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DYNAMICAL BEHAVIORS OF A STOCHASTIC SIRS EPIDEMIC MODEL

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Abstract. In this paper, we study the dynamical behavior of a stochastic SIRS epidemic model with specific nonlinear incidence rate and vaccination. We show the existence and positivity of the solution of the SIRS stochastic differential equation. We defined a number $\mathscr{R}$ and we prove the disease free equilibrium is almost sure exponentially stable if $\mathscr{R} < 1$. We studying the behavior around the endemic equilibrium E*. Numerical simulations presented our theoretical results.

Keywords: SIRS epidemic model; vaccination; incidence rate; white noise; stochastic stability.

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1. Introduction

Mathematical epidemiology play an important role in the study and control of the infectious diseases, the objective of this study is to implement measures to combat and terminate the

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spread of these diseases.

In recent years many authors have been developed the mathematics model for the transmission dynamics of infectious diseases, amid these models there are the classical deterministic SIR epidemic model [1, 2], the population are divide into three classes or S represents the number of the individuals susceptible, I represents the number of infective individuals, R represents the number of recovered individuals with temporary immunity acquired from a disease.

In the other hand the modeling of population dynamics of diseases have recognized the introduction of stochastic term into deterministic models witch do incorporate the effect of fluctuating environment. To formulate stochastic differential equation (SDE) there is many approach, D. Greenhalgh et al. [3] have used The technique of parameter perturbation by the white noise. They proved Almost sure exponential stability of the disease free equilibrium and stability in probability. Adnani et al.[4] and Lahrouz et al.[13] have utilized The technique of perturbations stochastic by withe noise around the endemic equilibrium state. They have proved the asymptotically mean square stable of stochastic linearized system. The case of a color noise was introduced by Lahrouz et al., and Gray et al.[5],[6]. They have made a full analysis on asymptotic behavior of an SIS epidemic model under a finite regimes-switching.

Men et al.[7], have studied the SIR Models with horizontal and vertical transmission described by the system of differential equations :

$$
\begin{aligned}
\frac{dS}{dt} &= -\beta SI - bS + (1-m)pdI + b(1-m)(S+R),\\
\frac{dI}{dt} &= \beta SI - (pd+r)I,\\
\frac{dR}{dt} &= rI - bR + dmpI + mb(S+R),
\end{aligned}
\tag{1}
$$

where β is the contact rate, b is the mortality rate in the susceptible and the recovered individuals, d is the mortality rate in the infective individuals, r is the recover rate in the infective individuals into recovered individuals, p is the proportion of the offspring of infective parents that are susceptible individuals, and q is the proportion of the offspring of infective parents that are infective individuals, p,q verify $p+q=1$, m is the successful vaccination proportion to

the newborn from S and R, Men et al. have defined the reproduction number of system (1) by $\mathscr{R}_0 = \frac{(1-m)\beta}{pd+r}$, and proved that If $\mathscr{R}_0 < 1$ then the infection-free equilibrium $E_0(1-m,0)$ is globally stable, else if $\mathscr{R}_0 > 1$ then the epidemic equilibrium $E^*(S^*,I^*)$ is globally asymptotically stable. For biological reasons, we assume that $b - pd > 0$. We consider the following SIRS model with non-linear incidence rate :

$$
\begin{aligned}
\frac{dS}{dt} &= -\frac{\beta SI}{f(S,I)} - bS + (1-m)pdI + b(1-m)(S+R) + \gamma R, \\
\frac{dI}{dt} &= \frac{\beta SI}{f(S,I)} - (pd+r)I, \\
\frac{dR}{dt} &= rI - bR + dmpI + mb(S+R) - \gamma R,
\end{aligned} \tag{2}
$$

where $f(S,I) = 1 + \alpha_1 S + \alpha_2 I + \alpha_3 SI$. The incidence rate $\beta SI/(1+\alpha_1 S + \alpha_2 I + \alpha_3 SI)$ have been introduced by Hattaf et al.[8], where $\alpha_1, \alpha_2, \alpha_3 \geq 0$ are constants, this incidence rate generalise the incidence rate existing in the literature, if $\alpha_1 = \alpha_2 = \alpha_3 = 0$ then we get the bilinear incidence rate βSI, if we put $\alpha_2 = \alpha_3 = 0$ then we have the saturated incidence rate $\beta SI/(1+\alpha_1 I)$, we get functional response of Crowley Martin [9] if $\alpha_3 = \alpha_1\alpha_2$, and if $\alpha_3 = 0$ we obtained Beddington-DeAnglis functional response [10],[11].

In this paper, we consider the stochastic version of SIRS model (2)with a general incidence rate find it by perturbing the parameter β by the white noise :

$$
\begin{aligned}
\frac{dS}{dt} &= \left(-\frac{\beta SI}{f(S,I)} - bS + (1-m)pdI + b(1-m)(S+R) + \gamma R\right)dt - \frac{\sigma SI}{f(S,I)}dB(t), \\
\frac{dI}{dt} &= \left(\frac{\beta SI}{f(S,I)} - (pd+r)I\right)dt + \frac{\sigma SI}{f(S,I)}dB(t), \\
\frac{dR}{dt} &= (rI - bR + dmpI + mb(S+R) - \gamma R)\,dt.
\end{aligned} \tag{3}
$$

The system is constant, so we normalized to unity $S(t) + I(t) + R(t) = 1$. therefor, we only need to consider the model defined as follows:

$$
\begin{aligned}
\frac{dS}{dt} &= -\frac{\beta SI}{f(S,I)} + b(1-m) + \gamma - (b+\gamma)S + [(1-m)(pd-b) - \gamma]I, \\
\frac{dI}{dt} &= \frac{\beta SI}{f(S,I)} - (pd+r)I.
\end{aligned} \tag{4}
$$

The stochastic version defined by :

(5)
$$\begin{aligned} dS &= \left[-\frac{\beta SI}{f(S,I)} + b(1-m) + \gamma - (b+\gamma)S + [(1-m)(pd-b) - \gamma]I\right]dt - \frac{\sigma SI}{f(S,I)}dB(t), \\ \frac{dI}{dt} &= \left[\frac{\beta SI}{f(S,I)} - (pd+r)I\right]dt + \frac{\sigma SI}{f(S,I)}dB(t). \end{aligned}$$

It is important to note that system (3) includes many special case existing in the literature. For example, If $\alpha_i = 0$, i=1,2,3 and $\gamma = 0$, $\sigma = 0$ we obtain the SIR epidemic model concerning pulse vaccination strategy presented by X. Meng et al.[7]. If $\alpha_i = 0$, i=1,2,3. we obtain the Stochastic SIR Model with Vertical Transmission and Vaccination presented by Zhang et al. [15] and Witbooi [16].

the remnant of the paper is organized as flows. In section 3 we prove that system (4) allows a unique global and positive solution starting from a initial value in Γ, in section 4 we proved Almost sure exponential stability of the disease free equilibrium, in section 5 we investigate its asymptotic behavior around the endemic equilibrium of system (5), in section 6 we show the numerical simulation to illustrate our theoretical result. Finally, the conclusion of our paper is in Section 5.

2. Preliminaries

Let $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathscr{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e. it is increasing and right continuous while $\mathscr{F}_0$ contains all P-null sets). Next, we consider the d-dimensional stochastic system :

$$dx(t) = f(x(t),t)dt + g(x(t),t)dB(t), \tag{2.1}$$

where $f(x,t)$ is a function in $\mathbb{R}^d$ defined in $[t_0, +\infty)$ and $g(x,t)$ is an $d \times m$ matrix, f and g are locally Lipschitz functions in x. $\{B(t)\}_{t\geq 0}$ is an d-dimensional standard Wiener process defined on the above probability space.

Let us suppose that $f(t,0) = g(t,0) = 0$ for all $t \geq 0$. We assume that $x = 0$ is a solution of the system (2.1).

Definition 2.1.[13] *The trivial solution $x = 0$ of system (2.1) is said to be almost surely exponentially stable if for all $x(0) = x_0 \in \mathbb{R}^d$:*

$$\limsup_{t\to\infty} \frac{1}{t} \ln|x(t,x_0)| < 0, \ \ almost \ \ surely \ \ (briefly \ \ a.s.).$$

Denote by $\mathscr{C}^{2,1}(\mathbb{R}^d \times [t_0,+\infty);\mathbb{R}_+)$ the family of all nonnegative functions $U(x,t)$ defined on $\mathbb{R}^d \times [t_0,+\infty)$ such that they are continuously twice differentiable in x and once in t. The differential operator $\mathscr{L}$ [17] associated with (2.1) is defined by

$$\mathscr{L} = \frac{\partial}{\partial t} + \sum_{i=1}^{d} f_i(x,t).\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{d} [g^T(x,t)g(x,t)]_{ij}.\frac{\partial}{\partial x_i \partial x_j}.$$

If the differential operator $\mathscr{L}$ acts on a function $V \in \mathscr{C}^{2,1}(\mathbb{R}^d \times [t_0,+\infty[;\mathbb{R}_+)$, then

$$\mathscr{L}V = V_t(x,t) + V_x(x,t)f(x,t) + \frac{1}{2}Trac[g(x,t)^T V_{xx}(x,t)g(x,t)]$$

where $V_t(x,t) = \frac{\partial V}{\partial t}$, $V_x(x,t) = \left(\frac{\partial V}{\partial x_1}, ..., \frac{\partial V}{\partial x_d}\right)$, $V_{xx}(x,t) = \left(\frac{\partial^2 V}{\partial x_i \partial x_j}\right)$.

lemma 2.1.(Strong Law of Large Numbers) *Let $M = \{M_t\}_{t\geq 0}$, $t \geq 0$ be a real-valued continuous local martingale vanishing at $t = 0$. Then*

$$(i) \quad \lim_{t\to\infty} \langle M,M\rangle_t = \infty \quad a.s. \Rightarrow \lim_{t\to\infty} \frac{M_t}{\langle M,M\rangle_t} = 0 \quad a.s.$$

And

$$(ii) \quad \limsup_{t\to\infty} \frac{\langle M,M\rangle_t}{t} < \infty \quad a.s. \Rightarrow \lim \frac{M_t}{t} = 0 \quad a.s.$$

In order to establish the conditions for the exponentially stability of the disease-free equilibrium of system (4), we need the following lemma

lemma 2.2. *For $k \in \mathbb{N}$, let $X(t) = (X_1(t), X_2(t), .., X_k(t))$ be a bounded $\mathbb{R}^k$-valued function. Let $(t_{0,n})$ be any increasing unbounded sequence of positive real numbers. Then there is a family of*

sequences $(t_{s,n})$ such that for each $s \in 1,2,..,k$, $(t_{s,n})$ is a subsequence of $(t_{s,n})$ and the sequence $X_s(t_{s,n})$ converges to the largest limit point of the sequence $X_s(t_{s-1,n})$.

3. Global existence and positivity

In this section, we will prove that model (4) has a unique global positive solution for any initial value in Γ. Where

$$\Gamma = \{(S,I,R) \in \mathbb{R}^3_+ : S > 0, I > 0, R > 0, \;\; S+I+R = 1\}.$$

Theorem 3.1. *For any given initial value $(S_0,I_0,R_0) \in \Gamma$, there is a unique positive solution $(S(t),I(t),R(t))$ of (3) on $t \geq 0$ and the solution will remain in Γ with probability 1, namely $(S(t),I(t),R(t)) \in \Gamma$ for all $t \geq 0$ almost surely.*

Proof. *Since the coefficients of system (3) are locally Lipschitz continuous, then for any initial value $(S_0,I_0,R_0) \in \Gamma$ there is a unique local solution $(S(t),I(t),R(t))$ on $t \in [0,\tau_e]$, where τ_e is the explosion time. To show that this solution is global, we only need to prove $\tau_e = \infty$ a.s. Define the stopping time :*

$$\tau = \inf\{t \in [0,\tau_e) : \;\; S(t) \leq 0, \;\; or \;\; I(t) \leq 0, \;\; or \;\; R(t) \leq 0\},$$

We set $\inf \emptyset = \infty$, as usual $\emptyset$ denotes the empty set. We have $\tau \leq \tau_e$, If $\tau = \infty$ a.s., then $\tau_e = \infty$ a.s. and $(S(t),I(t),R(t)) \in \Gamma$ for all $t \geq 0$. In addition, to complete the proof we only need to prove $\tau = \infty$. Assume that $\tau < \infty$, then there exists a $T > 0$ such that $\mathbb{P}(\tau < T) > 0$. Consider the $\mathscr{C}^2$-function Q, defined by the expression :

$$Q(X) = -\ln(SIR)$$

Using Itô's Formula, we have for all t

$$\begin{aligned} dQ = {} & \frac{-1}{S}\left[-\frac{\beta SI}{f(S,I)} - bS + (1-m)pdI + b(1-m)(S+R) + \gamma R\right]dt \\ & - \frac{1}{I}\left[\frac{\beta SI}{f(S,I)} - (pd+r)I\right]dt - \frac{1}{R}\left[rI - bR + dmpI + mb(S+R) - \gamma R\right]dt \\ & + \left[\frac{1}{2}\frac{\sigma^2 I^2}{f^2} + \frac{1}{2}\frac{\sigma^2 S^2}{f^2}\right]dt + \frac{\sigma I}{f(S,I)}dB(t) - \frac{\sigma S}{f(S,I)}dB(t). \end{aligned} \tag{6}$$

We have $f(S,I) \geq 0$, implies that

$$dQ(S,I,R) \leq G(S,I)dt + \frac{\sigma(I-S)}{f(S,I)}dB(t),$$

where

$$G(S,I) = \beta I + +2b + pd + r + \gamma + \frac{\sigma}{2}S^2 + \frac{\sigma}{2}I^2,$$

Integrating the above inequality, we obtain

$$Q(S(t),I(t),R(t)) \leq Q(S_0,I_0,R_0) + \int_0^t G(S,I)\mathrm{d}s + \int_0^t \frac{\sigma}{f(S,I)}(I-S)dB(s). \tag{3.1}$$

There is some element of $X(\tau)$ equal 0. Then

$$\lim_{t\to\tau} Q(S(t),I(t),R(t)) = +\infty$$

Letting $t \to \tau$ in (3.1), we have

$$+\infty \leq Q(X_0) + \int_0^\tau G(S(s),I(s))\mathrm{d}s + \int_0^\tau \frac{\sigma}{f(S(s),I(s))}(I(s)-S(s))dB(s) < \infty$$

Which contradicts our assumption. So we must therefore have $\tau = \infty$ a.s. This completes the proof of Theorem 3.1.

4. Exponential stability

In this section, we give a sufficient conditions for the exponentially stability of the disease-free equilibrium. We set $X(t) = (S(t),I(t))$. We define the following stochastic process $\Psi(t)$ and $U(X(t))$ as following

$$\Psi(t) = (S(t)-c)^2 + aI(t),$$

$$U(S(t),I(t)) = \ln(\Psi(t)).$$

where $c = \dfrac{b(1-m)+\gamma}{b+\gamma}$. We note that $\Psi(t) > 0$ a.s. for all t, also we defined the following invariant $\mathscr{R}$ with a constant $0 < k \leq 1$ and $l = 1 + \alpha_1 + \alpha_2 + \alpha_3$, we will employed in the main theorem of the stability :

$$\mathscr{R} = \frac{\beta - \frac{1}{2}\frac{k\sigma^2}{l}}{pd+r}$$

In the rest of this section we will shown that if $\mathscr{R} < 1$ and $k\sigma^2 < \min\{\beta l^2, 4(b+\gamma)l^2\}$ then the disease free equilibrium will almost sure exponentially stable.

Theorem 4.1. *Suppose that the following inequality holds:*

$$k\sigma^2 < \min\{\beta l^2, 4(b+\gamma)l^2\}$$

If $\mathscr{R} < 1$ then the disease-free equilibrium of the system (5) is almost surely exponentially stable.

Proof. *Step 1 : We prove that $\Psi(t)$ converges exponentially to zero a.s. By using Itô's formula we obtain*

$$dU(X(t)) = \mathscr{L}U(X(t))dt + \mathscr{M}(t),$$

integrating both sides from 0 to t yields that

$$U(X(t) = U(X_0) + \int_0^t LU(X(s))ds + \mathscr{M}(t).$$

Where

$$\mathscr{M}(t) = \int_0^t \left(\frac{a\sigma S(s)I(s)}{f(S(s),I(s))\Psi(s)} - \frac{2(S(s)-c)\sigma S(s)I(s)}{f(S(s),I(s))\Psi(s)} \right) dB(s).$$

We have

$$\frac{a\sigma SI - 2(S-c)\sigma SI}{f(S,I\Psi(t)} \leq \sigma(a+2c),$$

it follows that

$$\frac{1}{t}\int_0^t \left(\frac{a\sigma S(s)I(s)}{f(S(s),I(s))\Psi(s)} - \frac{2(S(s)-c)\sigma S(s)I(s)}{f(S(s),I(s))\Psi(s)} \right)^2 \mathrm{d}s < \infty.$$

According to Lemma 2.1 we obtain

$$\lim_{t\to\infty} \frac{\mathscr{M}_t}{t} = 0 \qquad a.s.$$

Consequently,

$$\frac{1}{t}\limsup_{t\to\infty} U(X(t) = \frac{1}{t}\limsup_{t\to\infty} \int_0^t \mathscr{L}U(X(s))ds.$$

Step 2 : We prove that $\limsup_{t\to\infty} \mathscr{L}U(X(t)) < 0$ a.s. Applying the operator $\mathscr{L}$ to U we obtain :

$$\mathscr{L}U = \frac{2(S-c)}{\Psi}\left[-\frac{\beta SI}{f(S,I)} + b(1-m) + \gamma - (b+\gamma)S + \left[(1-m)(pd-b) - \gamma \right] I \right]$$

$$+ \frac{a}{\Psi}\left[\frac{\beta SI}{f(S,I)} - (pd+r)I \right] + \frac{1}{2}\left(\frac{\sigma SI}{f(S,I)\Psi} \right)^2 \left[-2(S-c)^2 + 2aI + 4(S-c)a - a^2 \right]. \tag{7}$$

Using the inequality

$$-2(S-c)^2+2aI+4(S-c)a-a^2 \le -2(S-c)^2+2a+4a-a^2 \le -a^2(1-\frac{6}{a}),$$

we get

$$\begin{aligned}\mathscr{L}U \le & \frac{2(S-c)}{\Psi}\left[-\frac{\beta SI}{x}-((1-m)(b-pd)+\gamma)I-(b+\gamma)(S-c)\right] \\ & +\frac{a}{\Psi}[\beta SI-(pd+r)I]-\frac{1}{2}\left(\frac{\sigma SI}{\Psi l}\right)^2 a^2\left(1-\frac{6}{a}\right). \qquad (8)\end{aligned}$$

In view of Lemma 2.2 we can define the following limits for a suitable increasing, unbounded sequence t_n as

$$\begin{aligned} z &= \lim_{n\to\infty}\frac{(S(t_n)-c)^2}{\Psi(t_n)}, \\ x &= \lim_{n\to\infty}\frac{I(t_n)}{\Psi(t_n)}, \\ y &= \lim_{n\to\infty} S(t_n). \end{aligned}$$

And with

$$\limsup_{t\to\infty}\mathscr{L}U(X(t)) = \lim_{n\to\infty}\mathscr{L}U(X(t_n).$$

In particular then we have $z+ax=1$, and $y\le 1$. We notice

$$\Pi = \limsup_{t\to\infty}\mathscr{L}U(X(t)).$$

Therefore, we can write (8) *as follows*

$$\begin{aligned}\Pi \le & 2(y-c)\left[\frac{-\beta yx}{l}-((1-m)(b-pd)+\gamma)x\right]-2(b+\gamma)z+a\left[\beta yx-(pd+r)x\right] \\ & -\frac{1}{2}\left(\frac{\sigma}{l}yx\right)^2 a^2(1-\frac{6}{a}) \\ =& 2(y-c)\left[-((1-m)(b-pd)+\gamma)x\right]-2(b+\gamma)z+\beta yx\left[-\frac{2(y-c)}{l}+a\right] \\ & -(pd+r)x-\frac{1}{2}\left(\frac{\sigma}{l}yx\right)^2 a^2\left(1-\frac{6}{a}\right),\end{aligned}$$

using the inequality

$$\beta yx\left[-\frac{2(y-c)}{l}+a\right] \le \beta yx\left(a-\frac{2}{l}\right) = a\beta yx\left(1-\frac{2}{al}\right) \le a\beta yx$$

we get

$$\Pi \leq ax\left[\beta y+\frac{2c}{a}[(1-m)(b-pd)+\gamma\right]-2(b+\gamma)z-\frac{1}{2}\left(\frac{\sigma}{x}yx\right)^2 a^2\left(1-\frac{6}{a}\right).$$

Hence, there exist $0<k\leq 1$ *such that the following inequality hold*

$$\begin{aligned}-\frac{1}{2}\left(\frac{\sigma}{l}yx\right)^2 a^2(1-\frac{6}{a}) &\leq -\frac{1}{2}\left(\frac{\sigma}{x}yx\right)^2 ka^2\left(1-\frac{6}{a}\right)\\ &\leq -\frac{1}{2}x\left(\frac{\sigma}{l}y\right)^2 ka\left(1-\frac{6}{a}\right)(1-z)\\ &\leq -ax\frac{1}{2}k(\frac{\sigma y}{l})^2\left(1-\frac{6}{a}\right)+\frac{1}{2}zk\left(\frac{\sigma y}{l}\right)^2. \qquad (9)\end{aligned}$$

From (9), yields that

$$\Pi \leq ax\left[\beta y+\frac{2c}{a}[(1-m)(b-pd)+\gamma\right]-2(b+\gamma)z-ax\frac{1}{2}k\left(\frac{\sigma y}{l}\right)^2\left(1-\frac{6}{a}\right)+\frac{1}{2}zk\left(\frac{\sigma y}{l}\right)^2$$

By the inequality $k\sigma^2<\beta l^2$*, we find*

$$\begin{aligned}\beta y-\frac{1}{2}k(\frac{\sigma y}{l})^2\left(1-\frac{6}{a}\right) &=\frac{1}{2}k\frac{\sigma^2}{l^2}\left(1-\frac{6}{a}\right)\left[\frac{2\beta al^2}{k\sigma^2(a-6)}y-y^2\right]\\ &\leq\frac{1}{2}c\frac{\sigma}{l}\left(1-\frac{6}{a}\right)\left[\frac{2\beta l^2 a}{k\sigma^2(a-6)}-1\right]\\ &=\beta-\frac{1}{2}k\frac{\sigma^2}{l^2}\left(1-\frac{6}{a}\right). \qquad (10)\end{aligned}$$

Using the inequality (10)*, we obtain*

$$\Pi \leq ax\left[\beta-(pd+r)-\frac{1}{2}k\frac{\sigma^2}{l^2}\left(1-\frac{6}{a}\right)+\frac{2c}{a}[(1-m)(b-pd)+\gamma]\right]+z\left[\frac{1}{2}k\frac{\sigma^2}{l^2}-2(b+\gamma)\right].$$

Since $k\sigma^2<4(b+\gamma)l^2$ *implide that the coefficients of z are negative. as though* $\mathscr{R}<1$ *we deduce that the coefficients of x are negative, gold :*

$$\beta-\frac{1}{2}\frac{k\sigma^2}{l}-(pd+r)<0,$$

we choose a number a such that the following inequality hold,

$$\beta-(pd+r)-\frac{1}{2}k\frac{\sigma^2}{l^2}\left(1-\frac{6}{a}\right)+\frac{2c}{a}\left[(1-m)(b-pd)+\gamma\right]<0$$

We have $z+ax=1$*, since the limits z, x cannot all be zero. Consequently, we obtain that* $\Pi<0$*, the proof of theorem is completed.*

5. Asymptotic Behavior Around the Endemic Equilibrium

By study epidemic dynamical systems, we are interested by extinction, and persistent in a population. In the deterministic models, the second problem is solved by showing that the endemic equilibrium of the model is globally asymptotic stable. But, there is none of endemic equilibrium in system (5). So in this section we study the behavior around the endemic equilibrium E^* to indicate that whether the disease will prevail. We find the following result.

Theorem 4.1. *Consider the stochastic system (5) with initial condition in* Γ*, If* $R_0 > 1$*, then we have*

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t (S(s)-S^*)^2 + (I(s)-I^*)^2 ds \leq \frac{aI^*\sigma^2}{\vartheta}$$

Where $\vartheta = \min\{2(b+\gamma), 2(\gamma+pd+r-(1-m)(pd-b))\}$

Proof. Consider the $\mathscr{C}^2$-function V defined by the expression

$$V(X) = a\left(I - I^* - I^*\log\left(\frac{I}{I^*}\right)\right) + (S-S^*+I-I^*)^2$$

where a is real constant to be chosen in the sequel. Using Itô's formula we get

$$dV(X(t)) = LV(X(t))dt + \frac{S(I-I^*)}{f(S,I)}dB(t), \tag{10}$$

where

$$LV = \left[b(1-m)+\gamma-(b+\gamma)S+((1-m)(pd-b)-\gamma)I-(pd+r)I\right]2(S-S^*+I-I^*)$$

$$\left[\frac{\beta SI}{f(S,I)}-(pd+r)I\right]a(I-I^*)+\frac{aI^*\sigma^2S^2}{f^2(S,I)}, \tag{11}$$

At the equilibrium state E^*, we have

$$\frac{\beta S^*I^*}{f(S^*,I^*)}-(pd+r)I^*=0, \tag{12}$$

$$-\frac{\beta S^*I^*}{f(S^*,I^*)}+b(1-m)+\gamma-(b+\gamma)S^*+((1-m)(pd-b)-\gamma)I^*=0. \tag{13}$$

Substituting (12) and (13) into (11), we get

$$
\begin{aligned}
LV =& \Big[-(b+\gamma)(S-S^*) + ((1-m)(pd-b)-\gamma)(I-I^*) - (pd+r)(I-I^*) \Big] \\
& \times 2(S-S^*+I-I^*) + a(I-I^*)\left[\beta S\left(\frac{1}{f(S,I)} - \frac{1}{f(S^*,I^*)}\right) + \frac{\beta}{f(S^*,I^*)}(S-S^*)\right] \\
& + \frac{aI^*\sigma^2S^2}{f^2(S,I)}. \qquad (14)
\end{aligned}
$$

Since $f(S,I) \geq 0$, we obtain

$$
\begin{aligned}
LV \leq& \Big[-(b+\gamma)(S-S^*) + ((1-m)(pd-b)-\gamma)(I-I^*) - (pd+r)(I-I^*) \Big] \\
& \times 2(S-S^*+I-I^*) + a(I-I^*)(S-S^*)\frac{\beta}{f(S^*,I^*)} + \frac{aI^*\sigma^2S^2}{f^2(S,I)} \\
=& -2(b+\gamma)(S-S^*)^2 - 2\Big(\gamma + pd + r - (1-m)(pd-b)\Big)(I-I^*)^2 \\
& + a(I-I^*)(S-S^*)\frac{\beta}{f(S^*,I^*)} - 2(I-I^*)(S-S^*)\Big[b+pd+r-(1-m)(pd-b)\Big] \\
& + \frac{aI^*\sigma^2S^2}{f^2(S,I)}, \\
\leq& -2(b+\gamma)(S-S^*)^2 - 2\Big(\gamma + pd + r - (1-m)(pd-b)\Big)(I-I^*)^2 \\
& + a(I-I^*)(S-S^*)\frac{\beta}{f(S^*,I^*)} - 2(I-I^*)(S-S^*)\Big[b+pd+r-(1-m)(pd-b)\Big] \\
& + aI^*\sigma^2,
\end{aligned}
$$

we can choose the number a such that

$$
a\frac{\beta}{f(S^*,I^*)} - 2\Big[b+pd+r-(1-m)(pd-b)\Big] = 0,
$$

then

$$
LV \leq -2(b+\gamma)(S-S^*)^2 - 2\Big(\gamma + pd + r - (1-m)(pd-b)\Big)(I-I^*)^2 + aI^*\sigma^2. \qquad (15)
$$

Therefore, $A_t = \int_0^t \frac{S(s)(I(s)-I^*)}{f(S(s),I(s))} dB(s)$ is continuous local martingale and $A(0) = 0$. Moreover, whose quadratic variation is

$$\langle A,A\rangle_t = \int_0^t \left(\frac{S(s)(I(s)-I^*)}{f(S,I)}\right)^2 ds \leq ct,$$

by Lemma 3.3, we get

$$\lim_{t\to\infty} \frac{A_t}{t} = 0 \quad a.s.$$

From(10) and (15), we obtain

$$dV \leq -2(b+\gamma)(S-S^*)^2 - 2\Big(\gamma + pd + r - (1-m)(pd-b)\Big)(I-I^*)^2 + aI^*\sigma^2 \frac{S(s)(I(s)-I^*)}{f(S,I)} dB(s),$$

integrating both sides from 0 to t yields that

$$V(t) - V(0) \leq -\vartheta \int_0^t (S(s)-S^*)^2 + (I(s)-I^*)^2 ds + aI^*\sigma^2 t + A_t,$$

consequently

$$\int_0^t (S(s)-S^*)^2 + (I(s)-I^*)^2 ds \leq \frac{V(0)}{\vartheta} + \frac{aI^*\sigma^2}{\vartheta} t,$$

finally, we deduce that

$$\limsup_{t\to\infty} \frac{1}{t} \int_0^t (S(s)-S^*)^2 + (I(s)-I^*)^2 ds \leq \frac{aI^*\sigma^2}{\vartheta} \quad a.s.$$

This completes the proof of Theorem 4.1.

6. Numerical simulations

In this section, we illustrate our theoretical results presented above, the system (5) is simulated for various sets of parameters and by using the Euler-Maruyama method. Fig. 1 illustrates that the almost sure exponential stabile of stochastic SIRS model (5), whenever the conditions of theorem is realised $\mathscr{R} = 0.9375 < 1$ and $k\sigma^2 < \min\{\beta l^2, 4(b+\gamma)l^2\}$. In Fig. 2 little values of intensity, will still render the free equilibrium unstable.

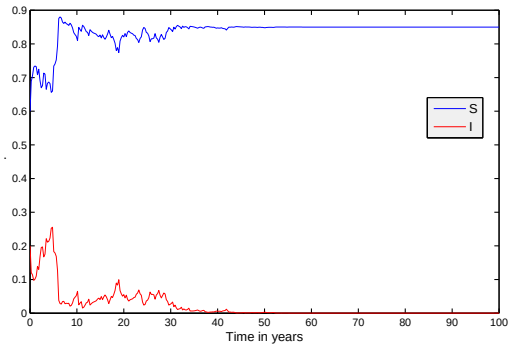

FIGURE 1. Stochastic trajectories of SIRS epidemic model (5) for the parameters: $\sigma = 0.6, \beta = 0.6, b = 0.2, m = 0.3, p = 0.6, d = 0.4, r = 0.2, \alpha_1 = 0.2, \alpha_2 = 0.2, \alpha_3 = 0.2; \gamma = 0.2$.

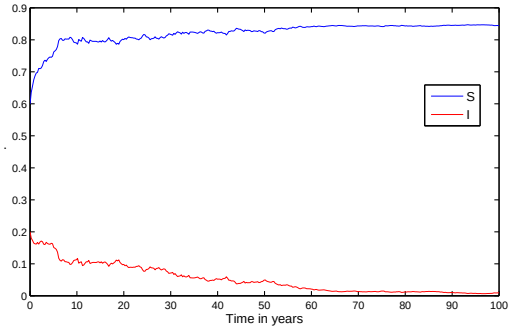

FIGURE 2. Stochastic trajectories of SIRS epidemic model (5) for the parameters: $\sigma = 0.12, \beta = 0.6, b = 0.2, m = 0.3, p = 0.6, d = 0.4, r = 0.2, \alpha_1 = 0.2, \alpha_2 = 0.2, \alpha_3 = 0.2, \gamma = 0.2$,

Conclusion

The introduction of stochastic effects into deterministic models gives us a more realistic way of modeling epidemic models. In this paper we have considerate a stochastic SIRS epidemic model with vaccination and non-linear incidence rate, we have studied the effects of the environmental fluctuation in SIRS epidemic model. we first proved the existence and positivity of solutions of our stochastic model which implies that the model is well posed. Then, we showed the stability exponentially almost surely of the disease free equilibrium E_0 as $\mathscr{R} < 1$. To indicate that whether the disease will prevail we study the behavior around the endemic equilibrium E^*. Finally we have simulated our theoretical result.

Conflict of Interests

The authors declare that there is no conflict of interests.

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