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THE GERBER-SHIU FUNCTION IN THE PERTURBED COMPOUND POISSON GAMMA OMEGA MODEL WITH A DIVIDEND BARRIER

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Abstract. In this paper, the perturbed compound Poisson Gamma Omega model with a barrier dividend strategy is studied. Using the strong Markov property and Taylor formula, the integro-differential equations for the Gerber-Shiu expected discounted penalty function are derived. The explicit solutions of the Gerber-Shiu expected discounted penalty function are also obtained when the claim size is exponentially distributed. Furthermore, a numerical example is presented to illustrate some properties of the function.

Keywords: Gamma Omega model; Strong Markov property; Barrier dividend strategy.

2010 AMS Subject Classification: 62P20, 91B30.

1. Introduction

Risk theory plays an important role in financial mathematics and insurance actuarial studies, and through the study of stochastic risk model in the insurance industry to deal with several actuarial variables, such as the time of ruin, the surplus immediately before ruin, the deficit at ruin, the ruin probability, the Gerber-Shiu expected discounted penalty function, the expected discounted dividend payments function, etc.

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Since the Gerber-Shiu expected discounted penalty function(simply called Gerber-Shiu function) was initially proposed by Gerber and Shiu[9], the function has been studied by many authors under more general models, such as compound Poisson risk model, renewal risk model, the perturbed risk model, Lévy risk model, etc. Import reference involved in Sabine[15], Chin and Yin[2], Claudio and Giovanni[3], Gao and Yin[8], Gao and Wu[7], etc.

The compound Poisson risk model perturbed by diffusion was initially proposed by Gerber[10], and has been further studied by many authors during the last few years. Dufresne and Gerber[5] studied the probability of ruin and derived the convolution of the probability of ruin. Li[13] investigated the expected discounted dividend payments function prior to ruin and obtained the explicit solutions of the function. Yuen and Wang[17] considered the Gerber-Shiu expected discounted penalty function with interest and a constant dividend barrier, then derived an integro-differential equation of the function and obtained the solution to the integro-differential equation which is in the form of an infinite series. Gao and Liu[6] studied the model with constant interest and a threshold dividend strategy, then derived the integro-differential equations with certain boundary conditions for the moment-generation function and the *n-th* moment of the present value of all dividends until ruin. In this model, the surplus of an insurance company at time t is given by

$$U(t) = u + ct - S(t) + \sigma W(t) = u + ct - \sum_{i=1}^{N(t)} X_i + \sigma W(t), \tag{1}$$

where $U(0) = u$ is the initial surplus, c is the premium rate, the total number of claims $\{N(t), t \geq 0\}$ is a homogeneous Poisson process with intensity λ, the claim sizes $\{X_1, X_2, ...\}$ form a sequence of positive independent identically distributed random variables with common distribution $P(x)$, density function $p(x)$ and mean value μ, $\{X_i, i \geq 1\}$ and $\{N(t), t \geq 0\}$ are mutually independent, the aggregate claims $\{S(t), t \geq 0\}$ is a compound Poisson process with intensity λ, $\{W(t), t \geq 0\}$ is a standard Wiener process with $W(t) \sim N(0,t)$, $\sigma > 0$ is a constant, which represents the diffusion volatility parameter. In order to guarantee a positive survival probability, it is assumed that $c = (1+\theta)\lambda\mu \geq 0$, which $\theta > 0$ is the relative security loading factor ensures that the ruin probability is less than 1.

In risk theory, a company goes out of business as soon as ruin occurs, that is, when the surplus is negative for the first time. However, in practical, companies can continue doing business even though they are technically ruined. The Omega model was introduced for a Wiener process in Albrecher, Gerber and Shiu[1], there was a distinction between ruin (negative surplus) and bankruptcy (going out of business). It was assumed that even with a negative surplus, the company can do business as usual until bankruptcy occurs and the probability of bankruptcy is concerned at a point of time. In addition, the dividend problems are received widespread attention, it was first introduced in De Finetti[4] for a binomial model to reflect more realistically the surplus cash flows in an insurance portfolio, and he found that the optimal strategy must be a barrier strategy. From then on, a great deal of papers have been devoted to study the barrier dividend strategy, such as Gerber[11], Taksar[16], Gerber and Shiu[12], Landriault[14], etc. In this paper, the perturbed compound Poisson Omega model with a barrier dividend strategy is studied, at the same time, it is assumed that dividends can only be paid at certain random times and thus constitute a discrete sequence of random variables, the interval times between successive dates when dividends can be paid are independent random variables with a common exponential distribution with parameter γ, that is, at any time the probability that a dividend can be paid within dt time units is γdt. The symbol for the dividend payments time interval as a exponential distribution with parameter γ leads to the name Gamma model in Albrecher, Gerber and Shiu[1], that is, the perturbed compound Poisson Gamma Omega model with a barrier dividend strategy is studied in this paper.

In this paper, the probability of bankruptcy is quantified by a bankruptcy rate function $\omega(u)$, where $u \leq 0$ is the value of the surplus at that time, it is a non-increasing function, that is, whenever the surplus is $u \leq 0$, $\omega(u)dt$ is the probability of bankruptcy within dt time units. However, it is unrealistic to assume that the surplus of a company can decrease without bounds, in this paper, it is assumed that $\omega(u)$ is infinite for $u \leq u_0 < 0$ and $\omega(u) > 0$ for $u_0 < u \leq 0$, bankruptcy occurs at the latest when the surplus drops to u_0. In other words, u_0 is the level of ”certain bankruptcy”.

If no dividends were paid, the surplus $U(t)$ can be described in (1). The company will pay dividends to its shareholders, for $t \geq 0$, let $D(t)$ denote the aggregate dividends paid by time t, then the modified surplus at time t is

$$U^*(t) = U(t) - D(t).$$

For a barrier dividend strategy, it is assumed that the company pays dividends according to the following strategy governed by parameter $b \geq 0$: whenever the modified surplus is below the level b, no dividends are paid, however, when the modified surplus is above the level b, dividends are paid out with $U(t) - b$ at a potential dividend payment time (until the next claim occurs). Let $\delta > 0$ be the force of interest for valuation, and let D denote the present value of all dividends until bankruptcy

$$D = \int_0^T e^{-\delta t} \mathrm{d}D(t),$$

where T is the bankruptcy time for the modified process. Let $\phi(u,b)$ denote the Gerber-Shiu function

$$\phi(u,b) = E[e^{-\delta T} w(U(T)) I(T < +\infty) \mid U^*(0) = u],$$

where $w = w(x)$ as a penalty function, be a nonnegative bounded measurable function, that $x = U(T) \leq 0$ is the value of the surplus at bankruptcy time, $I(E)$ is the indicator function of event E.

The remainder of the paper is organized as follows. In *section* 2, using the strong Markov property and Taylor formula, the integro-differential equations for the Gerber-Shiu function $\phi(u,b)$ are derived. In *section* 3, the explicit solutions of the Gerber-Shiu function $\phi(u,b)$ are obtained when the claim size is exponentially distributed. Furthermore, a numerical example is presented to illustrate some properties of the function in *section* 4. This result unifies and extends recent literature Albrecher, Gerber and Shiu[1] incorporating some of their results as special cases.

2. The integro-differential equations for $\phi(u,b)$

In this section, the perturbed compound Poisson Gamma Omega model with a barrier dividend strategy is studied. Using the strong Markov property and Taylor formula, the integro-differential equations for the Gerber-Shiu function $\phi(u,b)$ are derived.

Theorem 2.1. *The Gerber-Shiu function $\phi(u,b)$ satisfy the integro-differential equations*

$$(2)\quad \begin{cases} \frac{\sigma^2}{2}\phi''(u,b)+c\phi'(u,b)-(\lambda+\delta+\omega(u))\phi(u,b)+\omega(u)w(u) \\ \quad +\lambda\int_0^{u-u_0}\phi(u-x,b)p(x)dx+\lambda\int_{u-u_0}^{+\infty}w(u-x)p(x)dx=0, \quad (u_0<u\leq 0) \\ \frac{\sigma^2}{2}\phi''(u,b)+c\phi'(u,b)-(\lambda+\delta)\phi(u,b) \\ \quad +\lambda\int_0^{u-u_0}\phi(u-x,b)p(x)dx+\lambda\int_{u-u_0}^{+\infty}w(u-x)p(x)dx=0, \quad (0<u\leq b) \\ \frac{\sigma^2}{2}\phi''(u,b)+c\phi'(u,b)-(\lambda+\delta+\gamma)\phi(u,b)+\gamma\phi(b,b) \\ \quad +\lambda\int_0^{u-u_0}\phi(u-x,b)p(x)dx+\lambda\int_{u-u_0}^{+\infty}w(u-x)p(x)dx=0. \quad (u>b) \end{cases}$$

.

Proof. For $h>0$, the infinitesimal time interval $(0,h)$ is considered. By distinguishing whether or not the first claim occurs in the infinitesimal time interval, one can get

$$\begin{aligned} \phi(u,b) &= E[e^{-\delta T}w(U(T))I(T<+\infty)\mid U^*(0)=u] \\ &= E[I(T_1>h)e^{-\delta T}w(U(T))I(T<+\infty)\mid U^*(0)=u] \\ &\quad +E[I(T_1\leq h)e^{-\delta T}w(U(T))I(T<+\infty)\mid U^*(0)=u] \\ &= I+II. \end{aligned} \tag{3}$$

For $u_0<u\leq 0$, in the infinitesimal time interval $(0,h)$, which enables the surplus at time h does not exceed 0, that is, no dividends are paid in $(0,h)$, but potential bankruptcy and the first claim may occur. Note that the probability that the first claim occurs up to time h is $e^{-\lambda h}$, the probability that the first claim occurs between $(0,h)$ is $(1-e^{-\lambda h})$ and condition on the time $T_1\subset(0,h)$ and the amount $X_1=x$ of the first claim. Note that the probability of bankruptcy up to time h is $(1-\omega(u)h)$, the probability of bankruptcy between $(0,h)$ is $\omega(u)h$. It follows from $W(t)\sim N(0,t)$ that

$$E[W(t)]=0, \quad E[W^2(t)]=t.$$

Using the strong Markov property, one have

$$\begin{aligned} I &= E[I(T_1>h)e^{-\delta T}w(U(T))I(T<+\infty)\mid U^*(0)=u] \\ &= P(T_1>h)e^{-\delta h}E[\phi(U^*(h),b)] \\ &= e^{-(\lambda+\delta)h}E[\phi(U^*(h),b)]. \end{aligned}$$

By distinguishing whether or not bankruptcy in the infinitesimal time interval $(0,h)$ and $\phi(U^*(h),b) = e^{\delta h}w(u)$ when the bankruptcy occurs, correspondingly, one have

$$\begin{aligned} E[\phi(U^*(h),b)] &= E[I(T>h)\phi(U^*(h),b)] + E[I(T\le h)\phi(U^*(h),b)] \\ &= (1-\omega(u)h)E[\phi(U^*(h),b) \mid T>h] + \omega(u)he^{\delta h}w(u). \end{aligned}$$

It follows from $U^*(h) = u + ch + \sigma W(h), (T_1 > h)$ and the Taylor formula that

$$e^{-(\lambda+\delta)h} = 1-(\lambda+\delta)h + o(h),$$
$$E[\phi(U^*(h),b) \mid T>h] = \phi(u,b) + ch\phi'(u,b) + \frac{\sigma^2}{2!}h\phi''(u,b),$$

then

$$I = \phi(u,b) - (\lambda+\delta+\omega(u))h\phi(u,b) + ch\phi'(u,b) + \frac{\sigma^2}{2!}h\phi''(u,b) + \omega(u)w(u)h. \tag{4}$$

Using the similarly argument with I, one arrives that

$$II = \lambda h[\int_0^{u-u_0} \phi(u-x,b)p(x)\mathrm{d}x + \int_{u-u_0}^{\infty} w(u-x)p(x)\mathrm{d}x]. \tag{5}$$

Thus, following (4), (5), subtracting $\phi(u,b)$ from both sides of (3) and then divide by h and let $h \to 0$, simplifying yields (2).

For $0 < u \le b$, in the infinitesimal time interval $(0,h)$, which enables the surplus at time h does not exceed b, that is, in $(0,h)$ no dividends are paid, potential bankruptcy and the first claim may occur. According the above analysis, the I can be rewritten as

$$I = \phi(u,b) - (\lambda+\delta)h\phi(u,b) + ch\phi'(u,b) + \frac{\sigma^2}{2!}h\phi''(u,b). \tag{6}$$

Thus, following (5), (6), subtracting $\phi(u,b)$ from both sides of (3) and then divide by h and let $h \to 0$, simplifying yields (2).

For $u > b$, in the infinitesimal time interval $(0,h)$, which enables the surplus at time h does not drop to b, that is, in $(0,h)$ potential bankruptcy, dividends may paid, and the first claim may occur. Note that the probability of dividends payment up to time h is $e^{-\gamma h}$, the probability of dividends payment between $(0,h)$ is $(1-e^{-\gamma h})$ and condition on the time $T_1^* \subset (0,h)$ and the dividends can be paid with $D(h)$ at the time h, by distinguishing whether or not dividends payment in the infinitesimal time interval (0,h), the I can be rewritten as

$$\begin{aligned} I &= P(T_1 > h)[E[I(T_1^* > h)e^{-\delta T}w(U(T))I(T < +\infty) \mid U^*(0) = u] \\ &\qquad + E[I(T_1^* \leq h)e^{-\delta T}w(U(T))I(T < +\infty) \mid U^*(0) = u]] \\ &= e^{-\lambda h}(III + IV), \end{aligned}$$

using the strong Markov property and the Taylor formula, one have

$$\begin{aligned} III &= e^{-(\gamma+\delta)h}E[\phi(U^*(h),b) \mid T_1^* > h] \\ &= \phi(u,b) - (\gamma+\delta)h\phi(u,b) + ch\phi'(u,b) + \frac{\sigma^2}{2!}h\phi''(u,b), \\ IV &= (1-e^{-\gamma h})e^{-\delta h}E[\phi(U^*(h),b) \mid T_1^* \leq h] = \gamma h\phi(b,b), \end{aligned}$$

then

$$I = \phi(u,b) - (\lambda+\delta+\gamma)h\phi(u,b) + ch\phi'(u,b) + \frac{\sigma^2}{2!}h\phi''(u,b) + \gamma h\phi(b,b). \tag{7}$$

Thus, following (5), (7), subtracting $\phi(u,b)$ from both sides of (3) and then divide by h and let $h \to 0$,simplifying yields (2). This completes the proof.

Remark 2.1.

(1) With $\lambda = 0$, means no claims, the surplus process can be rewritten as

$$U(t) = u + ct + \sigma W(t),$$

the model is converted into the Gamma Omega model in Wiener surplus process with a barrier dividend strategy, then $\phi(u,b)$ satisfy the integro-differential equations can be rewritten as

$$\begin{cases} \frac{\sigma^2}{2}\phi''(u,b) + c\phi'(u,b) - (\delta + \omega(u))\phi(u,b) + \omega(u)w(u) = 0, & (u_0 < u \leq 0) \\ \frac{\sigma^2}{2}\phi''(u,b) + c\phi'(u,b) - \delta\phi(u,b) = 0, & (0 < u \leq b) \\ \frac{\sigma^2}{2}\phi''(u,b) + c\phi'(u,b) - (\delta+\gamma)\phi(u,b) + \gamma\phi(b,b) = 0, & (u > b) \end{cases}$$

the result coincides exactly with Section 1 in Gerber, Shiu and Yang[18].

(2) With $\gamma = 0$, means no dividend payments, the model is converted into the classical risk

model in compound Poisson surplus process perturbed by diffusion, then $\phi(u,b)=\phi(u)$ until bankruptcy satisfy the integro-differential equations can be rewritten as

$$\begin{cases} \frac{\sigma^2}{2}\phi''(u)+c\phi'(u)-(\lambda+\delta+\omega(u))\phi(u)+\omega(u)w(u) \\ \quad +\lambda\int_0^{u-u_0}\phi(u-x)p(x)dx+\lambda\int_{u-u_0}^{+\infty}w(u-x)p(x)dx=0, \qquad (u_0<u\le 0) \\ \frac{\sigma^2}{2}\phi''(u)+c\phi'(u)-(\lambda+\delta)\phi(u)+\lambda\int_0^{u-u_0}\phi(u-x)p(x)dx \\ \quad +\lambda\int_{u-u_0}^{+\infty}w(u-x)p(x)dx=0. \qquad (u>0) \end{cases}$$

(3) With $\omega(u)=0$, means the model is converted into the "extreme" model with the company can not bankruptcy, then $\phi(u,b)$ satisfy the integro-differential equations can be rewritten as

$$\begin{cases} \frac{\sigma^2}{2}\phi''(u,b)+c\phi'(u,b)-(\lambda+\delta)\phi(u,b)+\lambda\int_0^{+\infty}\phi(u-x,b)p(x)dx=0, \quad (u\le b) \\ \frac{\sigma^2}{2}\phi''(u,b)+c\phi'(u,b)-(\lambda+\delta+\gamma)\phi(u,b)+\gamma\phi(b,b) \\ \quad +\lambda\int_0^{+\infty}\phi(u-x,b)p(x)dx=0. \qquad (u>b) \end{cases}$$

3. An explicit formula of $\phi(u,b)$ for exponential claim amounts

In this section, the explicit solutions of the Gerber-Shiu function $\phi(u,b)$ are derived when the claim size is exponentially distributed $P(x)=1-e^{-\nu x}$, the bankruptcy rate function $\omega(u)=\omega$(constant value), the penalty function $w(u)=w$(constant value) and $u_0\to-\infty$ with a barrier dividend strategy.

In order to derive the explicit solutions of $\phi(u,b)$, according to the size of initial value, the function $\phi(u,b)$ is classified into the following three functions

$$\phi(u,b)=\begin{cases} \phi_l(u,b), & (u\le 0) \\ \phi_m(u,b), & (0<u\le b) \\ \phi_n(u,b), & (u>b) \end{cases}$$

thus $\phi(u,b)$ satisfy the integro-differential equations can be rewritten as

$$\begin{cases} \frac{\sigma^2}{2}\phi_l''(u,b)+c\phi_l'(u,b)-(\lambda+\delta+\omega(u))\phi_l(u,b)+\omega(u)w(u) & \\ \quad +\lambda\int_0^{u-u_0}\phi_l(u-x,b)p(x)\mathrm{d}x+\lambda\int_{u-u_0}^{+\infty}w(u-x)p(x)\mathrm{d}x=0, & (u_0<u\le 0) \\ \frac{\sigma^2}{2}\phi_m''(u,b)+c\phi_m'(u,b)-(\lambda+\delta)\phi_m(u,b)+\lambda\int_0^u\phi_m(u-x,b)p(x)\mathrm{d}x & \\ \quad +\lambda\int_u^{u-u_0}\phi_l(u-x,b)p(x)\mathrm{d}x+\lambda\int_{u-u_0}^{+\infty}w(u-x)p(x)\mathrm{d}x=0, & (0<u\le b) \\ \frac{\sigma^2}{2}\phi_n''(u,b)+c\phi_n'(u,b)-(\lambda+\delta+\gamma)\phi_n(u,b)+\gamma\phi_n(b,b)+\lambda\int_0^{u-b}\phi_n(u-x,b)p(x)\mathrm{d}x & (u>b) \\ \quad +\lambda\int_{u-b}^{u}\phi_m(u-x,b)p(x)\mathrm{d}x+\lambda\int_u^{u-u_0}\phi_l(u-x,b)p(x)\mathrm{d}x+\lambda\int_{u-u_0}^{+\infty}w(u-x)p(x)\mathrm{d}x=0. & \end{cases}$$

According the continuity of $\phi(u,b)$ and the continuity of $\phi'(u,b)$ at $u=0$, one have

$$\frac{\sigma^2}{2}\phi_l''(0-,b)-\omega(0)\phi_l(0,b)+\omega(0)w(0)=\frac{\sigma^2}{2}\phi_m''(0+,b), \tag{8}$$

$$\frac{\sigma^2}{2}\phi_m''(b-,b)+c\phi_m'(b-,b)=\frac{\sigma^2}{2}\phi_n''(b+,b)+c\phi_n'(b+,b). \tag{9}$$

With the substitution $z=u-x$, $\phi(u,b)$ satisfy the integro-differential equations can be rewritten as

$$\begin{cases} \frac{\sigma^2}{2}\phi_l''(u,b)+c\phi_l'(u,b)-(\lambda+\delta+\omega(u))\phi_l(u,b)+\omega(u)w(u) & \\ \quad +\lambda\int_{u_0}^{u}\phi_l(z,b)p(u-z)\mathrm{d}z+\lambda\int_{-\infty}^{u_0}w(z)p(u-z)\mathrm{d}z=0, & (u_0<u\le 0) \\ \frac{\sigma^2}{2}\phi_m''(u,b)+c\phi_m'(u,b)-(\lambda+\delta)\phi_m(u,b)+\lambda\int_0^u\phi_m(z,b)p(u-z)\mathrm{d}z & \\ \quad +\lambda\int_{u_0}^{0}\phi_l(z,b)p(u-z)\mathrm{d}z+\lambda\int_{-\infty}^{u_0}w(z)p(u-z)\mathrm{d}z=0, & (0<u\le b) \\ \frac{\sigma^2}{2}\phi_n''(u,b)+c\phi_n'(u,b)-(\lambda+\delta+\gamma)\phi_n(u,b)+\gamma\phi_n(b,b) & \\ \quad +\lambda\int_b^u\phi_n(z,b)p(u-z)\mathrm{d}z+\lambda\int_0^b\phi_m(z,b)p(u-z)\mathrm{d}z & \\ \quad +\lambda\int_{u_0}^{0}\phi_l(z,b)p(u-z)\mathrm{d}z+\lambda\int_{-\infty}^{u_0}w(z)p(u-z)\mathrm{d}z=0. & (u>b) \end{cases} \tag{10}$$

It is assume that the claim size density is given by $p(x)=ve^{-vx}$, $x>0$, $v>0$, applying the differential operator $(\mathrm{d}/\mathrm{d}u+\nu)$ to the above equations, then $\phi(u,b)$ satisfy the third-order differential equations

$$\begin{cases} \frac{\sigma^2}{2}\phi_l'''(u,b)+(\frac{\sigma^2}{2}\nu+c)\phi_l''(u,b)+[c\nu-(\lambda+\delta+\omega(u))]\phi_l'(u,b) & \\ \quad -[\nu(\delta+\omega(u))+\omega'(u)]\phi_l(u,b) & \\ \quad +(\omega'(u)w(u)+\omega(u)w'(u)+v\omega(u)w(u))=0, & (u_0<u\le 0) \\ \frac{\sigma^2}{2}\phi_m'''(u,b)+(\frac{\sigma^2}{2}\nu+c)\phi_m''(u,b)+[c\nu-(\lambda+\delta)]\phi_m'(u,b)-\delta\nu\phi_m(u,b)=0, & (0<u\le b) \\ \frac{\sigma^2}{2}\phi_n'''(u,b)+(\frac{\sigma^2}{2}\nu+c)\phi_n''(u,b)+[c\nu-(\lambda+\delta+\gamma)]\phi_n'(u,b) & \\ \quad -(\delta+\gamma)\nu\phi_n(u,b)+\gamma\nu\phi_n(b,b)=0. & (u>b) \end{cases} \tag{11}$$

For $u \le 0$, $\phi_l(u,b)$ satisfies the nonhomogeneous differential equation in (11), when the bankruptcy rate function $\omega(u) = \omega$(constant value), the penalty function $w(u) = w$(constant value) and $u_0 \to -\infty$, the special solution of $\phi_l(u,b)$ is

$$\phi_{l0}(u,b) = \frac{\omega w}{\delta+\omega},$$

then the general solution of $\phi_l(u,b)$ is

$$\phi_l(u,b) = \frac{\omega w}{\delta+\omega} + K_l h(u),$$

with the K_l is arbitrary coefficient and independent of u, $h(u)$ is the homogeneous solution of the third-order differential equation of $\phi_l(u,b)$ and satisfies the boundary condition $\lim\limits_{u\to-\infty} h(u) = 0$, that is, $h(u)$ satisfies the third-order differential equation

$$\frac{\sigma^2}{2}h'''(u) + (\frac{\sigma^2}{2}\nu + c)h''(u) + [c\nu - (\lambda+\delta+\omega)]h'(u) - \nu(\delta+\omega)h(u) = 0.$$

Using the characteristic roots methods, the solution of $h(u)$ is

$$h(u) = K_1^* e^{q_1 u} + K_2 e^{q_2 u} + K_3 e^{q_3 u},$$

with the K_1^*, K_2, K_3 are arbitrary coefficients and independent of u, $q_1 \ge 0$, $q_2 < 0$, $q_3 < 0$ being the roots of the characteristic equation

$$\frac{\sigma^2}{2}\xi^3 + (\frac{\sigma^2}{2}\nu + c)\xi^2 + [c\nu - (\lambda+\delta+\omega)]\xi - \nu(\delta+\omega) = 0.$$

Following the boundary condition $\lim\limits_{u\to-\infty} h(u) = 0$ that $K_2 = 0$, $K_3 = 0$, then the solution of $h(u)$ is

$$h(u) = K_1^* e^{q_1 u},$$

thus, let $K_1 = K_l K_1^*$, the solution of $\phi_l(u,b)$ can be rewritten as

$$\phi_l(u,b) = K_1 e^{q_1 u} + \frac{\omega w}{\delta+\omega}.$$

For $0 < u \le b$, $\phi_m(u,b)$ satisfies the homogeneous differential equation in (11), similarly, the solution of $\phi_m(u,b)$ is

$$\phi_m(u,b) = G_1 e^{s_1 u} + G_2 e^{s_2 u} + G_3 e^{s_3 u}, \tag{12}$$

with the G_1, G_2, G_3 are arbitrary coefficients and independent of u, $s_1 \geq 0$, $s_2 < 0$, $s_3 < 0$ being the roots of the characteristic equation

$$\frac{\sigma^2}{2}\xi^3 + (\frac{\sigma^2}{2}\nu + c)\xi^2 + (c\nu - (\lambda+\delta))\xi - \nu\delta = 0.$$

Substitution (12) into (10), with subsequent comparison of the coefficients of $e^{-\nu u}$ yields that

$$\frac{\nu K_1}{q_1+\nu} + \frac{\omega w}{\delta+\omega} = \frac{\nu G_1}{s_1+\nu} + \frac{\nu G_2}{s_2+\nu} + \frac{\nu G_3}{s_3+\nu}. \tag{13}$$

For $u > b$, $\phi(u,b)$ satisfies the nonhomogeneous differential equation in (11), the special solution of $\phi_n(u,b)$ is

$$\phi_{n_0}(u,b) = \frac{\gamma}{\gamma+\delta}\phi(b,b),$$

then the general solution is

$$\phi_n(u,b) = C_1 e^{r_1 u} + C_2 e^{r_2 u} + C_3 e^{r_3 u} + \frac{\gamma}{\gamma+\delta}\phi(b,b),$$

with the C_1, C_2, C_3 are arbitrary coefficients and independent of u, $r_1 \geq 0$, $r_2 < 0$, $r_3 < 0$ being the roots of the characteristic equation

$$\frac{\sigma^2}{2}\xi^3 + (\frac{\sigma^2}{2}\nu + c)\xi^2 + (c\nu - (\lambda+\delta+\gamma))\xi - (\delta+\gamma)\nu = 0.$$

Following the condition $\phi_n(u,b) \to 0$ for $u \to +\infty$ that $C_1 = 0$, then the solution of $\phi_n(u,b)$ can be rewritten as

$$\phi_n(u,b) = C_2 e^{r_2 u} + C_3 e^{r_3 u} + \frac{\gamma}{\gamma+\delta}\phi(b,b). \tag{14}$$

Substituting (14) into (10), with subsequent comparison of the coefficients of $e^{-\nu u}$ yields that

$$\frac{\nu C_2 e^{r_2 b}}{r_2+\nu} + \frac{\nu C_3 e^{r_3 b}}{r_3+\nu} + \frac{\gamma}{\delta+\gamma}\phi(b,b) = \nu G_1 \frac{e^{s_1 b}}{s_1+\nu} + \nu G_2 \frac{e^{s_2 b}}{s_2+\nu} + \nu G_3 \frac{e^{s_3 b}}{s_3+\nu}. \tag{15}$$

Therefore,

$$\phi(u,b) = \begin{cases} K_1 e^{q_1 u} + \frac{\omega w}{\delta+\omega}, & (u \leq 0) \\ G_1 e^{s_1 u} + G_2 e^{s_2 u} + G_3 e^{s_3 u}, & (0 < u \leq b) \\ C_2 e^{r_2 u} + C_3 e^{r_3 u} + \frac{\gamma}{\gamma+\delta}\phi(b,b), & (u > b) \end{cases} \tag{16}$$

with boundary conditions (13), (15).

Subject to the condition (8), the K_1, G_1, G_2 and G_3 satisfy

$$\frac{\sigma^2}{2}K_1q_1^2-\omega(K_1+\frac{\omega w}{\delta+\omega})+\omega w=\frac{\sigma^2}{2}[G_1s_1^2+G_2s_2^2+G_3s_3^2], \tag{17}$$

subject to the condition (9), the G_1, G_3, G_3 and C_2, C_3 satisfy

$$\begin{aligned}&\frac{\sigma^2}{2}(G_1s_1^2e^{s_1b}+G_2s_2^2e^{s_2b}+G_3s_3^2e^{s_3b})+c(G_1s_1e^{s_1b}+G_2s_2e^{s_2b}+G_3s_3e^{s_3b})\\ &=\frac{\sigma^2}{2}(C_2r_2^2e^{r_2b}+C_3r_3^2e^{r_3b})+c(C_2r_2e^{r_2b}+C_3r_3e^{r_3b}).\end{aligned} \tag{18}$$

Considering the continuity of $\phi(u,b)$, following (16), let $u=0$ and $u=b$, one can get

$$K_1+\frac{\omega w}{\delta+\omega}=G_1+G_2+G_3, \tag{19}$$

$$G_1e^{s_1b}+G_2e^{s_2b}+G_3e^{s_3b}=C_2e^{r_2b}+C_3e^{r_3b}+\frac{\gamma}{\gamma+\delta}\phi(b,b). \tag{20}$$

Considering the continuity of $\phi'(u,b)$ at $u=0$, one can get

$$K_1q_1=G_1s_1+G_2s_2+G_3s_3. \tag{21}$$

By solving the simultaneous equations from (13), (15) and (17)-(21), the unknown coefficients K_1, G_1, G_2, G_3, C_2, C_3, $\phi(b,b)$ are determined, then substitution them in (16), the explicit solutions of $\phi(u,b)$ can be derived.

Based on the above analysis, when some parameters are given by specific numerical values, the numerical results of $\phi(u,b)$ can be obtained in next section.

4. Numerical example

In this section, a numerical example is presented to verify the relationship between $\phi(u,b)$ and initial surplus u or parameter b with a barrier dividend strategy.

It is assumed that the claim size is exponentially distributed $P(x)=1-e^{-\nu x}$, for $\sigma=0.5$, $c=1.5$, $\lambda=1$, $\delta=0.5$, $\gamma=0.5$, $\omega(u)=\omega=0.1$, $\nu=1$, $w(u)=w=1$, the *Table* 1 provides numerical results for the expected sum of discounted penalty $\phi_l(u,b)$, the *Table* 2 provides numerical results for the expected sum of discounted penalty $\phi_m(u,b)$ and the *Table* 3 provides numerical results for the expected sum of discounted penalty $\phi_n(u,b)$.

TABLE 1. The expected sum of discounted penalty $\phi_l(u,b)$ with $\omega = 0.1$, $\nu = 1$.

$b\backslash u$	-0.9	-0.8	-0.7	-0.6	-0.5	-0.4	-0.3	-0.2	-0.1	0
0.1	0.41968	0.435962	0.453291	0.471736	0.491368	0.512263	0.534502	0.558173	0.583367	0.610183
0.2	0.18103	0.181954	0.182938	0.183985	0.1851	0.186286	0.187549	0.188892	0.190323	0.191845
0.3	0.120711	0.117753	0.114605	0.111255	0.107689	0.103894	0.0998546	0.0955552	0.090979	0.0861084
0.4	0.105228	0.101274	0.0970657	0.0925867	0.0878196	0.0827456	0.0773452	0.0715972	0.0654793	0.0589677
0.5	0.101056	0.0968337	0.0923399	0.0875568	0.082466	0.0770475	0.0712804	0.0651421	0.0586088	0.0516551
0.6	0.0997595	0.0954539	0.0908712	0.0859937	0.0808022	0.0752767	0.0693956	0.063136	0.0564737	0.0493825
0.7	0.0992124	0.0948716	0.0902515	0.085334	0.0801001	0.0745294	0.0686002	0.0622895	0.0555726	0.0484235
0.8	0.098877	0.0945146	0.0898715	0.0849296	0.0796697	0.0740713	0.0681126	0.0617705	0.0550202	0.0478356

TABLE 2. The expected sum of discounted penalty $\phi_m(u,b)$ with $\nu = 1$.

$b\backslash u$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0.1	0.63979									
0.2	0.195734	0.202089								
0.3	0.0834972	0.0832552	0.0838455							
0.4	0.054688	0.0527526	0.051576	0.0507225						
0.5	0.0469258	0.0445342	0.0428815	0.0415275	0.0403492					
0.6	0.0445135	0.0419801	0.0401794	0.03867	0.0373277	0.0361159				
0.7	0.0434956	0.0409023	0.0390392	0.0374641	0.0360527	0.0347679	0.0335966			
0.8	0.0428715	0.0402416	0.0383402	0.0367248	0.035271	0.0339415	0.0327231	0.0316092		
0.9	0.042385	0.0397265	0.0377953	0.0361486	0.0346617	0.0332974	0.0320422	0.0308895	0.029835	
1	0.0419677	0.0392847	0.0373278	0.0356542	0.0341391	0.0327448	0.0314581	0.0302722	0.0291826	0.0281855

TABLE 3. The expected sum of discounted penalty $\phi_n(u,b)$ with $\gamma = 0.5$, $\nu = 1$.

$b\backslash u$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0.1	0.63979	0.441255	0.386195	0.369396	0.362899	0.359269	0.356521	0.354122	0.351922	0.349874
0.2		0.202089	0.148326	0.13274	0.127379	0.124812	0.12306	0.121593	0.120266	0.119035
0.3			0.0838455	0.0686017	0.0635608	0.0612934	0.059822	0.0586175	0.0575358	0.0565353
0.4				0.0507225	0.0457657	0.043577	0.0421793	0.0410438	0.0400267	0.0390866
0.5					0.0403492	0.0381752	0.0367912	0.0356685	0.0346634	0.0337347
0.6						0.0361159	0.0347282	0.0336021	0.0325938	0.0316621
0.7							0.0335966	0.0324623	0.0314464	0.0305074
0.8								0.0316092	0.0305842	0.0296369
0.9									0.029835	0.0288789
1										0.0281855

Consequently, the number results show that the higher the initial surplus of the insurance company, the smaller the expected sum of discounted penalty prior to the time of bankruptcy for fixed b, and when $0.1 \leq b \leq 1$, it following $Table$ 1, $Table$ 2, $Table$ 3 that $\phi_l(u,b)$, $\phi_m(u,b)$ and $\phi_n(u,b)$ are decreasing with respect to b for fixed u. Furthermore, to compare the number results, the maximize or minimize the expected sum of discounted penalty until bankruptcy can be obtained through choosing b appropriately in this model.

Conflict of Interests

The authors declare that there is no conflict of interests.

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