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RAINBOW NUMBERS FOR SMALL CYCLES

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Abstract. The *rainbow number* $rb(G,H)$ is the minimum number k such that any k-edge-coloring of G contains a rainbow copy of H. In this paper, we determine the rainbow numbers of small cycles in the complete split graph and maximal outerplanar graph.

Keywords: rainbow number; rainbow cycle; complete split graph; planar graph.

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1. Introduction

An edge-colored graph is called *rainbow* if all of its edges have distinct colors. For two graphs G and H, the *anti-Ramsey number* $ar(G,H)$, introduced by Erdős et al. [3], is the maximum number of colors in an edge-coloring of G with no rainbow copy of H. The *rainbow number* $rb(G,H)$ is the minimum number k such that any k-edge-coloring of G contains a rainbow copy of H. Clearly, we have $rb(G,H) = ar(G,H) + 1$.

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In this paper, we consider the rainbow number of cycles. Erdős et al. [3] posed a conjecture on the anti-Ramsey number for cycles in complete graphs, which was later proved by Montellano et al. [8]. Axenovich et al. [1] determine the anti-Ramsey number of cycles in complete bipartite graphs. Jin et al. gave the anti-Ramsey numbers for graphs with independent cycles in [6]. Recently, the authors [5,7] present bounds for the rainbow number of cycles in plane triangulations. Note that the complete split graph contains the complete graph as a subclass. Gorgol et al. [4] determined the rainbow number of C_3 and C_3^+, a triangle with a pendant edge, in complete split graphs. In this paper, we determine the rainbow numbers of small cycles in complete split graphs and planar graphs.

2. Preliminaries

Let K_n, C_n, P_n be a complete graph, a cycle, a path on n vertices respectively. For a set S, we denote by $|S|$ the cardinality of S. We define that a uv-path is a path with first vertex u and last vertex v. For two graphs $G = (V(G), E(G))$ and $H = (V(H), E(H))$, the join of G and H is defined to be the graph by $G \cup H = (V(G) \cup V(H), E(G) \cup E(H))$. The sum of G and H, denoted by $G + H$, is defined to be the graph $G \cup H + \{uv : u \in V(G), v \in V(H)\}$. A complete split graph $K_n + \bar{K}_s$ is the sum of a complete graph K_n and an empty graph $\bar{K}_s$. Denote by $\mathscr{M}_n$ the class of all the maximal outerplanar graphs of order n. For two disjoint subsets $R, T \subseteq V(G)$, denote by $E_G[R, T]$ the set of all the edges between R and T in G. We use $G[R]$ to denote the subgraph induced by R in G when $R \subseteq V(G)$ or $R \subseteq E(G)$. Let c be an edge-coloring of G. We use $c(W)$ to denote the set of colors of $W \subseteq E(G)$. When $W = \{e\}$, we use $c(e)$ for short. A connected graph that has no cut vertices is called a block. A block of a graph is a subgraph that is a block and is maximal with respect to this property.

We need the following results:

Lemma 2.1. [5] *Let C_k with $k \geq 4$ be a rainbow cycle in an edge-colored graph G. If $G[V(C_k)]$ has a chord e, then there exists a rainbow cycle containing e in G of length smaller than k.*

Lemma 2.2. [2] *If the graph G does not contain any even cycles, then each block of G is either a K_1 or a K_2 or an odd cycle.*

3. Main results

Theorem 3.1. *If $n \geq 3$, then $rb(\mathscr{M}_n, C_3) = n$.*

Proof. First, we construct a maximal outerplanar graph $M_n \in \mathscr{M}_n$ for all $n \geq 3$ and its edge coloring with $n-1$ colors that does not contain any rainbow C_3.

Given a 2-edge-colored K_3, we construct a sequence of quadrangulations Q_r on $r \geq 3$ vertices starting with $Q_3 \cong K_3$. From Q_r we construct Q_{r+1} by choosing the outerface, inseting a new vertex in it and making it adjacent to two vertices of an arbitrary edge on the boundary of Q_r, all these two edges are colored with a new color. Then, we get Q_{r+1}. So Q_{r+1} has $r+1$ vertices, $2(r+1)-3$ edges, r faces and r colors.

In this way, we obtain a maximal outerplanar graph M_n whose edges are colored with $n-1$ colors. Finally, we observe that M_n does not contain any rainbow C_3, which proves that $rb(\mathscr{M}_n, C_3) \geq n$.

Now, we prove $rb(\mathscr{M}_n, C_3) \leq n$.

Let $M_n \in \mathscr{M}_n$. Color all the edges of M_n by n colors. Suppose that M_n does not contain any rainbow C_3. Let G be a rainbow spanning subgraph of M_n with $|E(G)| = n$. Since $|E(G)| = n$ and $|V(G)| = n$, G contains a rainbow cycle C_k for some $k \geq 3$. If $k = 3$, then we obtain a rainbow C_3 in M_n, a contradiction. So we have $k \geq 4$. Since M_n is a maximal outerplanar graph, G has a cycle $\widetilde{C} \cong C_k$ (for some $k \geq 4$) with no inner vertices. Clearly, $M_n[\widetilde{C}]$ is a maximal outerplanar graph. Let v be a vertex with degree 2 in $M_n[\widetilde{C}]$ and u, w be two neighbors of v. Then there is a path uvw on $\widetilde{C}$ with $uw \in E(M_n)$. Since M_n does not contain any rainbow C_3, by Lemma 2.1, the cycle $\widetilde{C} - \{v\} + uw$ is a rainbow cycle of order $k-1$ in M_n. In any case we obtain a shorter rainbow cycle. Hence there is a rainbow C_3 in M_n, a contradiction. This proves that $rb(\mathscr{M}_n, C_3) \leq n$.

The proof is completed.

Theorem 3.2. *If $n+s \geq 4$ and $n \geq 2$, then*

$$rb(K_n + \bar{K}_s, C_4) = \begin{cases} n+s+\lfloor \frac{n+s}{3} \rfloor, & \text{if } n \geq 2s; \\ \\ n+\lfloor \frac{n}{2} \rfloor + s, & \text{if } 2 \leq n < 2s. \end{cases}$$

Proof. Let $N = V(K_n)$, $S = V(K_s)$,

$$r = \begin{cases} n+s+\lfloor \frac{n+s}{3} \rfloor, & \text{if } n \geq 2s; \\ \\ n+\lfloor \frac{n}{2} \rfloor + s, & \text{if } 2 \leq n < 2s. \end{cases}$$

First, we present a $(r-1)$-edge-coloring of $K_n + \bar{K}_s$ which does not contain any rainbow C_4 as follows.

Take a set of maximum number of vertex disjoint triangles, denoted by $D_1, D_2, \cdots, D_t$, in $K_n + \bar{K}_s$ and denote by $\{D_{t+1}, \cdots, D_{n+s-2t}\}$ the set of the remaining vertices. Color all the triangles by distinct colors. Then lexically color the edges between D_i and D_j by the color j for $i < j$.

Clearly, $t = \lfloor \frac{n+s}{3} \rfloor$ for $n \geq 2s$ and $t = \lfloor \frac{n}{2} \rfloor$ for $2 \leq n < 2s$. So we can find that the coloring constructed above contains exactly $r-1$ colors, which proves $rb(K_n + \bar{K}_s, C_4) \geq r$.

Now we prove the upper bound $rb(K_n + \bar{K}_s, C_4) \leq r$. Given a r-edge-coloring of $K_n + \bar{K}_s$, we need to show that $K_n + \bar{K}_s$ contains a rainbow C_4. By the contradiction, assume that $K_n + \bar{K}_s$ does not contain any rainbow C_4.

Claim 1. *For any C_k of $K_n + \bar{K}_s$, $k \geq 5$, C_k contains a path of order four, say $uvwx$, such that $ux \in E(K_n + \bar{K}_s)$.*

Proof. Let $P = u_1u_2u_3u_4$ be a path on the cycle C_k. If $u_1 \in N$ or $u_4 \in N$, then the result holds clearly. So we have $u_1, u_4 \in S$. Let u_5 be another neighbour of u_1 on the cycle C_k. Then $u_5 \in N$. Hence $u_5u_3 \in E(K_n + \bar{K}_s)$ and the path $u_5u_1u_2u_3$ is a path as desired.

Claim 2. *If $K_n + \bar{K}_s$ contains a rainbow even cycle C_{2a+4} for $a \geq 1$, then it contains a rainbow cycle of order $2a+2$.*

Proof. Let C_{2a+4} be a rainbow cycle in $K_n + \bar{K}_s$. By Claim 1, there is a path $P = uvwx$ on C_{4+2a} with $ux \in E(K_n + \bar{K}_s)$. Since $K_n + \bar{K}_s$ does not contain any rainbow C_4, by Lemma 2.1, we have that $C_{2a+4} - \{v, w\} + ux$ is a rainbow cycle of order $2a+2$ in $K_n + \bar{K}_s$.

From Claim 2, we have that $K_n + \bar{K}_s$ does not contain any rainbow even cycle.

Claim 3. *Let C, C' be two distinct rainbow triangles in $K_n+\bar{K}_s$. If $V(C)\cap V(C')\neq\emptyset$ and all the edges of $C\cup C'$ are colored by distinct colors, then $K_n+\bar{K}_s$ contains a rainbow C_4.*

Proof. First, we consider that $|V(C)\cap V(C')|=2$. Let $C=u_1u_2u_3u_1$ and $C'=u_2u_3u_4u_2$. It is easy to see that $u_1u_2u_3u_4u_1$ is a rainbow C_4 in $K_n+\bar{K}_s$.

Next, we consider that $|V(C)\cap V(C')|=1$. Let $C=u_1u_2u_3u_1$ and $C'=u_3u_4u_5u_3$. In the graph $K_n+\bar{K}_s$, every C_3 has at least two vertices in N. Let $u_1,u_4\in N$. Then $u_1u_4\in E(K_n+\bar{K}_s)$. Since the cycle $u_1u_3u_5u_4u_1$ is a C_4 in $K_n+\bar{K}_s$ and $K_n+\bar{K}_s$ does not contain any rainbow C_4, we have $c(u_1u_4)\in c(\{u_1u_3,u_3u_5,u_4u_5\})$. Then the cycle $u_1u_2u_3u_4u_1$ is a rainbow C_4 in $K_n+\bar{K}_s$.

Let G be a rainbow spanning subgraph of $K_n+\bar{K}_s$ with $|E(G)|=r$. Then G does not contain any even cycle.

Claim 4. *Let C, C' be two distinct odd cycles in G. Then $V(C)\cap V(C')=\emptyset$.*

Proof. Since G does not contain any even cycle, by Lemma 2.2, each block of G is a K_1 or a K_2 or an odd cycle. Suppose that $V(C)\cap V(C')\neq\emptyset$.

First, we consider that $|V(C)\cap V(C')|\geq 2$. Since C and C' are two cycles, it is easy to see that $C\cup C'$ does not contain any cut vertices. Then there is a block B of G containing $C\cup C'$. So B is not a K_1 or a K_2 or an odd cycle, a contradiction.

Now we consider that $|V(C)\cap V(C')|=1$. If C and C' are triangles, then by Claim 3, there is a rainbow C_4 in $K_n+\bar{K}_s$, a contradiction. Next, we distinguish the following 2 cases to complete the proof.

Case 1. C is a triangle, C' is not a triangle or C is not a triangle, C' is a triangle.

Without loss of generality, we only consider that C is a triangle and C' is not a triangle. Let $V(C)\cap V(C')=\{u_1\}$. Since G does not contain any rainbow even cycle, we have $|E(C')|\geq 5$.

If $u_1\in N$, then we let $u_1u_2u_3$ be a path on C'. Clearly, we have that $u_1u_3\in E(K_n+\bar{K}_s)$. Since C' is an odd cycle and $u_1u_2u_3$ has length 2, the cycle $C'-\{u_2\}+u_1u_3$ is an even cycle and it is not rainbow. Then by Lemma 2.1, the cycle $u_1u_2u_3u_1$ is a rainbow triangle in $K_n+\bar{K}_s$ and we have $c(u_1u_3)\in c(E(C'-u_2))$. Note that C is also a rainbow triangle in G. Then it is easy to see that $c(E(C))\cap c(E(u_1u_2u_3u_1))=\emptyset$ and $V(u_1u_2u_3u_1)\cap V(C)=\{u_1\}$. Therefore, by Claim 3, there is a rainbow C_4 in $K_n+\bar{K}_s$, a contradiction.

If $u_1 \in S$, then let u_4, u_5 be 2 neighbors of u_1 in C'. So we have that $u_4, u_5 \in N$ and $u_4u_5 \in E(K_n + \bar{K}_s)$. Since C' is an odd cycle and $|E(u_4u_1u_5)| = 2$, the cycle $C' - \{u_1\} + u_4u_5$ is an even cycle and it is not rainbow. By Lemma 2.1, the cycle $u_4u_1u_5u_4$ is a rainbow triangle in $K_n + \bar{K}_s$ and we have $c(u_4u_5) \in c(E(C' - u_1))$. Note that C is a rainbow triangle in G. Clearly, we have that all the edges of $C \cup u_4u_1u_5u_4$ are colored by distinct colors and $V(u_4u_1u_5u_4) \cap V(C) = \{u_1\}$. Thus, by Claim 3, there is a rainbow C_4 in $K_n + \bar{K}_s$, a contradiction.

Case 2. C and C' are not triangles.

Let $V(C) \cap V(C') = \{u_1\}$. Since G does not contain any rainbow even cycle, we have $|E(C')| \geq 5$ and $|E(C)| \geq 5$.

If $u_1 \in N$, then let $u_1u_2u_3$ be a path on C and $u_1u_4u_5$ be a path on C'. Clearly, we have $u_1u_3, u_1u_5 \in E(K_n + \bar{K}_s)$. Because C is an odd cycle and the length of the path $u_1u_2u_3$ equals two, the cycle $C - \{u_2\} + u_1u_3$ is an even cycle and it is not rainbow. By Lemma 2.1, the cycle $u_1u_2u_3u_1$ is a rainbow triangle in $K_n + \bar{K}_s$ and we have $c(u_1u_3) \in c(E(C - u_2))$. We also have C' is an odd cycle and $|E(u_1u_4u_5)| = 2$. Then $C' - \{u_4\} + u_1u_5$ is an even cycle and it is not rainbow. By Lemma 2.1, the cycle $u_1u_4u_5u_1$ is also a rainbow triangle in $K_n + \bar{K}_s$ and we have $c(u_1u_5) \in c(E(C' - u_4))$. Now $K_n + \bar{K}_s$ contains two rainbow triangles $u_1u_2u_3u_1$ and $u_1u_4u_5u_1$ with $c(E(u_1u_2u_3u_1)) \cap c(E(u_1u_4u_5u_1)) = \emptyset$. By Claim 3, there is a rainbow C_4 in $K_n + \bar{K}_s$, a contradiction.

If $u_1 \in S$, then let u_6, u_7 be 2 neighbors of u_1 in C and u_8, u_9 be 2 neighbors of u_1 in C'. Then we have $u_6, u_8 \in N$ and $u_6u_7, u_8u_9 \in E(K_n + \bar{K}_s)$. Since C is an odd cycle and the path $u_6u_1u_7$ has length 2, the cycle $C - \{u_1\} + u_6u_7$ is an even cycle and it is not rainbow. Then by Lemma 2.1, the cycle $u_1u_6u_7u_1$ is a rainbow triangle in $K_n + \bar{K}_s$ and we have $c(u_6u_7) \in c(E(C - u_1))$. We also have C' is an odd cycle and $|E(u_8u_1u_9)| = 2$. Then $C' - \{u_1\} + u_8u_9$ is an even cycle and it is not rainbow. By Lemma 2.1, the cycle $u_1u_8u_9u_1$ is also a rainbow triangle in $K_n + \bar{K}_s$ and we have $c(u_8u_9) \in c(E(C' - u_1))$. So $K_n + \bar{K}_s$ contains two rainbow triangles $u_1u_6u_7u_1$ and $u_1u_8u_9u_1$ with $c(E(u_1u_6u_7u_1)) \cap c(E(u_1u_8u_9u_1)) = \emptyset$. By Claim 3, there is a rainbow C_4 in $K_n + \bar{K}_s$, a contradiction.

This proves the Claim.

Let m be the number of odd cycles in G. Denote T by the graph obtained from G by deleting one edge in each odd cycle. Since G does not contain any even cycle, T does not contain any cycle. So $|E(T)| \leq n+s-1$. From Claim 4, we have $|E(G)| = |E(T)|+m \leq n+s+m-1$. It is easy to see that

$$m \leq \begin{cases} \lfloor \frac{n+s}{3} \rfloor, & \text{if } n \geq 2s; \\ \\ \lfloor \frac{n}{2} \rfloor, & \text{if } 2 \leq n < 2s. \end{cases}$$

So we have that $|E(G)| \leq r-1 < r$, a contradiction to the fact that $|E(G)| = r$.

The proof is completed.

Conflict of Interests

The authors declare that there is no conflict of interests.

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