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ON POSITIVE PERMUTATION BRAID MOTIVATED BY A PUMP-MODULATED ND-DOPED FIBER LASER

M. ANIS

Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt

Abstract. In this paper we constructd the positive permutation braid of pump-modulated Nd-doped fiber laser dynamics and denoted it by $\beta^{+}_{n,k,r}$. We study this braid as a permutation, then calculate its inversion matrix. The braid representatives $\beta^{+}_{n,k,r}$ of the orbits associated to a modulated Nd-doped fiber laser as a permutation are given. It is shown that pump-modulated Nd-doped fiber laser knots and links are positive permutation braid.

Keywords: laser dynamics; knots and links; braid groups; positive permutation braid; topological invariants; inversion matrix.

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1. INTRODUCTION

1.1. **Knots, links and braids.** A subset K of R^3 $\big($ or $S^3\big)$ that is homeomorphic to a circle S^1 is called a knot, and a knot is a link if it is homeomorphic to a disjoint union, $S^1 \cup S^1 \cup ... \cup S^1$, of circles. The link of m circles is called a link with multiplicity m. In figure (1), there are some examples of knots and links.[1]

E-mail address: mona_anis1985@yahoo.com

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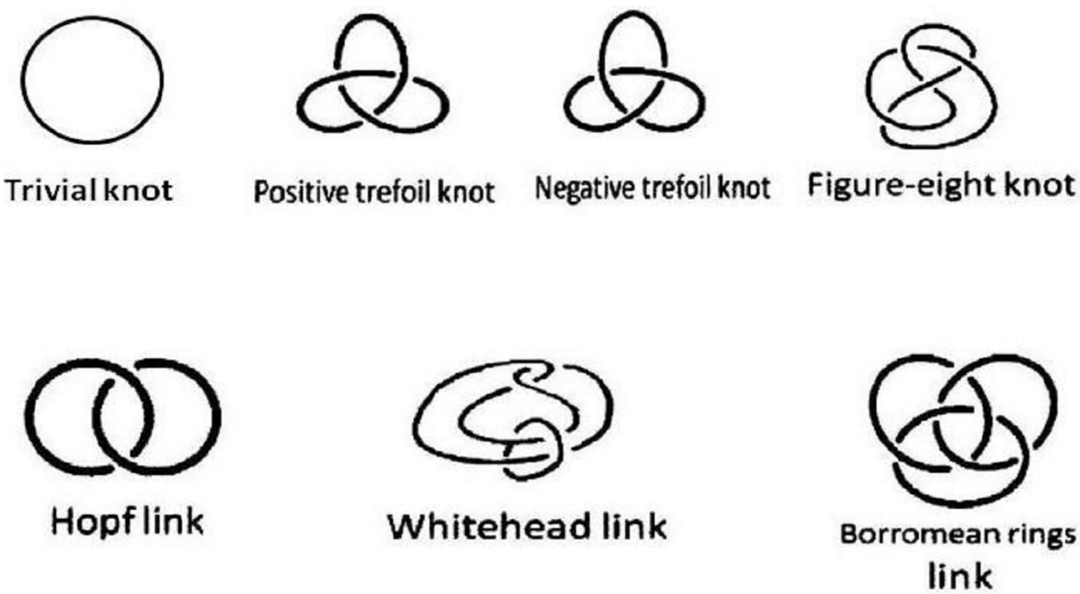

FIGURE 1. There are some examples of knots and links.

Braid theory is very useful in studying knot theory since any oriented knot can be represented as a closed braid. The braid group Bn can be defined via the following presentation, known as the braid presentation or Artin presentation [2]:

$$B_n = \left\{ \begin{array}{c} \sigma_i,\ i=1,2,\ldots,n-1:\ \sigma_i\sigma_j\ =\ \sigma_j\sigma_i\ \text{ if } |i-j|>1, \\ \sigma_i\sigma_{i+1}\sigma_i\ \ =\sigma_{i+1}\sigma_i\sigma_{i+1} \text{ if } i=1,2,\ldots,n-2 \end{array} \right\}$$

Where σ_i and σ_i^{-1} as in figure (2). Each closed braid is a knot or link, and conversely, as in figure (3).

A braid β consisting of an ordered sequence of the generators only, in which no inverse of any generator occurs will be called a positive braid and denoted by B_n^+. In braid group B_n, the braid which is accomplished by holding the top of the braid fixed and attaching the string bottoms to a rod which is then turned over once (in positive sequence), is known as a half twist positive braid and denoted by Δ_n

$$\Delta_n = (\sigma_1)(\sigma_2\sigma_1)(\sigma_3\sigma_2\sigma_1)\ldots(\sigma_{n-1}\ldots\sigma_1)$$

A positive permutation braid (In short PPB) is a positive braid where each pair of its strings cross at most once.

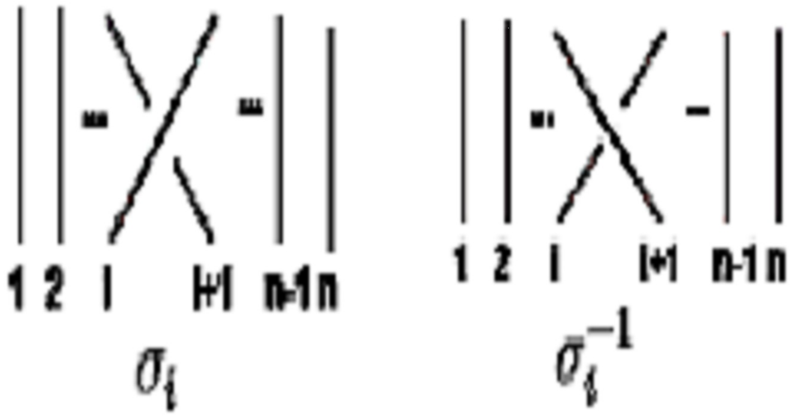

FIGURE 2. Generators of braid group.

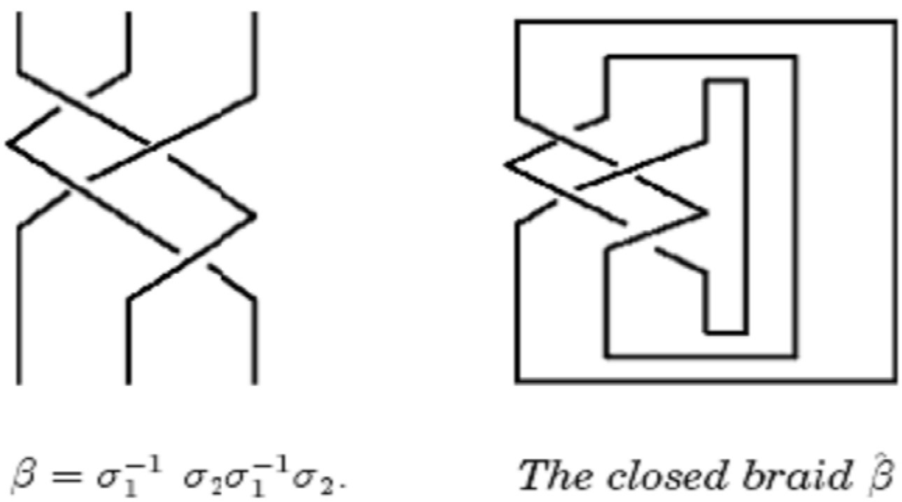

FIGURE 3. Trefoil knot as a closed braid.

The set of these braids in B_n is denoted by S_n^+,[3] . Hence

$$S_n^+ \subset B_n^+ \subset B_n.$$

1.2. **Inversion matrix.** Elements of the braid group B_n can be represented by matrices. A permutation matrix is a square binary matrix that has exactly one entry 1 in each row and each column and 0*s* elsewhere. In the i $\underline{th}$ row, the entry $\alpha(i)$ equals 1, for a permutation α. An inversion matrix of $\alpha \in S_n$, is the matrix

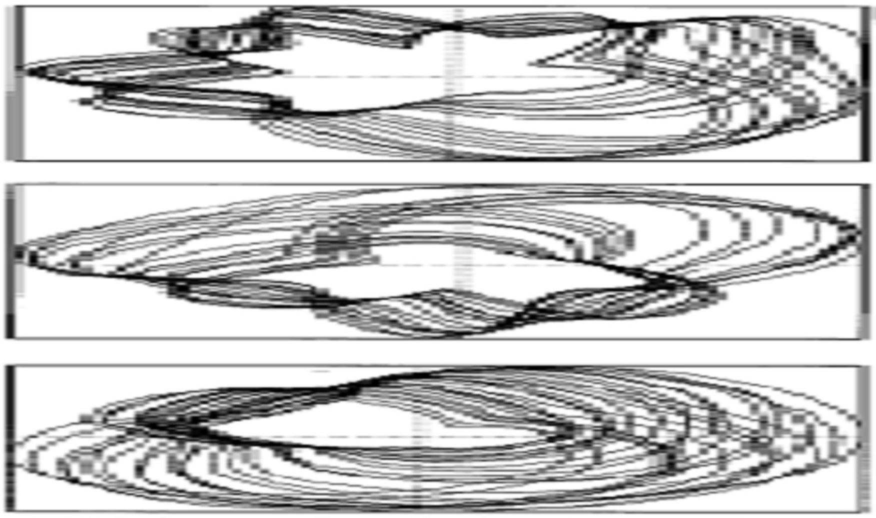

FIGURE 4. Some of the templates observed in the fiber laser.

$$M(\alpha) = (m_{ij})_{n\times n} = \begin{Bmatrix} 1 & \text{if } i<j \text{ and } \alpha(i)>\alpha(j) \\ 0 & \text{otherwise} \end{Bmatrix}$$

The construct a group of inversion matrix is $M_n(F) = \{M(\alpha) : \alpha \in S_n\} \cong S_n$, over the field $F = \{0,1\}$ with addition *mod* 2), which is isomorphic to the symmetric group [4]. For example,

$$M(\sigma_1\sigma_2\sigma_3\sigma_2\sigma_1) = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad M(\Delta = \sigma_1\sigma_2\sigma_1\sigma_3\sigma_2\sigma_1) = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

1.3. **Laser dynamics.** Lasers are virtually everywhere around us, for example, optical telecommunication and medical applications. It is taken from the phrase ”light amplification by stimulated emission of radiation”. Topological analysis can be used in order to study and compare the attractors of a single system (A modulated Nd-doped fiber laser), at different values of a control parameter (The modulation frequency w). We do this in an experimental situation where chaotic behavior is observed when w lies near the subharmonics $1/2, 1/3$, and $1/4$ of the natural relaxation frequency w_r . In the templates associated to this fiber laser system, it is observed that the global torsion increases systematically from one tongue to the next, as in figure(4).[5]

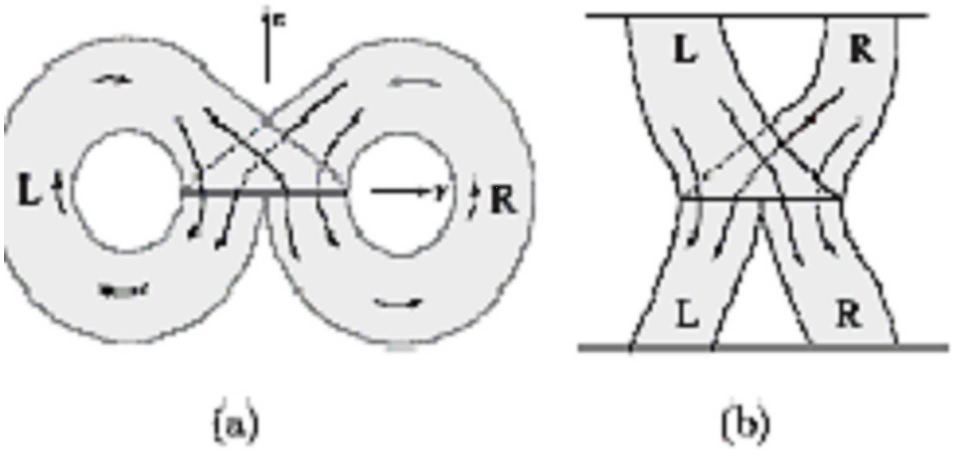

FIGURE 5. a: Lorenz template, b: Lorenz braid template.

In a three dimensional flow, periodic orbits defines knots, according to the uniqueness of solutions of ordinary differential equations, a Lorenz template or Lorenz attractor is an embedded branched surface in $R^3$with a semiflow. Indeed, the Lorenz template can be viewed as a branched two dimensional manifold. While the braid representatives of Lorenz knots and links can be embedded in a template so called Lorenz braid template, as in figure (5). The braid is called a Lorenz braid of type (k,r) and denoted by $\beta(k,r)$.[6].

2. POSITIVE PERMUTATION BRAID MOTIVATED BY PUMP-MODULATED ND-DOPED FIBER LASER DYNAMICS

The closed periodic orbits associated to a modulated Nd-doped fiber laser, with the attractor in the subharmonic region $w_{\frac{1}{n}}$ have the braid representatives

$$\beta_{n,k,r} = \Delta_k^{2(n-1)}\Delta_r L(r,k-r)\; n,k,r \in \mathbb{N}, r < k.$$

and it is in the left canonical form [7].

In this paper we introduced the positive permutation braid of pump-modulated Nd-doped fiber laser dynamics and denoted it by $\beta^+_{n,k,r}$. Our focus in this study will be on a subgroup of the braid , where we will study this braid as a permutation, then calculate its inversion matrix.

Theorem 1. *The closed periodic orbits associated to a modulated Nd-doped fiber laser, with the attractor in the subharmonic region $w_{\frac{1}{n}}$ a permutation have the braid representatives*

$$\beta^{+}_{n,k,r} = \Delta_r L(r,k-r)\ n,k,r \in \mathbb{N}, r < k.$$

Proof. The half twist positive braid is

$$\Delta_k = (\sigma_1)(\sigma_2\sigma_1)(\sigma_3\sigma_2\sigma_1)\ldots(\sigma_{k-1}\ldots\sigma_1)$$

$$\Delta_k = (\sigma_1\ldots\sigma_{k-1})$$

, then

$$\Delta_k^m = (\sigma_1\ldots\sigma_{k-1})^m, \qquad m \text{ is even number}$$

, then we have $(\sigma_1\ldots\sigma_{k-1})^m$ is equal the identity permutation as geometrically. This can be proved by induction. So $\Delta_k^{2(n-1)} = id$, and the permutation have the braid representation $\Delta_r L(r,k-r)\ n,k,r \in \mathbb{N}, r < k$,and is denoted it by $\beta^{+}_{n,k,r}$.

Lemma 2. *The braid representatives $\beta^{+}_{n,k,r} = \Delta_r L(r,k-r)$ of the orbits associated to a modulated Nd-doped fiber laser as apermutation is a positive permutation braid .*

□

Proof. The Lorenz word $L(r,k-r)$ is a positive braid word. So no self-intersection in each band of strands. Hence the word $\Delta_r L(r,k-r)$ is a one positive permutation braid.

Then the braid $\beta^{+}_{n,k,r} = \Delta_r L(r,k-r)$ is a positive permutation braid as a permutation. □

In [4] Elrifai and Anis were able to express the positive permutation braid as inversion matrix .So, we can also express $\beta^{+}_{n,k,r}$ as an inversion matrix .

Corollary 3. *There is one to one correspondence between $\beta^{+}_{n,k,r}$ and its inversion matrix .*

We can modify the Algorithm in [4] to write down a positive permutation braid word of $\beta^{+}_{n,k,r}$ from its inversion matrix as follow:

Algorithm 4. *For a given associated PPB of $\beta^{+}_{n,k,r}$, and its associated inversion matrix $M_{\beta^{+}_{n,k,r}}$.*

1) Each row will be translated to a braid word.

2) The row whose all its entries are zeros will contribute by the identity braid $e.(\Delta_k^{2(n-1)} = id)$

3) If the number of ones in the entries of the i^{th} row is k, then the corresponding braid word will be $\beta_i = \sigma_{i-1}\sigma i ... \sigma_{k-1}$.

4) Then write $\alpha^+ = w_k w_{k-1} ... w_1$.

Example 5. *From theorem* (1)*, and for different choices of the integers k and r, we have infinite number of knots and links. For instance, if $n = 2, k = 4$and $r = 2$ the braid*

$$\begin{aligned} \beta^+_{2,4,2} &= \Delta_2 L(2,2) = \sigma_1(\sigma_2\sigma_1\sigma_3\sigma_2) \\ &= \sigma_1\sigma_2\sigma_1\sigma_3\sigma_2. \end{aligned}$$

$$\beta^+_{2,4,2} = (1423) \text{ As a permutation.}$$

and its inversion matrix is,

$$M(\sigma_1\sigma_2\sigma_1\sigma_3\sigma_2) = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\underset{M_{\beta^+_{2,4,2}}}{\begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}} \quad \underset{\textit{stairs of } \beta^+_{2,4,2}}{\begin{bmatrix} \sigma_1\sigma_2\sigma_3 \\ \sigma_2\sigma_3 \\ e \\ e \end{bmatrix}}$$

, from algorithm (4) we have the braid $\beta^+_{2,4,2}$ from its inversion matrix as follow:

$$
\begin{aligned}
\beta^{+}_{2,4,2} &= e\; e\; \sigma_2\sigma_3\sigma_1\sigma_2\sigma_3 \\
&= \sigma_2\sigma_1\sigma_3\sigma_2\sigma_3 \\
&= \sigma_2\sigma_1\sigma_2\sigma_3\sigma_2 \\
&= \sigma_1\sigma_2\sigma_1\sigma_3\sigma_2
\end{aligned}
$$

Conflict of Interests

The authors declare that there is no conflict of interests.

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