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CONVERGENCE AND STABILITY ANALYSIS OF ITERATIVE ALGORITHM FOR A GENERALIZED SET-VALUED MIXED EQUILIBRIUM PROBLEM

F.A. KHAN[1,*], K.R. KAZMI[2], F.M. ALHARBI[1], M. DILSHAD[1]

[1]Department of Mathematics, University of Tabuk, Tabuk-71491, KSA

[2]Department of Mathematics, A.M.U., Aligarh-202002, India

Abstract. In this paper, we consider a generalized set-valued mixed equilibrium problem (in short, GSMEP) in real Hilbert space. Related to GSMEP, we consider a generalized Wiener-Hopf equation problem (in short, GWHEP) and show an equivalence relation between them. Further, we give a fixed-point formulation of GWHEP and construct an iterative algorithm for GWHEP. Furthermore, we extend the notion of stability given by Harder and Hick [3] and prove the existence of a solution of GWHEP and discuss the convergence and stability analysis of the iterative algorithm. Our results can be viewed as a refinement and improvement of some known results in the literature.

Keywords: generalized set-valued mixed equilibrium problem; generalized Wiener-Hopf equation problem; regularized operator; Yosida approximation; maximal strongly η-monotone mappings; cocoercive mappings; mixed-Lipschitz continuous mappings.

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1. Introduction

*Corresponding author
E-mail address: faizan_math@yahoo.com
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Equilibrium problems, as the important extension of variational inequalities, have been widely studied in recent years. One of the most interesting and important problems in the theory of equilibrium problems is the development of an efficient and implementable iterative algorithm. Various kinds of iterative schemes have been proposed for solving equilibrium problems and variational inequalities, see for example [1-9,11-13,15]. In early 1990's, Robinson [14] and Shi [15] initially used Wiener-Hopf equation to study the variational inequalities. In 2002, Moudafi [8] has studied the convergence analysis for a mixed equilibrium problem involving single-valued mappings.

Recently, many authors given in [2,4-6,11,12,15] used various generalizations of Wiener-Hopf equations to develop the iterative algorithms for solving various classes of variational inequalities and mixed equilibrium problems involving single and set-valued mappings.

Inspired by the works given in [2,4-6,8,11,12,15], in this paper, we consider a generalized set-valued mixed equilibrium problem (GSMEP) in real Hilbert space. Related to GSMEP, we consider a generalized Wiener-Hopf equation problem (GWHEP) and show an equivalence relation between them. Further, we give a fixed-point formulation of GWHEP and construct an iterative algorithm for GWHEP. Furthermore, we extend the notion of stability given by Harder and Hick [3] and prove the existence of a solution of GWHEP and discuss the convergence and stability analysis of the iterative algorithm. By exploiting the technique of this paper, one can generalize and improve the results given in [1-9,11-13,15].

2. Preliminaries

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively; let K be a nonempty, closed and convex subset of H and let $CB(H)$ be the family of all nonempty, closed and bounded subsets of H. The Hausdorff metric $\mathscr{H}(\cdot,\cdot)$ on $CB(H)$ is defined by

$$\mathscr{H}(A,B) = \max\left\{\sup_{x\in A}\inf_{y\in B} d(x,y),\ \sup_{y\in B}\inf_{x\in A} d(x,y)\right\},\ A,B\in CB(H).$$

We need the following known concepts and results.

Definition 2.1[6]. Let $\eta : H \times H \to H$ be a mapping. A set-valued mapping $M : H \to 2^H$ is said to be:

(i) s-strongly monotone if there exists a constant $s > 0$ such that

$$\langle u - v, \eta(x,y)\rangle \geq s||x-y||^2, \ \forall x,y \in H, \ u \in M(x), \ v \in M(y);$$

(ii) maximal strongly η-monotone if M is strongly η-monotone and $(I+\rho M)(H) = H$ for any $\rho > 0$, where I stands for identity mapping.

Definition 2.2[8]. A mapping $T : H \to H$ is said to be γ-cocoercive if there exists a constant $\gamma > 0$ such that

$$\langle T(x) - T(y), x-y\rangle \geq \gamma||Tx - Ty||^2, \ \forall x,y \in H.$$

Definition 2.3[6]. A set-valued mapping $T : H \to CB(H)$ is said to be μ-$\mathscr{H}$-Lipschitz continuous if there exists a constant $\mu > 0$ such that

$$\mathscr{H}(T(x), T(y)) \leq \mu\|x-y\|, \ \forall x,y \in H.$$

Theorem 2.1[6,10]. (i) Let $T : H \to CB(H)$ be a set-valued mapping. Then for any given $\varepsilon > 0$ and for any given $x, y \in H$ and $u \in T(x)$, there exists $v \in T(y)$ such that

$$\|u - v\| \leq (1+\varepsilon) \ \mathscr{H}(T(x), T(y));$$

(ii) If $T : H \to C(H)$, then the above inequality holds for $\varepsilon = 0$.

Definition 2.4[1]. A real valued bifunction $F : K \times K \to \mathbb{R}$ is said to be:

(i) monotone if

$$F(x,y) + F(y,x) \leq 0, \ \forall x,y \in K;$$

(ii) strictly monotone if

$$F(x,y) + F(y,x) < 0, \ \forall x,y \in K \text{ with } x \neq y;$$

(iii) α-strongly monotone if there exists a constant $\alpha > 0$ such that

$$F(x,y) + F(y,x) \leq -\alpha||x-y||^2, \ \forall x,y \in K;$$

(iv) upper-hemicontinuous if

$$\limsup_{t\to 0} F(tz+(1-t)x,y) \le F(x,y),\ \forall x,y,z \in K.$$

Theorem 2.2[1]. If the following conditions hold:

(i) F is monotone and upper hemicontinuous;

(ii) $F(x,.)$ is convex and lower-semicontinuous for each $x \in K$;

(iii) There exists a compact subset B of H and there exists $y_0 \in B \cap K$ such that $F(x,y_0) < 0$ for each $x \in K \setminus B$.

Then the set of solutions to the following equilibrium problem (in short, EP): Find $x^* \in K$ such that

$$F(x^*,y) \ge\ 0,\ \forall y \in K, \tag{2.1}$$

is nonempty, convex and compact.

Remark 2.1[1,8]. If F is strictly monotone, then the solution of EP (2.1) is unique.

Definition 2.5[3,13]. Let $G : H \to 2^H$ be a set-valued mapping and $x_0 \in H$. Assume that $x_{n+1} \in f(G,x_n)$ defines an iteration procedure which yields a sequence of points $\{x_n\}$ in H. Suppose that $F(G) = \{x \in H :\ x \in G(x)\} \neq \emptyset$ and $\{x_n\}$ converges to some $x \in G(x)$. Let $\{y_n\}$ be an arbitrary sequence in H and $\varepsilon_n = \|y_{n+1} - x_{n+1}\|$.

(i) If $\lim\limits_{n\to\infty} \varepsilon_n = 0$ implies that $\lim\limits_{n\to\infty} y_n = x$, then the iteration procedure $x_{n+1} \in f(G,x_n)$ is said to be G-stable.

(ii) If $\sum\limits_{n=0}^{\infty} \varepsilon_n < 0$ implies that $\lim\limits_{n\to\infty} y_n = x$, then the iteration procedure $x_{n+1} \in f(G,x_n)$ is said to be almost G-stable.

Remark 2.2. Definition 2.5 can be viewed as an extension of the concept of stability of the iteration procedure given by Harder and Hick [3].

Theorem 2.3[5,6]. Let $\{a_n\},\ \{b_n\}$ and $\{c_n\}$ be nonnegative real sequences satisfying

$$a_{n+1} = (1-\lambda_n)a_n + \lambda_n b_n + c_n,\ \forall n \ge 0,$$

where $\sum\limits_{n=0}^{\infty} \lambda_n = \infty$; $\{\lambda_n\} \subset [0,1]$; $\lim\limits_{n\to\infty} b_n = 0$ and $\sum\limits_{n=0}^{\infty} c_n < \infty$. Then $\lim\limits_{n\to\infty} a_n = 0$.

3. Formulation of the problems

Let $g : K \to K,\ \eta : H \times H \to H,\ N : H \times H \times H \to H$ be nonlinear mappings and $F : K \times K \to \mathbb{R}$ be a bifunction such that $F(x,x) = 0, \forall x \in K$. Let $T, B, S : H \to CB(H)$ be three non-monotone set-valued mappings with non-compact values, then we consider the following generalized set-valued mixed equilibrium problem (in short, GSMEP):

Find $x \in K,\ u \in T(x),\ v \in B(x),\ w \in S(x)$ such that

$$F(g(x),y) + \langle N(u,v,w), \eta(y,g(x))\rangle \geq 0,\ \forall y \in K. \tag{3.1}$$

We remark that for appropriate choices of the mappings g, F, N, T, B, S, and the space H, one can obtain many known classes of mixed equilibrium problems and variational inequalities from GSMEP (3.1), see similar type of problems in [1-9,11-13,15].

We need the following concepts and results.

Definition 3.1[8,9]. For $\rho > 0$ and a given bifunction F, the associated Yosida approximation, F_ρ, over K and the corresponding regularized operator, A^F_ρ, are defined as follows:

$$F_\rho(x,y) = \Big\langle \frac{1}{\rho}(x - J^F_\rho(x)), \eta(y,x)\Big\rangle \text{ and } A^F_\rho(x) = \frac{1}{\rho}\big(x - J^F_\rho(x)\big), \tag{3.2}$$

in which $J^F_\rho(x) \in K$ is the unique solution of

$$\rho F(J^F_\rho(x), y) + \langle J^F_\rho(x) - x, \eta\big(y, J^F_\rho(x)\big)\rangle \geq 0,\ \forall y \in K. \tag{3.3}$$

Remark 3.1. If F satisfies all conditions of Theorem 2.2 and Remark 2.1, and η is continuous and affine then the problem (3.3) has an unique solution.

Remark 3.2. If $K \equiv H$ and $F(x,y) = \sup\limits_{\xi \in M(x)} \langle \xi, \eta(y,x)\rangle,\ \forall x,y \in K$, where M is a maximal strongly η-monotone operator, then it directly yields

$$J^F_\rho(x) = (I + \rho M)^{-1}(x) \text{ and } A^F_\rho(x) = M_\rho(x),$$

where $M_\rho := \frac{1}{\rho}(I - (I + \rho M)^{-1})$ is the Yosida approximation of M. In this case J^F_ρ generalizes the concept of resolvent mapping for single-valued maximal strongly monotone mapping given in Li and Feng [7].

Theorem 3.1[6]. Let the bifunction $F : K \times K \to \mathbb{R}$ be α-strongly monotone and satisfy the conditions of Theorem 2.2, and let the mapping $\eta : H \times H \to H$ be δ-strongly monotone and τ-Lipschitz continuous with $\eta(x,y) + \eta(y,x) = 0,\ \forall x,y \in H$, then the mapping J_ρ^F is $\frac{\tau}{\delta+\rho\alpha}$-Lipschitz continuous and A_ρ^F is c-cocoercive for $c = \frac{1}{\max\{1+(\frac{\tau}{\delta+\rho\alpha})^2, 2\}}$.

Now, related to GSMEP (3.1), we consider the following generalized Wiener-Hopf equation problem (in short, GWHEP):

Find $z \in H, x \in K, u \in T(x), v \in B(x), w \in S(x)$ such that

$$N(u,v,w) + A_\rho^F(z) \ = \ 0 \text{ and } g(x) \ = \ J_\rho^F(z). \tag{3.4}$$

Lemma 3.1. GSMEP (3.1) has a solution (x,u,v,w) with $x \in K, u \in T(x), v \in B(x), w \in S(x)$ if and only if (x,u,v,w) satisfies the relation

$$g(x) \ = \ J_\rho^F[g(x) - \rho N(u,v,w)].$$

Proof. The proof directly follows from the definition of J_ρ^F given by (3.3).

4. Iterative algorithm

The following lemma, which will be used in the sequel, is an equivalence between the solutions of GSMEP (3.1) and GWHEP (3.4).

Lemma 4.1. GSMEP (3.1) has a solution (x,u,v,w) with $x \in K, u \in T(x), v \in B(x), w \in S(x)$ if and only if GWHEP (3.4) has a solution (z,x,u,v,w) with $z \in H$, where

$$g(x) \ = \ J_\rho^F(z) \tag{4.1}$$

and

$$z \ = \ g(x) - \rho N(u,v,w). \tag{4.2}$$

Proof. The proof immediately follows from the definition of A_ρ^F and Lemma 3.1.

The GWHEP (3.4) can be written as

$$A_\rho^F(z) = -N(u,v,w),$$

which implies that

$$\begin{aligned} z &= J_\rho^F(z) - \rho N(u,v,w), \\ &= g(x) - \rho N(u,v,w), \text{ by } (4.1). \end{aligned}$$

Using this fixed point formulation, we construct the following iterative algorithm.

Iterative Algorithm 4.1. For a given $z_0 \in H,\ x_0 \in K,\ u_0 \in T(x_0),\ v_0 \in B(x_0),\ w_0 \in S(x_0)$ and $g(x_0) = J_\rho^F(z_0)$, using induction principle, we can compute an approximate solution $(z_n, x_n, u_n, v_n, w_n)$ given by the following iterative scheme:

$$g(x_n) = J_\rho^F(z_n), \tag{4.3}$$

$$u_n \in T(x_n) :\ ||u_{n+1} - u_n|| \leq \left(1 + (1+n)^{-1}\right) \mathscr{H}(T(x_{n+1}), T(x_n)), \tag{4.4}$$

$$v_n \in B(x_n) :\ ||v_{n+1} - v_n|| \leq \left(1 + (1+n)^{-1}\right) \mathscr{H}(B(x_{n+1}), B(x_n)), \tag{4.5}$$

$$w_n \in S(x_n) :\ ||w_{n+1} - w_n|| \leq \left(1 + (1+n)^{-1}\right) \mathscr{H}(S(x_{n+1}), S(x_n)), \tag{4.6}$$

$$z_{n+1} = (1-\lambda) z_n - \lambda \left[g(x_n) - \rho N(u_n, v_n, w_n)\right], \tag{4.7}$$

where $n = 0, 1, 2, ...;\ \rho > 0$ is a constant and $0 < \lambda < 1$ is a relaxation parameter.

5. Existence of solution, convergence and stability analysis

We prove the existence of a solution of GWHEP (3.4) and discuss the convergence and stability analysis of the Iterative Algorithm 4.1.

Theorem 5.1. Let K be a nonempty, closed and convex subset of H; let the mapping $\eta : H \times H \to H$ be δ-strongly monotone and τ-Lipschitz continuous with $\eta(x,y) + \eta(y,x) = 0,\ \forall x, y \in H$; let the bifunction $F : K \times K \to \mathbb{R}$ be α-strongly monotone and satisfy the assumptions of Theorem 2.2; let the mapping $N : H \times H \times H \to H$ be $(\sigma_1, \sigma_2, \sigma_3)$-mixed Lipschitz continuous; let the mappings $T, B, S : H \to CB(H)$ be μ_1-$\mathscr{H}$-Lipschitz continuous, μ_2-$\mathscr{H}$-Lipschitz and μ_3-$\mathscr{H}$-Lipschitz continuous, respectively; let the mapping $g : K \to K$ be γ-strongly monotone and ξ-Lipschitz continuous. Suppose that there exists a constant $\rho > 0$ such that the following conditions hlod:

$$\frac{\tau}{\delta + \rho\alpha}\left[1 + \frac{\rho\ e}{b}\right] < 1\ ; \tag{5.1}$$

$$\xi > \sqrt{2\gamma - 1}\ ;\ \tau e < \alpha b\ ;\ b > 0, \tag{5.2}$$

where $e := (\sigma_1\mu_1 + \sigma_2\mu_2 + \sigma_3\mu_3)$ and $b := 1 - \sqrt{1 - 2\gamma + \xi^2}$. Then the sequences $\{z_n\}, \{x_n\}, \{u_n\}, \{v_n\}, \{w_n\}$ generated by Iterative Algorithm 4.1 strongly converge to $z \in H, x \in K, u \in T(x), v \in B(x), w \in S(x)$, respectively, and (z, x, u, v, w) is a solution of GWHEP (3.4).

Proof. From Iterative Algorithm 4.1, we have

$$\begin{aligned} ||z_{n+2} - z_{n+1}|| \leq\ & (1-\lambda)||z_{n+1} - z_n|| + \lambda||g(x_{n+1}) - g(x_n)|| \\ & + \lambda\rho||N(u_{n+1}, v_{n+1}, w_{n+1}) - N(u_n, v_n, w_n)||. \end{aligned} \tag{5.3}$$

Since N is $(\sigma_1, \sigma_2, \sigma_3)$-mixed Lipschitz continuous; T is μ_1-$\mathscr{H}$-Lipschitz continuous; B is μ_2-$\mathscr{H}$-Lipschitz continuous and S is μ_3-$\mathscr{H}$-Lipschitz continuous, we have

$$\begin{aligned} & ||N(u_{n+1}, v_{n+1}, w_{n+1}) - N(u_n, v_n, w_n)|| \\ & \leq \sigma_1||u_{n+1} - u_n|| + \sigma_2||v_{n+1} - v_n|| + \sigma_3||w_{n+1} - w_n|| \\ & \leq \left(1 + (1+n)^{-1}\right) \left[\sigma_1\mathscr{H}(T(x_{n+1}), T(x_n)) + \sigma_2\mathscr{H}(B(x_{n+1}), B(x_n)) + \sigma_3\mathscr{H}(S(x_{n+1}), S(x_n))\right] \\ & \leq \left(1 + (1+n)^{-1}\right) (\sigma_1\mu_1 + \sigma_2\mu_2 + \sigma_3\mu_3)||x_{n+1} - x_n||. \end{aligned} \tag{5.4}$$

By using Theorem 3.1 and (4.3), we have

$$\begin{aligned} ||g(x_{n+1}) - g(x_n)|| &= ||J_\rho^F(z_{n+1}) - J_\rho^F(z_n)|| \\ &\leq \frac{\tau}{\delta + \rho\alpha}||z_{n+1} - z_n||. \end{aligned} \tag{5.5}$$

Using γ-strongly monotonicity and ξ-Lipschitz continuity of g and (5.5), we have

$$\begin{aligned} ||x_{n+1} - x_n|| &= ||x_{n+1} - x_n - (g(x_{n+1}) - g(x_n)) + J_\rho^F(z_{n+1}) - J_\rho^F(z_n)|| \\ &\leq ||x_{n+1} - x_n - (g(x_{n+1}) - g(x_n))|| + \frac{\tau}{\delta + \rho\alpha}\ ||z_{n+1} - z_n|| \\ &\leq \sqrt{1 - 2\gamma + \xi^2}\ ||x_{n+1} - x_n|| + \frac{\tau}{\delta + \rho\alpha}\ ||z_{n+1} - z_n||, \end{aligned}$$

which implies that

$$||x_{n+1} - x_n|| \ \leq\ \frac{\tau}{(1 - \sqrt{1 - 2\gamma + \xi^2})\ (\delta + \rho\alpha)}\ ||z_{n+1} - z_n||. \tag{5.6}$$

From (5.3)-(5.6), we have the following estimate:

$$\begin{aligned}||z_{n+2}-z_{n+1}|| &\leq (1-\lambda)||z_{n+1}-z_n|| \\ &\quad +\lambda\ \frac{\tau}{\delta+\rho\alpha}\Big[1+\frac{\rho(\sigma_1\mu_1+\sigma_2\mu_2+\sigma_3\mu_3)(1+(1+n)^{-1})}{(1-\sqrt{1-2\gamma+\xi^2})}\Big]||z_{n+1}-z_n|| \\ &= (1-\lambda\Big(1-\theta_n)\Big)||z_{n+1}-z_n||, \end{aligned} \tag{5.7}$$

where

$$\theta_n \ := \ \frac{\tau}{\delta+\rho\alpha}\Big[1+\frac{\rho(\sigma_1\mu_1+\sigma_2\mu_2+\sigma_3\mu_3)(1+(1+n)^{-1})}{(1-\sqrt{1-2\gamma+\xi^2})}\Big].$$

Letting $n\to\infty$, we see that $\theta_n\to\theta$, where

$$\theta \ := \ \frac{\tau}{\delta+\rho\alpha}\Big[1+\frac{\rho(\sigma_1\mu_1+\sigma_2\mu_2+\sigma_3\mu_3)}{(1-\sqrt{1-2\gamma+\xi^2})}\Big]. \tag{5.8}$$

Since $\theta<1$ by conditions (5.1), (5.2), then $(1-\lambda\big(1-\theta_n)\big)<1$ for sufficiently large n. It follows from (5.7) that $\{z_n\}$ is Cauchy sequence and hence there is a $z\in H$ such that $z_n\to z$. Similarly, by (5.6), we observe that $x_n\to x\in K$ as $n\to\infty$, since K is closed. Also, from (4.4)-(4.6) and the Lipschitz continuity of T,B,S, we have $u_n\to u,\ v_n\to v$ and $w_n\to w$ in H.

Next, we claim that $u\in T(x)$. Since $u_n\in T(x_n)$, we have

$$\begin{aligned} d(u,T(x)) \ &\leq \ ||u-u_n||+d(u_n,T(x_n))+\mathscr{H}(T(x_n),T(x)) \\ &\leq ||u-u_n||+\mu_1||x_n-x||\to 0 \text{ as } n\to\infty. \end{aligned}$$

Hence $u\in T(x)$, since $T(x)\in CB(H)$. In similar way, we can observe that $v\in B(x)$ and $w\in S(x)$. Finally, continuity of N,T,B,S,g,J_ρ^F, and Iterative Algorithm 4.1 ensure that (z,x,u,v,w) is a solution of GWHEP (3.4).

Theorem 5.2(Stability). Let the mappings g,η,F,N,T,B,S be same as in Theorem 5.1 and conditions (5.1), (5.2) of Theorem 5.1 hold with $e=(1+\varepsilon)(\sigma_1\mu_1+\sigma_2\mu_2+\sigma_3\mu_3)$. Let $\{q_n\}$ be any sequence in H and define $\{a_n\}\subset[0,\infty)$ by

$$g(y_n)=J_\rho^F(q_n), \tag{5.9}$$

$$\overline{u}_n\in T(y_n)\ :\ ||\overline{u}_{n+1}-\overline{u}_n||\leq\ \big(1+(1+n)^{-1}\big)\ \mathscr{H}(T(y_{n+1}),T(y_n)), \tag{5.10}$$

$$\overline{v}_n\in B(y_n)\ :\ ||\overline{v}_{n+1}-\overline{v}_n||\ \leq\big(1+(1+n)^{-1}\big)\ \mathscr{H}(B(y_{n+1}),B(y_n)), \tag{5.11}$$

$$\overline{w}_n \in S(y_n) \ : \ ||\overline{w}_{n+1} - \overline{w}_n|| \leq \big(1 + (1+n)^{-1}\big)\mathscr{H}(S(y_{n+1}), S(y_n)), \tag{5.12}$$

$$a_n \ = \ ||q_{n+1} - (1-\lambda)q_n - \lambda[g(y_n) - \rho N(\overline{u}_n, \overline{v}_n, \overline{w}_n)]||, \tag{5.13}$$

where $n = 0, 1, 2, ...$; $\rho > 0$ is a constant and $0 < \lambda < 1$ is a relaxation parameter. Then $\lim_{n\to\infty}(q_n, y_n, \overline{u}_n, \overline{v}_n, \overline{w}_n) = (z, x, u, v, w)$ if and only if $\lim_{n\to\infty} a_n = 0$, where (z, x, u, v, w) is a solution of GWHEP (3.4).

Proof. By Theorem 5.1, GWHEP (3.4) has a solution (z, x, u, v, w), that is,

$$z = (1-\lambda)z + \lambda[g(x) - \rho N(u, v, w)].$$

Now, we assume that $\lim_{n\to\infty} a_n = 0$, we have

$$\begin{aligned}||q_{n+1} - z|| &\leq ||(1-\lambda)q_n + \lambda[g(y_n) - \rho N(\overline{u}_n, \overline{v}_n, \overline{w}_n)] - z|| \\ &\quad + ||q_{n+1} - (1-\lambda)q_n - \lambda[g(y_n) - \rho N(\overline{u}_n, \overline{v}_n, \overline{w}_n)]|| \\ &\leq (1-\lambda)||q_n - z|| + \lambda||g(y_n) - g(x)|| + \rho\lambda||N(\overline{u}_n, \overline{v}_n, \overline{w}_n) - N(u, v, w)|| + a_n.\end{aligned}$$

By Theorem 2.1 and Theorem 3.1, the preceding inequality reduces to

$$\begin{aligned}||q_{n+1} - z|| &\leq (1-\lambda)||q_n - z|| + \lambda\frac{\tau}{\delta+\rho\alpha}||q_n - z|| \\ &\quad + \lambda\rho(1+\varepsilon)\big(\sigma_1\mathscr{H}(T(y_n), T(y)) + \sigma_2\mathscr{H}(B(y_n), B(y)) + \sigma_3\mathscr{H}(S(y_n), S(y))\big) \\ &\leq (1-\lambda)||q_n - z|| + \lambda\frac{\tau}{\delta+\rho\alpha}||q_n - z|| + \lambda\rho(1+\varepsilon)(\sigma_1\mu_1 + \sigma_2\mu_2 + \sigma_3\mu_3)||y_n - x|| + a_n.\end{aligned} \tag{5.14}$$

Next, we estimate $||y_n - x||$:

$$\begin{aligned}||y_n - x|| &\leq ||y_n - x - (g(y_n) - g(x))|| + ||J_\rho^F(q_n) - J_\rho^F(z)|| \\ &\leq \sqrt{1 - 2\gamma + \xi^2}\ ||y_n - x|| + \frac{\tau}{\delta+\rho\alpha}\ ||q_n - z||.\end{aligned}$$

It follows that

$$||y_n - x|| \leq \frac{\tau}{(1 - \sqrt{1-2\gamma+\xi^2})(\delta+\rho\alpha)}\ ||q_n - z||. \tag{5.15}$$

Hence, from (5.14) and (5.15), we have the following estimate:

$$||q_{n+1} - z|| \leq \big(1 - \lambda(1-\theta_\varepsilon)\big)||q_n - z|| + a_n, \tag{5.16}$$

where $\theta_\varepsilon := \frac{\tau}{\delta+\rho\alpha}\Big[1 + \frac{\rho(1+\varepsilon)(\sigma_1\mu_1+\sigma_2\mu_2+\sigma_3\mu_3)}{(1-\sqrt{1-2\gamma+\xi^2})}\Big]$.

Setting: $b_n = ||q_n - z||$; $\lambda_n = \lambda(1-\theta_\varepsilon)$; $\beta_n = \lambda^{-1}(1-\theta_\varepsilon)^{-1}a_n$; $\gamma_n = 0$, $\forall n$.

By conditions (5.1), (5.2), it follows that $\theta_\varepsilon < 1$, and hence $\lambda_n \in [0,1]$, $\forall n$ and $\sum \lambda_n = \infty$. Since $\lim\limits_{n\to\infty} a_n = 0$, then $\lim\limits_{n\to\infty} \beta_n = 0$. Hence, by Theorem 2.3 and (5.16), it follows that $b_n \to 0$ as $n \to 0$, that is, $q_n \to z$ as $n \to \infty$. Also, from (5.10)-(5.12), (5.15) and the Lipschitz continuity of N, T, B, S, we observe that $y_n \to x$, $\overline{u}_n \to u$, $\overline{v}_n \to v$ and $\overline{w}_n \to w$ as $n \to \infty$. Thus, $\lim\limits_{n\to\infty}(q_n, y_n, \overline{u}_n, \overline{v}_n, \overline{w}_n) = (z, x, u, v, w)$.

Conversely, assume that $\lim\limits_{n\to\infty}(q_n, y_n, \overline{u}_n, \overline{v}_n, \overline{w}_n) = (z, x, u, v, w)$. Then (5.13) implies that

$$a_n \leq ||q_{n+1} - z|| + ||(1-\lambda)q_n + \lambda[g(y_n) - \rho N(\overline{u}_n, \overline{v}_n, \overline{w}_n)] - z||$$

$$\leq ||q_{n+1} - z|| + (1-\lambda)||q_n - z|| + \lambda||g(y_n) - g(x)|| + \rho\lambda||N(\overline{u}_n, \overline{v}_n, \overline{w}_n) - N(u, v, w)||$$

$$\to 0 \text{ as } n \to \infty.$$

This completes the proof.

Remark 5.1. For $\rho > 0$, it is clear that $\gamma \leq \xi$; $\delta \leq \tau$; $\xi \geq \sqrt{2\gamma - 1}$; $\tau e < \alpha b$, $b > 0$. Further, $\theta \in (0,1)$ and conditions (5.1),(5.2) of Theorem 5.1 hold for some suitable values of constants, for example,

$$(\alpha = 4,\ \gamma = \xi = 1.5,\ \delta = 1,\ \tau = 1.5,\ \varepsilon = .1,\ \sigma_1 = \sigma_2 = \sigma_3 = \mu_1 = \mu_2 = \mu_3 = .5,\ b = .5).$$

Conflict of Interests

The authors declare that there is no conflict of interests.

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