polynomial sequences.

Keywords: Jacobsthal sequence; Jacobsthal-Lucas sequence; matrix sequences; bivariate polynomial sequences.

2010 AMS Subject Classification: 11B83, 11B39, 15A24, 15B36.

1. Introduction

Integer sequences, such as Fibonacci, Lucas, Jacobsthal, Jacobsthal Lucas, Pell charm us with their abundant applications in science and art, and very interesting properties [1-3]. For instance, it is well known that computers use conditional directives to change the flow of execution of a program. In addition to branch instructions, some microcontrollers use skip instructions

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Received December 14, 2017

which conditionally bypass the next instruction.This brings out being useful for one case out of the four possibilities on 2 bits, 3 cases on 3 bits, 5 cases on 4 bits, 11 cases on 5 bits, 21 cases on 6 bits,..., which are exactly the Jacobsthal numbers [2]. Many properties of Jacobsthal and Jacobsthal Lucas sequences are deduced directly from elementary matrix algebra. For example F. Koken and D. Bozkurt in [4] defined a Jacobsthal matrix of the type nxn and using this matrix derived some properties of Jacobsthal numbers. Of course the oldest known integer sequence is made of Fibonacci numbers which are very important because of golden section. H. Civciv and R. Turkmen, in [5,6] defined (s,t)-Fibonacci and (s,t)-Lucas matrix sequences by using (s,t)-Fibonacci and (s,t)-Lucas sequences. Ş. Uygun, in [7] defined (s,t)-Jacobsthal and (s,t)-Jacobsthal Lucas sequences. Also K. Uslu and Ş. Uygun, in [8] defined (s,t)-Jacobsthal and (s,t)-Jacobsthal Lucas matrix sequences by using (s,t)-Jacobsthal and (s,t)-Jacobsthal Lucas sequences. In [9,10] M. Catalani gave some properties of bivariate Fibonacci and Lucas polynomials. S. Halici, in[11] give some sum formulas for bivariate Fibonacci and Lucas polynomials. N. Tuglu, E. Kocer, A. Stakhov found some properties of the bivariate Fibonacci like $p-$ polynomials in [12].

In this study, firstly we define bivariate Jacobsthal and bivariate Jacobsthal Lucas polynomial sequences, then by using these sequences, we also define bivariate Jacobsthal and bivariate Jacobsthal Lucas matrix polynomial sequences. We derive numerous interesting properties of these sequences. Then we investigate some relationship between bivariate Jacobsthal and bivariate Jacobsthal Lucas matrix polynomial sequences.

The Jacobsthal and Jacobsthal Lucas sequences are defined in [13] recurrently by

$$\begin{aligned} \hat{j}_n &= \hat{j}_{n-1} + 2\hat{j}_{n-2}, \quad (\hat{j}_0 = 0,\ \hat{j}_1 = 1) \\ \hat{c}_n &= \hat{c}_{n-1} + 2\hat{c}_{n-2}, \quad (\hat{c}_0 = 2,\ \hat{c}_1 = 1) \end{aligned}$$

where $n \geq 1$ any integer.

2. Bivariate Jacobsthal and Bivariate Jacobsthal Lucas Matrix Polynomial Sequences

Firstly, let us first consider the following definition of bivariate Jacobsthal sequence which will be needed for the definition of bivariate Jacobsthal matrix polynomial sequence.

Definition 2.1. Let be $n \in \mathbb{N}$, any integer. Then bivariate Jacobsthal polynomial sequence $\{\hat{\jmath}_n\}_{n\in\mathbb{N}}$ is defined by the following equation:

$$\hat{\jmath}_n(x,y) = xy\hat{\jmath}_{n-1}(x,y) + 2y\hat{\jmath}_{n-2}(x,y) \tag{2.1}$$

with initial conditions $\hat{\jmath}_0(x,y) = 0,\ \hat{\jmath}_1(x,y) = 1$.[13]

First few terms of the bivariate Jacobsthal number polynomial sequences are

$$\begin{aligned}
\hat{\jmath}_0(x,y) &= 0, \quad \hat{\jmath}_1(x,y) = 1, \quad \hat{\jmath}_2(x,y) = xy, \quad \hat{\jmath}_3(x,y) = x^2y^2 + 2y, \\
\hat{\jmath}_4(x,y) &= x^3y^3 + 4xy^2, \qquad \hat{\jmath}_5(x,y) = x^4y^4 + 6x^2y^3 + 4y^2.
\end{aligned}$$

Definition 2.2. Let be $n \in \mathbb{N}$, any integer. Then bivariate Jacobsthal Lucas polynomial sequence $\{\hat{c}_n\}_{n\in\mathbb{N}}$ is defined by the following equation:

$$\hat{c}_n(x,y) = xy\hat{c}_{n-1}(x,y) + 2y\hat{c}_{n-2}(x,y) \tag{2.2}$$

with initial conditions $\hat{c}_0(x,y) = 2,\ \hat{c}_1(x,y) = xy$.[13]

$$\begin{aligned}
\hat{c}_0(x,y) &= 2, \quad \hat{c}_1(x,y) = xy, \\
\hat{c}_2(x,y) &= x^2y^2 + 4y, \quad \hat{c}_3(x,y) = x^3y^3 + 6xy^2, \\
\hat{c}_4(x,y) &= x^4y^4 + 8x^2y^3 + 8y^2, \\
\hat{c}_5(x,y) &= x^5y^5 + 10x^3y^4 + 20xy^3.
\end{aligned}$$

Lemma 2.1. *For $n \geq 0$ any integer, the Binet formulas for nth bivariate Jacobsthal polynomial and nth bivariate Jacobsthal Lucas polynomial are given by*

$$\hat{\jmath}_n = \frac{r_1^n - r_2^n}{r_1 - r_2},$$

and

$$\hat{c}_n = r_1^n + r_2^n$$

respectively where $r_1 = \frac{xy+\sqrt{x^2y^2+8y}}{2}$ *and* $r_2 = \frac{xy-\sqrt{x^2y^2+8y}}{2}$, *are the roots of the characteristic equation* $r^2 = xyr + 2y$ *associated to the recurrence relation defined in (2.1)*

We can see easily

$$r_1.r_2 = -2y, \quad r_1 + r_2 = xy, \quad r_1 - r_2 = \sqrt{x^2y^2 + 8y}$$

Definition 2.3. The $n.th$ element of bivariate Jacobsthal matrix polynomial sequences is defined as

$$\hat{J}_n(x,y) = xy\hat{J}_{n-1}(x,y) + 2y\hat{J}_{n-2}(x,y) \tag{2.3}$$

with initial conditions $\hat{J}_0(x,y) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\hat{J}_1(x,y) = \begin{pmatrix} xy & 2 \\ y & 0 \end{pmatrix}$

and similarly the $n.th$ element of bivariate Jacobsthal Lucas matrix polynomial sequences is defined as

$$\hat{C}_n(x,y) = xy\hat{C}_{n-1}(x,y) + 2y\hat{C}_{n-2}(x,y) \tag{2.4}$$

with initial conditions $\hat{C}_0(x,y) = \begin{pmatrix} xy & 4 \\ 2y & -xy \end{pmatrix}$, $\hat{C}_1(x,y) = \begin{pmatrix} x^2y^2 + 4y & 2xy \\ xy^2 & 4y \end{pmatrix}$

where $x \neq 0,\ y \neq 0$ and $x^2y^2 + 8y > 0$.

Bivariate Jacobsthal $\left\{\hat{J}_n\right\}_{n\in\mathbb{N}}$ and bivariate Jacobsthal Lucas $\left\{\hat{C}_n\right\}_{n\in\mathbb{N}}$ matrix polynomial sequences are defined by carrying to matrix theory bivariate Jacobsthal and bivariate Jacobsthal Lucas polynomial sequences.

The following theorem shows us the nth general term of the bivariate Jacobsthal matrix polynomial sequence given in (2.3).

Theorem 2.1. *For* n *is any positive integer, we have*

$$\hat{J}_n = \begin{pmatrix} \hat{j}_{n+1} & 2\hat{j}_n \\ y\hat{j}_n & 2y\hat{j}_{n-1} \end{pmatrix} \tag{2.5}$$

Proof. Let us consider $n = 1$ in (5). We clearly know that $\hat{j}_0 = 0$, $\hat{j}_1 = 1$, $\hat{j}_2 = xy$, so

$$\hat{J}_1 = \begin{pmatrix} \hat{j}_2 & 2\hat{j}_1 \\ y\hat{j}_1 & 2y\hat{j}_0 \end{pmatrix} = \begin{pmatrix} xy & 2 \\ y & 0 \end{pmatrix}.$$

By iterating this procedure and considering induction steps, let us assume that the equality in (2.5) holds for all $m \leq n \in \mathbb{Z}^+$. To end up the proof, we have to show that the case also holds for $n+1$. Therefore, we get

$$\begin{aligned}
\hat{J}_{n+1} &= \hat{J}_n + 2\hat{J}_{n-1} \\
&= xy\begin{pmatrix} \hat{j}_{n+1} & 2\hat{j}_n \\ y\hat{j}_n & 2y\hat{j}_{n-1} \end{pmatrix} + 2y\begin{pmatrix} \hat{j}_n & 2\hat{j}_{n-1} \\ y\hat{j}_{n-1} & 2y\hat{j}_{n-2} \end{pmatrix} \\
&= \begin{pmatrix} xy\hat{j}_{n+1} + 2y\hat{j}_n & 2(xy\hat{j}_n + 2y\hat{j}_{n-1}) \\ y(xy\hat{j}_n + 2y\hat{j}_{n-1}) & 2y(xy\hat{j}_{n-1} + 2y\hat{j}_{n-2}) \end{pmatrix} \\
&= \begin{pmatrix} \hat{j}_{n+2} & 2\hat{j}_{n+1} \\ y\hat{j}_{n+1} & 2y\hat{j}_n \end{pmatrix}
\end{aligned}$$

Therefore the proof is completed.

Theorem 2.2. *For $n \in \mathbb{N}$, we have*

$$\hat{J}_{m+n} = \hat{J}_m . \hat{J}_n \tag{2.6}$$

Proof. It's proven by induction. We can easily see the truth of the hypothesis for $n = 0$. Let us suppose that the equality in (2.6) holds for all $p \leq n \in \mathbb{Z}^+$.After that ,we want to show that the equality is true for $p = n + 1$.

$$\begin{aligned}
\hat{J}_{m+N+1} &= xy\hat{J}_{m+N} + 2y\hat{J}_{m+N-1} \\
&= xy\hat{J}_m\hat{J}_N + 2y\hat{J}_m\hat{J}_{N-1} \\
&= \hat{J}_m(xy\hat{J}_N + 2y\hat{J}_{N-1}) \\
&= \hat{J}_m\hat{J}_{N+1}
\end{aligned}$$

Theorem 2.3. *For any integer $n \geq 1$, we get*

$$\hat{J}_n = \hat{J}_1^n \tag{2.7}$$

Proof. It can be proven easily by using induction method.

Theorem 2.4. ***(Catalan Identity)*** *For any integer $n, r \geq 1$*

$$\hat{J}_{n+r}\hat{J}_{n-r} - \hat{J}_n^2 = 0$$

Proof. It's proved by using the equality (2.7).

Theorem 2.5. ***(d'Ocaqne's Identity)*** *For any integer* $n, m \geq 1$

$$\hat{J}_n \hat{J}_{m+1} - \hat{J}_m \hat{J}_{n+1} = 0$$

Proof. It's also proved by using the equality (2.7).

Theorem 2.6. ***(Honsberger Identity)*** *For any integer* $n, m \geq 1$

$$\hat{J}_{m+1} \hat{J}_n - 2y \hat{J}_m \hat{J}_{n-1} = xy \hat{J}_{m+n}$$

Proof.

$$\begin{aligned}
\hat{J}_{m+1}\hat{J}_n - 2y\hat{J}_m\hat{J}_{n-1} &= \hat{J}_1^{m+1}\hat{J}_1^n - 2y\hat{J}_1^m\hat{J}_1^{n-1} \\
&= \hat{J}_1^{m+n+1} - 2y\hat{J}_1^{m+n-1} \\
&= \hat{J}_1^{m+n-1}(\hat{J}_1^2 - 2yI) \\
&= \hat{J}_1^{m+n-1}(\hat{J}_2 - 2yI) \\
&= \hat{J}_1^{m+n-1}\begin{pmatrix} x^2y^2 & 2xy \\ xy^2 & 0 \end{pmatrix} \\
&= xy\hat{J}_1^{m+n} = xy\hat{J}_{m+n}
\end{aligned}$$

Theorem 2.7. ***(Generating Function)***

$$\tilde{J} = \sum_{k=1}^{n} \hat{J}_k t^k = \frac{1}{1 - xyt - 2yt^2} \begin{pmatrix} 1 & 2 \\ yt & 1 - xyt \end{pmatrix}$$

Proof.

$$\begin{aligned}
\tilde{J} &= \hat{J}_0 + \hat{J}_1 t + \hat{J}_2 t^2 + \ldots \\
-xyt\tilde{J} &= -xyt\hat{J}_0 - xy\hat{J}_1 t^2 - xy\hat{J}_2 t^3 - \ldots \\
-2yt^2\tilde{J} &= -2yt^2\hat{J}_0 - 2yt^3\hat{J}_1 - 2yt^4\hat{J}_2 - \ldots
\end{aligned}$$

By adding these equalities, we obtain

$$\begin{aligned}(1-xyt-2yt^2)\tilde{J} &= \hat{J}_0+\hat{J}_1t-xy\hat{J}_0+t^2(\hat{J}_2-xy\hat{J}_1-2y\hat{J}_0)+... \\ &= I+\hat{J}_1t-xytI+0 \\ \tilde{J} &= \frac{I(1-xyt)+\hat{J}_1t}{1-xyt-2yt^2}=\frac{1}{1-xyt-2yt^2}\begin{pmatrix} 1 & 2 \\ yt & 1-xyt \end{pmatrix}\end{aligned}$$

Theorem 2.8. *For n is any positive integer, we have*

$$\hat{C}_n=\begin{pmatrix} \hat{c}_{n+1} & 2\,\hat{c}_n \\ y\,\hat{c}_n & 2y\,\hat{c}_{n-1} \end{pmatrix} \tag{2.8}$$

Proof. We use the method of induction again. For $n=1$, we have

$$\hat{C}_1(x,y)=\begin{pmatrix} x^2y^2+4y & 2xy \\ xy^2 & 4y \end{pmatrix}$$

Let us suppose that the equality in (2. 8) holds for all $m\leq n\in\mathbb{Z}^+$. To end up the proof, we have to show that the case also holds for $n+1$. We get

$$\begin{aligned}\hat{C}_{n+1} &= xy\hat{C}_n+2y\hat{C}_{n-1} \\ &= xy\begin{pmatrix} \hat{c}_{n+1} & 2\hat{c}_n \\ y\hat{c}_n & 2y\hat{c}_{n-1} \end{pmatrix}+2y\begin{pmatrix} \hat{c}_n & 2\hat{c}_{n-1} \\ y\hat{c}_{n-1} & 2y\hat{c}_{n-2} \end{pmatrix} \\ &= \begin{pmatrix} xy\hat{c}_{n+1}+2y\hat{c}_n & 2(xy\hat{c}_n+2y\hat{c}_{n-1}) \\ y(xy\hat{c}_n+2y\hat{c}_{n-1}) & 2y(xy\hat{c}_{n-1}+2y\hat{c}_{n-2}) \end{pmatrix} \\ &= \begin{pmatrix} \hat{c}_{n+2} & 2\hat{c}_{n+1} \\ y\hat{c}_{n+1} & 2y\hat{c}_n \end{pmatrix}.\end{aligned}$$

Theorem 2.9. *For $n>0$ any integer, we have*

$$\hat{C}_n=xy\hat{J}_n+4y\hat{J}_{n-1}.$$

Proof. For $n=1$, it is obvious that the claim is true:

$$\hat{C}_1=\begin{pmatrix} x^2y+4y & 2xy \\ xy & 4y \end{pmatrix}=xy\begin{pmatrix} xy & 2 \\ y & 0 \end{pmatrix}+4y\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}=xy\hat{J}_1+4y\hat{J}_0.$$

By using the above equality, we have

$$\hat{C}_{n+1}=\hat{C}_1\hat{J}_n=(xy\hat{J}_1+4y\hat{J}_0)\hat{J}_n=xy\hat{J}_{n+1}+4y\hat{J}_n.$$

Theorem 2.10. *For $n\geq 0$ any integer, we have*

$$\hat{C}_{n+1}=\hat{C}_1\hat{J}_n.$$

Proof. For $n=0$ it can be easily seen the truth of the hypothesis due to product of identity matrix. For $n=1$, it is obvious from $\hat{C}_1=\begin{pmatrix} x^2y^2+4y & 2xy \\ xy^2 & 4y \end{pmatrix}$ and $\hat{J}_1=\begin{pmatrix} xy & 2 \\ y & 0 \end{pmatrix}$

$$\begin{aligned}
\hat{C}_2 &= \hat{C}_1\hat{J}_1 \\
&= \begin{pmatrix} x^2y^2+4y & 2xy \\ xy^2 & 4y \end{pmatrix}.\begin{pmatrix} xy & 2 \\ y & 0 \end{pmatrix} \\
&= \begin{pmatrix} x^3y^3+6xy^2 & 2x^2y^2+8y \\ x^2y^3+4y^2 & 2xy^2 \end{pmatrix} \\
&= \begin{pmatrix} \hat{c}_3 & 2\hat{c}_2 \\ y\hat{c}_2 & 2y\hat{c}_1 \end{pmatrix}
\end{aligned}$$

We assume that it is true for all integers $m\leq n$. Now we show that it is true for $m=n+1$:

$$\begin{aligned}
\hat{C}_1.\hat{J}_{n+1} &= \hat{C}_1.\hat{J}_n\hat{J}_1 \\
&= \hat{C}_{n+1}.\hat{J}_1 \\
&= \begin{pmatrix} \hat{c}_{n+2} & 2\hat{c}_{n+1} \\ y\hat{c}_{n+1} & 2y\hat{c}_n \end{pmatrix}.\begin{pmatrix} xy & 2 \\ y & 0 \end{pmatrix} \\
&= \begin{pmatrix} \hat{c}_{n+3} & 2\hat{c}_{n+2} \\ \hat{c}_{n+2} & 2\hat{c}_{n+1} \end{pmatrix}=\hat{C}_{n+2}.
\end{aligned}$$

Theorem 2.11. *For $n > 0$ any integer, we have*

$$\hat{C}_n = \hat{J}_{n+1} + 2y\hat{J}_{n-1}$$

Theorem 2.12. *For $n, m \geq 0$ any integers, we have the commutative property*

$$\hat{J}_m \hat{C}_{n+1} = \hat{C}_{n+1} \hat{J}_m.$$

Proof.

$$\begin{aligned} \hat{J}_m \hat{C}_{n+1} &= \hat{J}_m \hat{C}_1 \hat{J}_n = \hat{J}_m \left[xy\hat{J}_1 + 4y\hat{J}_0 \right] \hat{J}_n = xy\hat{J}_{n+m+1} + 4y\hat{J}_{n+m} \\ &= \left[xy\hat{J}_1 + 4y\hat{J}_0 \right] \hat{J}_{n+m} = \hat{C}_1 \hat{J}_n \hat{J}_m = \hat{C}_{n+1} \hat{J}_m. \end{aligned}$$

Theorem 2.13. *For $n \geq 0$ any integer, we have*

$$\hat{C}^2_{n+1} = \hat{C}^2_1 . \hat{J}_{2n} \tag{2.9}$$

$$\hat{C}^2_{n+1} = \hat{C}_1 . \hat{C}_{2n+1} \tag{2.10}$$

$$\hat{C}_{2n+1} = \hat{J}_n \hat{C}_{n+1} \tag{2.11}$$

Proof. Their proofs are seen easily by using some algebraic operations

Corollary 2.1. For $n \geq 0$ any integer, we have some relations between bivariate Jacobsthal and bivariate Jacobsthal Lucas polynomial sequences

i): $\hat{c}^2_{n+2} + 2y\hat{c}^2_{n+1} = (x^2 + 8y)\,\hat{j}_{2n+3}$

ii): $\hat{c}^2_{n+2} + 2y\hat{c}^2_{n+1} = xy\hat{c}_{2n+4} + 2y\hat{c}_{2n+2}$

iii): $\hat{c}_{2n} = \hat{j}_n \hat{c}_{n+1} + 2y\hat{c}_n \hat{j}_{n-1}$

Proof. For the proof of i), we use the equality (2.9)

$$\hat{C}_{n+1}^{2} = \hat{C}_{1}^{2}\hat{J}_{2n} = \begin{pmatrix} x^2y^2+4y & 2xy \\ xy^2 & 4y \end{pmatrix}^2 \begin{pmatrix} \hat{j}_{2n+1} & 2\hat{j}_{2n} \\ y\hat{j}_{2n} & 2y\hat{j}_{2n-1} \end{pmatrix}$$

$$\begin{pmatrix} \hat{c}_{n+2} & 2\hat{c}_{n+1} \\ \hat{c}_{n+1} & 2\hat{c}_{n} \end{pmatrix}^2 = \begin{pmatrix} x^4y^4+16y^2+10x^2y^3 & 2x^3y^3+16xy^2 \\ x^3y^4+8xy^3 & 2x^2y^3+16y^2 \end{pmatrix} \begin{pmatrix} \hat{j}_{2n+1} & 2\hat{j}_{2n} \\ \hat{j}_{2n} & 2\hat{j}_{2n-1} \end{pmatrix}.$$

From the equality of the matrices, we have

$$\begin{aligned} \hat{c}_{n+2}^2+2y\hat{c}_{n+1}^2 &= \left(x^4y^4+16y^2+10x^2y^3\right)\hat{j}_{2n+1}+\left(2x^3y^4+16xy^3\right)\hat{j}_{2n} \\ &= x^3y^3\left(xy\hat{j}_{2n+1}+2y\hat{j}_{2n}\right)+8xy^2\left(xy\hat{j}_{2n+1}+2y\hat{j}_{2n}\right)+16y^2\hat{j}_{2n+1}+2x^2y^3\hat{j}_{2n+1} \\ &= x^3y^3\hat{j}_{2n+2}+8xy^2\hat{j}_{2n+2}+16y^2\hat{j}_{2n+1}+2x^2y^3\hat{j}_{2n+1} \\ &= x^2y^2(xy\hat{j}_{2n+2}+2y\hat{j}_{2n+1})+8y(xy\hat{j}_{2n+2}+2y\hat{j}_{2n+1}) \\ &= x^2y^2\hat{j}_{2n+3}+8y\hat{j}_{2n+3}=\left(x^2y^2+8y\right)\hat{j}_{2n+3}. \end{aligned}$$

For the proof of ii), we use the equality (2.10)

$$\hat{C}_{n+1}^{2}=\hat{C}_{1}.\hat{C}_{2n+1}=\begin{pmatrix} x^2y^2+4y & 2xy \\ xy^2 & 4y \end{pmatrix}\begin{pmatrix} \hat{c}_{2n+2} & 2\hat{c}_{2n+1} \\ \hat{c}_{2n+1} & 2\hat{c}_{2n} \end{pmatrix}.$$

From the equality of the matrices, we obtain

$$\hat{c}_{n+2}^2+2y\hat{c}_{n+1}^2=(x^2y+4y)\hat{c}_{2n+2}+2xy\hat{c}_{2n+1}=x\hat{c}_{2n+3}+4y\hat{c}_{2n+2}.$$

For the proof of iii), we use the equality (2.11)

$$\hat{C}_{2n+1}=\begin{pmatrix} \hat{c}_{2n+2} & 2\hat{c}_{2n+1} \\ y\hat{c}_{2n+1} & 2y\hat{c}_{2n} \end{pmatrix}=\hat{J}_{n}\hat{C}_{n+1}=\begin{pmatrix} \hat{j}_{n+1} & 2\hat{j}_{n} \\ y\hat{j}_{n} & 2y\hat{j}_{n-1} \end{pmatrix}\begin{pmatrix} \hat{c}_{n+2} & 2\hat{c}_{n+1} \\ y\hat{c}_{n+1} & 2y\hat{c}_{n} \end{pmatrix}.$$

From the equality of the matrices, we obtain

$$\begin{aligned} 2y\hat{c}_{2n} &= 2y\hat{j}_{n}\hat{c}_{n+1}+4y^2\hat{c}_{n}\hat{j}_{n-1} \\ \hat{c}_{2n} &= \hat{j}_{n}\hat{c}_{n+1}+2y\hat{c}_{n}\hat{j}_{n-1}. \end{aligned}$$

Theorem 2.14. *For $n \geq 0$, we get*

$$\hat{J}_n = \left(\frac{\hat{J}_1 - r_2 \hat{J}_0}{r_1 - r_2}\right) r_1^n - \left(\frac{\hat{J}_1 - r_1 \hat{J}_0}{r_1 - r_2}\right) r_2^n.$$

Proof.

$$\begin{aligned}
\hat{J}_n &= \left(\frac{\hat{J}_1 - r_2 \hat{J}_0}{r_1 - r_2}\right) r_1^n - \left(\frac{\hat{J}_1 - r_1 \hat{J}_0}{r_1 - r_2}\right) r_2^n \\
&= \frac{r_1^n}{r_1 - r_2} \begin{pmatrix} xy - r_2 & 2 \\ y & -r_2 \end{pmatrix} - \frac{r_2^n}{r_1 - r_2} \begin{pmatrix} xy - r_1 & 2 \\ y & -r_1 \end{pmatrix} \\
&= \frac{1}{r_1 - r_2} \begin{pmatrix} xy\left(r_1^n - r_2^n\right) - r_1 r_2 \left(r_1^{n-1} - r_2^{n-1}\right) & 2\left(r_1^n - r_2^n\right) \\ y(r_1^n - r_2^n) & -r_1 r_2 \left(r_1^{n-1} - r_2^{n-1}\right) \end{pmatrix} \\
&= \begin{pmatrix} \hat{j}_{n+1} & 2\hat{j}_n \\ y\hat{j}_n & 2y\hat{j}_{n-1} \end{pmatrix}.
\end{aligned}$$

Theorem 2.15. *The partial sum of bivariate Jacobsthal matrix polynomial sequence for $xy + 2y \neq 1$ is given in the following*

$$\sum_{k=1}^{n} \hat{J}_k = \begin{bmatrix} \hat{j}_{n+2} - xy + 2y\hat{j}_{n+1} - 2t & 2(\hat{j}_{n+1} + 2y\hat{j}_n - 1) \\ y(\hat{j}_{n+1} + 2y\hat{j}_n - 1) & 2y(\hat{j}_n + 2y\hat{j}_{n-1} - 1) \end{bmatrix}$$

Proof. Let $S_n = \sum_{k=1}^{n} \hat{J}_k$. By multiplying $\hat{J}_1$ two sides of the equality, we get

$$S_n \hat{J}_1 = \hat{J}_2 + \hat{J}_3 + ... + \hat{J}_{n+1}.$$

By adding $\hat{J}_1$ two sides of the equality, we get

$$\begin{aligned}
S_n \hat{J}_1 + \hat{J}_1 &= \hat{J}_1 + \hat{J}_2 + \hat{J}_3 + ... + \hat{J}_{n+1} \\
S_n \hat{J}_1 - S_n &= J_{n+1} - \hat{J}_1 \\
S_n(\hat{J}_1 - \hat{J}_0) &= \hat{J}_{n+1} - \hat{J}_1.
\end{aligned}$$

The inverse of $\hat{J}_1 - \hat{J}_0$ is available for $\det(\hat{J}_1 - \hat{J}_0) = 1 - xy - 2y \neq 0$. Then we get

$$S_n = \left(\hat{J}_{n+1} - \hat{J}_1\right) (\hat{J}_1 - \hat{J}_0)^{-1}.$$

By using following equalities $\hat{J}_{n+1}-\hat{J}_1=\begin{bmatrix}\hat{j}_{n+2}-xy & 2\hat{j}_{n+1}-2\\ yj_{n+1}-y & 2y\hat{j}_n\end{bmatrix}$, $\hat{J}_1-\hat{J}_0=\begin{bmatrix}xy-1 & 2\\ y & -1\end{bmatrix}$ and $(\hat{J}_1-\hat{J}_0)^{-1}=\frac{1}{xy+2y-1}\begin{bmatrix}1 & 2\\ y & 1-xy\end{bmatrix}$, we get

$$
\begin{aligned}
S_n &= \frac{1}{xy+2y-1}\begin{bmatrix}\hat{j}_{n+2}-xy & 2\hat{j}_{n+1}-2\\ yj_{n+1}-y & 2y\hat{j}_n\end{bmatrix}\begin{bmatrix}1 & 2\\ y & 1-xy\end{bmatrix}\\
&= \frac{1}{xy+2y-1}\begin{bmatrix}\hat{j}_{n+2}+2y\hat{j}_{n+1}-xy-2y & 2(\hat{j}_{n+1}+2y\hat{j}_n-1)\\ y(\hat{j}_{n+1}+2y\hat{j}_n-1) & 2y\left(\hat{j}_n+2y\hat{j}_{n-1}-1\right)\end{bmatrix}
\end{aligned}
$$

Corollary 2.2. The partial sums of bivariate Jacobsthal polynomial sequence for $xy+2y\neq 1$ are given in the following:

$$\sum_{k=1}^{n}\hat{j}_{k+1}=\frac{\hat{j}_{n+2}-xy+2y\hat{j}_{n+1}-2y}{xy+2y-1},$$

and

$$\sum_{k=1}^{n}\hat{j}_k=\frac{\hat{j}_{n+1}+2y\hat{j}_n-1}{xy+2y-1}.$$

Theorem 2.16. *The partial sum of bivariate Jacobsthal Lucas matrix polynomial sequence for $xy+2y\neq 1$ is given in the following* $\sum_{k=1}^{n}\hat{C}_{k+1}=(a_{ij})$

$$
\begin{aligned}
a_{11} &= \frac{1}{xy+2y-1}\left(\hat{j}_{n+4}+2y\hat{j}_{n+3}+2y\hat{j}_{n+2}+4y^2\hat{j}_{n+1}-x^2y^2(xy+2y)-2y(3xy+4y)\right)\\
a_{12} &= \frac{2}{xy+2y-1}\left(\hat{j}_{n+3}+2y\hat{j}_{n+2}+2y\hat{j}_{n+1}+4y^2\hat{j}_n-x^2y^2-4y-2xy^2\right)\\
a_{21} &= \frac{y}{xy+2y-1}\left(\hat{j}_{n+3}+2y\hat{j}_{n+2}+2y\hat{j}_{n+1}+4y^2\hat{j}_n-2xy^2-x^2y^2-4y\right)\\
a_{22} &= \frac{2y}{xy+2y-1}\left(\hat{j}_{n+2}+2y\hat{j}_{n+1}+2y\hat{j}_n+4y^2\hat{j}_{n-1}-xy-4y\right)
\end{aligned}
$$

Proof. By using $\hat{C}_{k+1} = \hat{C}_1 \hat{J}_k$ and Theorem (2.10) we get

$$\begin{aligned}\sum_{k=1}^{n} \hat{C}_{k+1} &= \sum_{k=1}^{n} \hat{C}_1 \hat{J}_k = \\ &= \frac{1}{xy+2y-1} \begin{bmatrix} x^2y^2+4y & 2xy \\ xy^2 & 4y \end{bmatrix} \\ &\quad \cdot \begin{bmatrix} \hat{j}_{n+2}+2y\hat{j}_{n+1}-xy-2y & 2(\hat{j}_{n+1}+2y\hat{j}_n-1) \\ y(\hat{j}_{n+1}+2y\hat{j}_n-1) & 2y\left(\hat{j}_n+2y\hat{j}_{n-1}-1\right) \end{bmatrix}.\end{aligned}$$

If the product of matrices is made the desired result is found.

Corollary 2.3. he partial sums of bivariate Jacobsthal Lucas polynomial sequence for $xy+2y \neq 1$ are given in the following:

$$\sum_{k=1}^{n} \hat{c}_k = \frac{1}{xy+2y-1}\left(\hat{j}_{n+2}+2y\hat{j}_{n+1}+2y\hat{j}_n+4y^2\hat{j}_{n-1}-xy-4y\right).$$

and

$$\sum_{k=1}^{n} \hat{c}_{k+1} = \frac{1}{xy+2y-1}\left(\hat{j}_{n+3}+2y\hat{j}_{n+2}+2y\hat{j}_{n+1}+4y^2\hat{j}_n-x^2y^2-4y-2xy^2\right).$$

Conflict of Interests

The authors declare that there is no conflict of interests.

References

[1] T. Koshy, Jacobsthal and Jacobsthal Lucas Numbers with Applications, John Wiley and Sons Inc., NY (2001).

[2] A . F. Horadam, Jacobsthal Representation Numbers, Fibonacci Q. 34 (1) (1996), 40-54.

[3] A. F. Horadam, Jacobsthal Representation Polynomials, Fibonacci Q. 35 (2) (1997), 137-148.

[4] F. Köken, D. Bozkurt, On the Jacobsthal Numbers by Matrix Methods, Int. J. Contemp. Math Sci. 3 (13) (2008), 605-614.

[5] H. Civciv, R. Turkmen, On the (s,t) Fibonacci and Fibonacci matrix sequences, Ars Comb. 87 (2008), 161-173.

[6] H. Civciv, R. Turkmen, Notes on the (s,t) Lucas and Lucas matrix sequences, Ars Comb. 89 (2008), 271-285.

[7] Ş. Uygun, The (s,t)-Jacobsthal and (s,t)-Jacobsthal Lucas sequences, Appl. Math. Sci. 70 (9) (2015), 3467-3476.

[8] K. Uslu, S. Uygun, The (s,t) Jacobsthal and (s,t) Jacobsthal-Lucas Matrix sequences, Ars Comb. 108 (2013), 13-22.

[9] M. Catalani, Some Formulae for Bivariate Fibonacci and Lucas Polynomials, arXiv:math/0406323 [math.CO], 2004.

[10] M. Catalani, Generalized Bivariate Fibonacci Polynomials, arXiv:math/0211366 [math.CO], 2004.

[11] Halıcı, S., Akyüz Z., On Some Formulae for Bivariate Pell Polynomials, Far East J. Appl. Math. 41 (2) (2010), 101-110.

[12] N. Tuglu, E. Gökcen Kocer, A. Stakhov, Bivariate Fibonacci like $p-$ polynomials, Appl. Math. Comput. 217 (2011), 10239-10246.

[13] S. Uygun, The Bivariate Jacobsthal and Jacobsthal Lucas Polynomial Sequences (to appear).