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ITERATIVE APPROXIMATION FOR THE COMMON SOLUTIONS OF A INFINITE VARIATIONAL INEQUALITY SYSTEM FOR INVERSE-STRONGLY ACCRETIVE MAPPINGS

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Abstract. The aim of this paper is to introduce and study a system of the infinite variational inequalities for inverse-strongly accretive mappings by using relaxed extradient method. Results proved in this paper may be viewed as an improvement and refinement of the recent results of X.Qin[1] and Aoyama,K[2]

Keywords: variational inequalities; viscosity approximation; inverse-strongly accretive mapping; fixed point.

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1. Introduction

Let H be a real Hilbert space with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$, C be a nonempty closed convex subset of H and A be a operator from C into H . The classical variational inequality problem is formulated as finding a point $u \in C$ such that

$$\langle Au, v-u\rangle \geq 0$$

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for all $v \in C$. Such a point $u \in C$ is called a solution of the problem. Variational inequalities were initially studied by Stampacchia [3, 4] and ever since have been widely studied. The set of solutions of the variational inequality problem is denoted by VI(C,A). For given $z \in H, u \in C$, we see that the following inequality holds

$$\langle u - z, v - u \rangle \geq 0$$

if and only if $u = P_C z : \|P_C z - z\| = inf_{v \in C} \|v - z\|$. It is known that projection operator P_C is nonexpansive. It is also know that P_C satisfies

$$\langle x - y, P_C x - P_C y \rangle \geq ||P_C x - P_C y||^2, \forall x, y \in H.$$

One can see that the variational inequality is equivalent to a fixed point problem. An element $x^* \in C$ is a solution of the variational inequality if and only if $x^* \in C$ is a fixed point of the mapping $P_C(I - \lambda A)$, where I is the identity mapping and $\lambda > 0$ is a constant.This alternative equivalent formulation has played a significant role in the studies of variational inequalities and related optimization problems.

In this paper, let C be a nonempty closed convex subset of a real Banach space E . Let A, B be two inverse-strongly accretive mappings. We consider the following problem of finding $(\widetilde{x}, \widetilde{y}) \in C \times C$ such that

$$\begin{cases} \langle \lambda_n A\widetilde{y} + \widetilde{x} - \widetilde{y}, J(x - \widetilde{x}) \rangle \geq 0, \forall x \in C, \\ \langle \mu_n B\widetilde{x} + \widetilde{y} - \widetilde{x}, J(x - \widetilde{y}) \rangle \geq 0, \forall x \in C, \end{cases} \tag{1}$$

which is called a general system of infinite variational inequalities, where $\{\lambda_n\}, \{\mu_n\} \subset (0, \infty)$. In particular, if $A = B, \lambda_n = \mu_n = \lambda$, then problem reduces to finding $(\widetilde{x}, \widetilde{y}) \in C \times C$ such that

$$\begin{cases} \langle \lambda A\widetilde{y} + \widetilde{x} - \widetilde{y}, J(x - \widetilde{x}) \rangle \geq 0, \forall x \in C, \\ \langle \lambda A\widetilde{x} + \widetilde{y} - \widetilde{x}, J(x - \widetilde{y}) \rangle \geq 0, \forall x \in C, \end{cases} \tag{2}$$

which is defined by Verma [5] and is called the new system of variational inequalities. Further, if we add up the requirement that $\widetilde{x} = \widetilde{y}$, then problem (1) reduces to the classical variational inequality VI(A , C).

Recently, many authors studied the problem of finding a common element of the fixed point set of nonexpansive mappings and the solution set of variational inequalities for α-inverse-strongly monotone mappings in the framework of Banach space.In 2006, Aoyama, Iiduka and Takahashi [2] obtained a weak Theorem about weak convergence of an iterative sequence for accretive operators in a uniformly convex and 2-uniformly smooth Banach space. In2009,X.Qin[1],et al.consider the problem of strong convergence of an iterative algorithm for systems of variational inequalities and nonexpansive mapping with applications.

In this paper, motivated by [1,2,6,7], let E be a uniformly convex and q-uniformly smooth Banach space, C be a nonempty closed convex subset of E. We introduce a general iterative algorithm for the system of infinite variational inequality (1) and a sunny nonexpansive mapping.

$$(3) \qquad \begin{cases} x_1 = u \in C \\ y_n = Q_C(x_n - \mu_n B x_n) \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n(\delta T x_n + (1-\delta) Q_C(y_n - \lambda_n A y_n), \ n \geq 0. \end{cases}$$

The problem (1) is proven to be equivalent to a fixed point problem of nonexpansive mapping. By using a relaxed extradient methods, we prove that under some conditions the iterative sequence $\{x_n\}$ converges strongly to $\widetilde{x} \in C$ and $(\widetilde{x}, \widetilde{y})$ is a solution of the problem(1),where $\widetilde{y} = Q_C(\widetilde{x} - \mu_n B\widetilde{x})$. The results here improve and extend the related results of other authors, such as [1,2,6].

2. Preliminaries

Recall that a mapping T of C into itself is called nonexpansive, if

$$\|Tx - Ty\| \leq \|x - y\|.$$

for all $x, y \in C$. We denote by $F(T)$ the set of fixed points of T.

For $\alpha > 0$, an operator A of C into E is said to be α-inverse strongly accretive if

$$\langle Ax - Ay, J(x-y)\rangle \geq \alpha \|Ax - Ay\|^2.$$

for all $x, y \in C$. It is obviously that

$$\|Ax - Ay\| \leq \frac{1}{\alpha}||x - y||.$$

Let D be a subset of C and Q be a mapping of C into D, then Q is said to be sunny if

$$Q(Qx + t(x - Qx)) = Qx,$$

whenever $Qx + t(x - Qx) \in C$ for $x \in C$ and $t \geq 0$. A mapping Q of C into itself is called a retraction if $Q^2 = Q$. If a mapping Q of C into itself is a retraction, then $Qz = z$ for every $z \in R(Q)$, where $R(Q)$ is the range of Q. A subset D of C is called a sunny nonexpansive retract of C if there exists a sunny nonexpansive retraction from C into D.

Assume E be a real Banach space, C be a nonempty closed convex subset of E. Let $U = \{x \in E : x = 1\}$, A Banach space E is said to be uniformly convex, if for each $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that for any $x, y \in U$, $\|x - y\| \leq \epsilon,\ which\ implies \frac{\|x-y\|}{2} \leq 1 - \delta$. It is known that a uniformly convex Banach space is reflexive and strictly convex.

A Banach space E is said to be smooth if the limit $\lim_{t \to 0} \frac{\|x-ty\|-\|y\|}{t}$ exists for all $x, y \in U$. It is also said to be uniformly smooth if the limit is attained uniformly for $x, y \in U$. The norm of E is said to be *Frećhet* differentiable if for each $x \in U$, the limit is attained uniformly for $y \in U$. And we define a function $\rho : [0, \infty) \to [0, \infty)$ called the modulus of smoothness of E as follows:

$$\rho(t) = \sup\{\frac{1}{2}(\|x + y\| + \|x - y\|) - 1 : \|x\| = 1, \|y\| = t\}.$$

It is known that E is uniformly smooth if and only if $\lim_{t \to 0} \frac{\rho(t)}{t} = 0$. Let q be a fixed real number with $1 < q \leq 2$. Then a Banach space E is said to be q-uniformly smooth if there exists a constant $c > 0$ such that $\rho(t) \leq ct^q$ for all $t > 0$. We could obtain the following lemma.

Lemma 2.1.[8,9] Let q be a real number with $1 < q \leq 2$ and let E be a Banach space. Then E is q-uniformly smooth if and only if there exists a constant $K \geq 1$ such that

$$\frac{1}{2}(||x + y||^q + ||x - y||^q) \leq ||x||^q + ||Ky||^q$$

for all $x, y \in E$.

The best constant K in Lemma 2.1 is called the q-uniformly smoothness constant of E. Let q be a given real number with $q > 1$. The (generalized) duality mapping J_q from E into 2^{E^*} is defined by

$$J_q(x) = \{x^* \in E^* : \langle x, x^* \rangle = ||x||^q, ||x^*|| = ||x||^{q-1}\}$$

for all $x \in E$. In particular, $J = J_2$ is called the normalized duality mapping. It is known that J_q

$$J_q(x) = ||x||^{q-2} J(x)$$

Lemma 2.2.[10] Let q be a given real number with $1 < q \leq 2$ and let E be a q-uniformly smooth Banach space. Then

$$||x + y||^q \leq ||x||^q + q\langle y, J_q(x)\rangle + 2||Ky||^q$$

for all $x, y \in E$, where J_q is the generalized duality mapping of E and K is the q-uniformly smoothness constant of E.

Lemma 2.3.[1] Let C be a nonempty closed convex subset of a real Hilbert space H. Let T_1 and T_2 be two nonexpansive mappings from C into itself with a common fixed point. Define a mapping $T : C \to C$ by $Tx = \delta T_1 x + (1 - \delta)T_2 x$, where $\delta \in (0, 1)$. Then T is nonexpansive and $F(T) = F(T_1) \bigcap F(T_2)$.

Lemma 2.4.[11] In a Banach space E,there holds the inequality

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y)\rangle, \ \forall x, y \in C, wherej(x + y) \in J(x + y).$$

Lemma 2.5.[15] Let C be a nonempty closed con- vex subset of a smooth Banach space E. Let Q_C be a sunny nonexpansive retraction from E onto C and let A be an accretive operator of C into E. Then,for all $\lambda > 0$,

$$\Omega = F(Q_C(I - \lambda A)).$$

Lemma 2.6.[15] Let E be a uniformly convex Banach space, C a nonempty closed convex subset of E and $T : K \to K$ a nonexpansive mapping. Then $I - T$ is demi-closed at zero.

Lemma 2.7.[12,13] Let $\{\alpha_n\}_{n=0}^{\infty}$ be a sequence of nonnegative real numbers satisfying the property

$$\alpha_{n+1} \leq (1-\gamma_n)\alpha_n + \gamma_n\delta_n, n \geq 0,$$

where $\{\gamma_n\} \subset (0,1)$ and $\{\delta_n\}$ are such that

$$(1) lim_{n\to\infty}\gamma_n = 0, \Sigma\gamma_n = \infty; (2) \limsup_{n\to\infty} \frac{\delta_n}{\gamma_N} \leq 0 (or \Sigma|\delta_n| < \infty).$$

then $lim_{n\to\infty}\alpha_n = 0$.

Lemma 2.8.[14]Let $\{x_n\} and \{y_n\}$ be bounded sequences in a Banach space X and Let $\{\alpha_n\} \subset [0,1]$ with $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$, $n \geq 0$,such that

$$(1) x_{n+1} = \alpha_n x_n + (1-\alpha_n)y_n; \ (2) \limsup_{n\to\infty}(||y_{n+1} - y_n|| - ||x_{n+1} - x_n||) \leq 0.$$

then $lim_{n\to\infty}||y_n - x_n|| = 0$.

Lemma 2.9. For given $(\widetilde{x}, \widetilde{y}) \in C \times C$, where $\widetilde{y} = Q_C(\widetilde{x} - \mu_n B\widetilde{x})$, $\widetilde{x}, \widetilde{y}$ is a solution of problem(1),if and only if $\widetilde{x}$ is a common fixed point of the mapping $S_n : C \to C$ defined by

$$S_n(x) = Q_C[Q_C(x - \mu_n Bx) - \lambda_n A Q_C(x - \mu_n Bx)], \forall n \in N,$$

where $\{\lambda_n\}, \{\mu_n\} \subset (0,1)$ and Q_C is a sunny nonexpansive retraction from E onto C.

Proof.

$$(4) \qquad \begin{cases} \langle \lambda_n A\widetilde{y} + \widetilde{x} - \widetilde{y}, J(x - \widetilde{x})\rangle \geq 0, \forall x \in C, \\ \langle \mu_n B\widetilde{x} + \widetilde{y} - \widetilde{x}, J(x - \widetilde{y})\rangle \geq 0, \forall x \in C, \end{cases}$$

$\Leftrightarrow$

$$(5) \qquad \begin{cases} \widetilde{x} = Q_C(\widetilde{y} - \lambda_n A\widetilde{y}) \\ \widetilde{y} = Q_C(\widetilde{x} - \mu_n B\widetilde{x}) \end{cases}$$

$\Leftrightarrow \widetilde{x} = Q_C(Q_C(\widetilde{x} - \mu_n B\widetilde{x}) - \lambda_n A Q_C(\widetilde{x} - \mu_n B\widetilde{x}))$

3. Main results

Theorem 3.1 *Let E be a uniformly convex and q-uniformly smooth Banach space with the best smooth constant K, C a nonempty closed convex subset of E. Let $Q_C : E \to C$ be a sunny nonexpansive retraction and $A, B : C \to E$ be α-inverse-strongly accretive mapping and β-inverse-strongly accretive mapping. Let $T : C \to C$ be a nonexpansive mapping with a fixed point and assume that $F = F(T) \cap (\cap_{n=1}^{\infty} F(S_n)) \neq \varnothing$, where S_n is defined as Lemma 2.9. Suppose $\{\lambda_n\} \subset [a, \sqrt[q-1]{\frac{q\alpha^{q-1}}{2K^q}}]$, $\{\mu_n\} \subset [a, \sqrt[q-1]{\frac{q\beta^{q-1}}{2K^q}}]$, $a > 0, \delta \in (0,1)$. If the sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ in $[0,1]$ satisfy the following conditions:*

$(C1)\alpha_n + \beta_n + \gamma_n = 1;$

$(C2)\Sigma\alpha_n = \infty, lim_{n\to\infty}\alpha_n = 0;$

$(C3)0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1;$

$(C4)lim_{n\to\infty}(\lambda_{n+1} - \lambda_n) = 0, lim_{n\to\infty}(\mu_{n+1} - \mu_n) = 0.$

Then the sequence $\{x_n\}$ defined by (3) convergence strongly to $\widetilde{x} = Q_F u$, and $(\widetilde{x}, \widetilde{y})$ is a solution of the problem(1),where $\widetilde{y} = Q_C(\widetilde{x} - \mu_n B\widetilde{x})$.

Proof. Step1 We show that F is closed and convex.

Since A is an α -inverse-strongly accretive mapping, applying lemma2.1,2.2and $\{\lambda_n\} \subset [a, \sqrt[q-1]{\frac{q\alpha^{q-1}}{2K^q}}]$, we get

$$\begin{aligned}
\|(I-\lambda_n A)x - (I-\lambda_n A)y\|^q &= \|(x-y) - \lambda_n(Ax - Ay)\|^q \\
&\leq \|x-y\|^q - q\lambda_n\langle Ax - Ay, J_q(x-y)\rangle + 2||K\lambda_n(Ax-Ay)||^q \\
&= \|x-y\|^q - \lambda_n q||x-y||^{q-2}\langle Ax - Ay, J(x-y)\rangle \\
&+ 2K^q\lambda_n^q||Ax-Ay||^q \\
&\leq ||x-y||^q - \lambda_n q\alpha^{q-1}||Ax-Ay||^q + 2K^q\lambda_n^q||Ax-Ay||^q \\
&= ||x-y||^q + \lambda_n(2K^q\lambda_n^{q-1} - q\alpha^{q-1})||Ax-Ay||^q
\end{aligned}$$

which implies that $I - \lambda_n A$ is nonexpansive, so is $I - \mu_n B$. From lemma 2.9, we obtain that

$$\begin{aligned}
S_n &= Q_C(Q_C(I-\mu_n B) - \lambda_n A Q_C(I-\mu_n B)) \\
&= Q_C(I-\lambda_n A)Q_C(I-\mu_n B)
\end{aligned}$$

S_n is nonexpansive. Consequently, $F = (\cap_{n=1}^{\infty} F(S_n)) \cap F(T)$ is closed and convex.

Step 2 We observe $\{x_n\}$ is bounded.

Indeed ,taking a fixed point $\overline{x}$ of F ,we have $\overline{x} = Q_C(Q_C(\overline{x}-\mu_n B\overline{x})-\lambda_n AQ_C(\overline{x}-\mu_n B\overline{x}))$

Let $\overline{y} = Q_C(\overline{x} - \mu_n B\overline{x})$, then $\overline{x} = Q_C(\overline{y} - \lambda_n A\overline{y})$. And let $l_n = \delta Tx_n + (1-\delta)Q_C(y_n - \lambda_n Ay_n)$, we get

$$\begin{aligned}\|l_n - \overline{x}\| &= \delta_n\|Tx_n - \overline{x}\| + (1-\delta_n)\|Q_C(y_n - \lambda_n Ay_n) - \overline{x}\| \\ &\leq \delta_n\|x_n - \overline{x}\| + (1-\delta_n)\|Q_C(y_n - \lambda_n Ay_n) - Q_C(\overline{y} - \lambda_n A\overline{y})\| \\ &\leq \delta_n\|x_n - \overline{x}\| + (1-\delta_n)\|y_n - \overline{y}\| \\ &\leq \delta_n\|x_n - \overline{x}\| + (1-\delta_n)\|Q_C(x_n - \mu_n Bx_n) - Q_C(\overline{x} - \mu_n B\overline{x})\| \\ &\leq \|x_n - \overline{x}\|\end{aligned}$$

Then we arrive at

$$\begin{aligned}||x_{n+1} - \overline{x}|| &= ||\alpha_n u + \beta_n x_n + \gamma_n l_n - \overline{x}|| \\ &= \alpha_n||u - \overline{x}|| + \beta_n||x_n - \overline{x}|| + \gamma_n\|l_n - \overline{x}\| \\ &\leq (1-\alpha_n)||x_n - \overline{x}|| + \alpha_n||u - \overline{x}|| \\ &\leq max\{||x_n - \overline{x}||, ||u - \overline{x}||\}. \\ &\leq ||u - \overline{x}||\end{aligned}$$

Hence $\{x_n\}$ is bounded, so are the sets $\{y_n\} and \{l_n\}$.

According to step 1 and by (4), we observe that

$$\begin{aligned}||l_{n+1} - l_n|| &\leq \delta\|Tx_{n+1} - Tx_n\| + (1-\delta)\|Q_C(y_{n+1} - \lambda_{n+1}Ay_{n+1}) - Q_C(y_n - \lambda_n Ay_n)\| \\ &\leq \delta\|x_{n+1} - x_n\| + (1-\delta)\|(y_{n+1} - \lambda_{n+1}Ay_{n+1}) - (y_n - \lambda_n Ay_n)\| \\ &\leq \delta\|x_{n+1} - x_n\| + (1-\delta)\|(y_{n+1} - \lambda_{n+1}Ay_{n+1}) - (y_n - \lambda_{n+1}Ay_n) \\ &+ (\lambda_n - \lambda_{n+1})Ay_n\| \\ &\leq \delta\|x_{n+1} - x_n\| + (1-\delta)\|y_{n+1} - y_n\| + |\lambda_n - \lambda_{n+1}|\|Ay_n\| \\ &\leq \delta\|x_{n+1} - x_n\| + (1-\delta)\|Q_C(x_{n+1} - \mu_{n+1}Bx_{n+1}) - Q_C(x_n - \mu_n Bx_n)\| \\ &+ |\lambda_n - \lambda_{n+1}|\|Ay_n\| \\ &\leq \|x_{n+1} - x_n\| + |\mu_n - \mu_{n+1}|\|Bx_n\| + |\lambda_n - \lambda_{n+1}|\|Ay_n\|\end{aligned}$$

Step 3 We prove that $lim_{n\to\infty}||x_{n+1}-x_n||=0$.

Define $x_{n+1}=\beta_n x_n+(1-\beta_n)h_n$, observe that

$$\begin{aligned} h_{n+1}-h_n &= \frac{x_{n+2}-\beta_{n+1}x_{n+1}}{1-\beta_{n+1}}-\frac{x_{n+1}-\beta_n x_n}{1-\beta_n} \\ &= \frac{\alpha_{n+1}u+\gamma_{n+1}l_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n u+\gamma_n l_n}{1-\beta_n} \\ &= \frac{\alpha_{n+1}u}{1-\beta_{n+1}}+\frac{(1-\alpha_{n+1}-\beta_{n+1})l_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n u}{1-\beta_n}-\frac{(1-\alpha_n-\beta_n)l_n}{1-\beta_n} \\ &= \frac{\alpha_{n+1}}{1-\beta_{n+1}}(u-l_{n+1})-\frac{\alpha_n}{1-\beta_n}(u-l_n)+(l_{n+1}-l_n). \end{aligned}$$

Applying the conclusion of step 1, we have

$$\begin{aligned} ||h_{n+1}-h_n||-||x_{n+1}-x_n|| \le \frac{\alpha_{n+1}}{1-\beta_{n+1}}||u-l_{n+1}||+\frac{\alpha_n}{1-\beta_n}||u-l_n|| \\ +|\mu_n-\mu_{n+1}|\|Bx_n\|+|\lambda_n-\lambda_{n+1}|\|Ay_n\| \end{aligned}$$

Since $\{y_n\}$ and $\{l_n\}$ are bounded, by(C2),(C3)and(C4),we obtain that $lim_{n\to\infty}sup(||h_{n+1}-h_n||-||x_{n+1}-x_n||)\le 0$.

Hence by lemma 2.8,we have $lim_{n\to\infty}||h_n-x_n||=0$.

Consequently $lim_{n\to\infty}||x_{n+1}-x_n||=lim_{n\to\infty}(1-\beta_n)||h_n-x_n||=0$.

On the other hand, from $x_{n+1}=\alpha_n u+\beta_n x_n+\gamma_n l_n$, we have

$x_{n+1}-x_n=\alpha_n(u-x_n)+\gamma_n(l_n-x_n)$, then $lim_{n\to\infty}||l_n-x_n||=0$.

Step 4 We claim that $\limsup_{n\to\infty}\langle u-\widetilde{x},J(x_n-\widetilde{x})\rangle\le 0$. where $\widetilde{x}=Q_F u$ Define a mapping $W_n:C\to C$ by $W_n x=\delta Tx+(1-\delta)Q_C(I-\lambda_n A)Q_C(I-\mu_n B)x,\forall x\in C$, which implies that $W_n x_n=l_n$.

We choose a sequence $\{x_{n_i}\}$ of $\{x_n\}$ that converges weakly to x such that

$$\limsup_{n\to\infty}\langle u-\widetilde{x},J(x_n-\widetilde{x})\rangle=\limsup_{i\to\infty}\langle u-\widetilde{x},J(x_{n_i}-\widetilde{x})\rangle$$

Since $\{\lambda_n\}\subset[a,\sqrt[q-1]{\frac{q\alpha^{q-1}}{2K^q}}]$, $\{\mu_n\}\subset[a,\sqrt[q-1]{\frac{q\beta^{q-1}}{2K^q}}]$, $a>0$, it follows that $\{\lambda_{n_i}\},\{\mu_{n_i}\}$ are bounded. So there exists a subsequence $\{\lambda_{n_i}\}$ of $\{\lambda_{n_i}\}$ which converges to $\{\lambda_0\}\subset$

$[a, \sqrt[q-1]{\frac{q\alpha^{q-1}}{2K^q}}]$, and a subsequence $\{\mu_{n_i}\}$ of $\{\mu_{n_i}\}$ which converges to $\{\mu_0\} \subset [a, \sqrt[q-1]{\frac{q\alpha^{q-1}}{2K^q}}]$. Without loss of generality, we assume that $\{\lambda_{n_i}\} \to \lambda_0, \{\mu_{n_i} \to \mu_0\}$, then

$$
\begin{aligned}
S_0 &= Q_C(Q_C(I - \mu_0 B) - \lambda_0 A Q_C(I - \mu_0 B)) \\
&= Q_C(I - \lambda_0 A) Q_C(I - \mu_0 B)
\end{aligned}
$$

S_n is nonexpansive.

Since Q_C is nonexpansive, it follows from $l_n = \delta T x_n + (1-\delta) Q_C(y_n - \lambda_n A y_n)$, then

$$
\begin{aligned}
\|W_0 x_{n_i} - x_{n_i}\| &\leq \|\delta T x_{n_i} + (1-\delta) Q_C(y_{n_i} - \lambda_0 A y_{n-i}) - l_{n_i}\| + \|l_{n_i} - x_{n_i}\| \\
&\leq \|\delta T x_{n_i} + (1-\delta) Q_C(y_{n_i} - \lambda_0 A y_{n-i}) - \delta T x_{n_i} - (1-\delta) Q_C(y_{n_i} \\
&- \lambda_n A y_{n-i})\| + \|l_{n_i} - x_{n_i}\| \\
&\leq (1-\delta)|\lambda_{n-i} - \lambda_0| \|A y_{n-i}\| + \|l_{n_i} - x_{n_i}\|
\end{aligned}
$$

It follows from lemma2.6 that $x \in F(W_0)$. By using lemma2.5 and same as[15], we can obtain that $x \in F(W_0) = Q_F u$.

We have $\limsup_{n\to\infty}\langle u - \widetilde{x}, J(x_n - \widetilde{x})\rangle \leq 0 = \limsup_{n\to\infty}\langle u - \widetilde{x}, J(x_{n+1} - \widetilde{x})\rangle \leq 0$ holds.

Step 5 We show that $lim_{n\to\infty}||x_n - \widetilde{x}|| = 0$.

$$
\begin{aligned}
||x_{n+1} - \widetilde{x}||^2 &= \alpha_n \langle u - \widetilde{x}, J(x_{n+1} - \widetilde{x})\rangle + \beta_n \langle x_n - \widetilde{x}, J(x_{n+1} - \widetilde{x})\rangle + \gamma_n \langle l_n - \widetilde{x}, J(x_{n+1} - \widetilde{x})\rangle \\
&= \alpha_n \langle u - \widetilde{x}, J(x_{n+1} - \widetilde{x})\rangle + \beta_n \|x_n - \widetilde{x}\| \|x_{n+1} - \widetilde{x}\| + \gamma_n \|l_n - \widetilde{x}\| \|x_{n+1} - \widetilde{x}\| \\
&\leq (1-\alpha_n) \|x_n - \widetilde{x}\| \|x_{n+1} - \widetilde{x}\| + \alpha_n \langle u - \widetilde{x}, J(x_{n+1} - \widetilde{x})\rangle \\
&= \frac{1-\alpha_n}{2}(\|x_n - \widetilde{x}\|^2 + \|x_{n+1} - \widetilde{x}\|^2) + \alpha_n \langle u - \widetilde{x}, J(x_{n+1} - \widetilde{x})\rangle
\end{aligned}
$$

Then

$$
\begin{aligned}
||x_{n+1} - \widetilde{x}||^2 &\leq \frac{1-\alpha_n}{1+\alpha_n}||x_n - \widetilde{x}||^2 + \frac{2\alpha_n}{1+\alpha_n}\langle u - \widetilde{x}, J(x_{n+1} - \widetilde{x})\rangle \\
&= (1 - \frac{2\alpha_n}{1+\alpha_n})||x_n - \widetilde{x}||^2 + \frac{2\alpha_n}{1+\alpha_n}\langle u - \widetilde{x}, J(x_{n+1} - \widetilde{x})\rangle
\end{aligned}
$$

Where $\gamma_n = \frac{2\alpha_n}{1+\alpha_n}, \sigma_n = \langle u - \widetilde{x}, J(x_{n+1} - \widetilde{x})\rangle$.

Since by (C2),step 3, we have

$$
lim_{n\to\infty}\gamma_n = 0, \sum \gamma_n = \infty, \limsup_{n\to\infty} \sigma_n \leq 0.
$$

applying lemma 2.7,we deduce that $lim_{n\to\infty}||x_n - \widetilde{x}|| = 0$.

The proof of Theorem 3.1 is completes .

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