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COMPARING METHOD OF UNDETERMINED COEFFICIENTS WITH MODIFICATION RULE AND METHOD OF VARIATION OF PARAMETERS

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Abstract. In this research article we have used the method of undetermined coefficients with modification rule and method of variation of parameters to solve linear ordinary differential equations and found that the method of variation of parameters is the easiest method to use.

Keywords: method of undetermined coefficients; method of variation of parameters; linear ordinary differential equations.

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1. Introduction

Solving nonhomogeneous second or higher order linear ordinary differential equation (ODE) involve finding first the solution of the homogeneous differential equation and then finding the solution of the nonhomogeneous part [8, 9]. There are basically two methods for determining the solution of the particular integral, the method of undetermined coefficients and variation of

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parameters [5, 7, 3, 4, 1, 6, 2]. In this review article we have determined the appropriate method to use in case where the particular solution is part of the homogeneous solution.

2. Material and Methods

A general linear nth-order nonhomogeneous ODE is

$$a_n(x)y^{(n)}+a_{n-1}(x)y^{(n-1)}+\cdots+a_1(x)y'+a_0(x)y=G(x). \tag{1}$$

where the coefficients $a_n(x),\ a_{n-1}(x),\ \cdots,\ a_1(x),\ a_0$ are all constants or variables with the same degree as their derivatives (Euler-Cauchy equations).

The general form of second order nonhomogeneous ODE with constant coefficients is

$$a_2y''+a_1y'+a_0y=G(x) \tag{2}$$

The general form of second order nonhomogeneous ODE with variable coefficients (Euler-Cauchy equation) is

$$a_2x^2y''+a_1xy'+a_0y=G(x), \quad x>0 \tag{3}$$

Solution of the homogeneous ODE

The general homogeneous ODE is

$$a_n(x)y^{(n)}+a_{n-1}(x)y^{(n-1)}+\cdots+a_1(x)y'+a_0(x)y=0 \tag{4}$$

Theorem

Fundamental Theorem for the Homogeneous Linear ODE (4)

For a homogeneous linear ODE (4), sums and constant multiples of solutions on some open interval I are again solutions on I.

If $G(x)=0$ in Equation (2), then the homogeneous equation has the following solutions (complementary function):

$$y_c=Ae^{m_1x}+Be^{m_2x},\ y_c=Axe^{mx}+Be^{mx} \text{ and } y_c=e^{\alpha x}(A\cos\beta x+B\sin\beta x)$$

for distinct, identical and complex roots of the auxiliary equations, respectively.

If $G(x) = 0$ in Equation (3), then the homogeneous equation has the following solutions:

$$y_c = Ax^{m_1} + Bx^{m_2},\ y_c = Ax^m + Bx^m \ln x \text{ and } y_c = x^{\alpha}\left[A\cos\left(\beta \ln x\right) + B\sin\left(\beta \ln x\right)\right]$$

for distinct, identical and complex roots of the auxiliary equations, respectively.

2.1. **Determination of the particular solution by Method of Undetermined Coefficients.** The particular solution, y_p of the nonhomogeneous part is assumed to take the same form as $G(x)$. The y_p and its derivative are substituted in the differential equation and the unknown constants are determined. If a term in the choice of y_p is part of the complementary function, then y_p is modified by multiplying by x. If the new y_p and its derivatives does not work we multiply by x^2.

Generally, the following general rule is applied

If any $y_{p_i}, i = 1,2,3,\cdots,n$ contains terms that duplicate terms in y_c, then that y_{p_i} must be multiplied by x^n, where n is the smallest positive integer that eliminates that duplication [9, 7].

2.2. **Determination of the particular solution by Method of Variation of Parameters.** Solving second order linear equations by method of variation of parameters.

We know that the complementary function for the homogeneous Equation (2) is

$$y_c = Ay_1(x) + By_2(x) \tag{5}$$

where y_1 and y_2 are linearly independent solutions. Let's replace the constants (parameters) A and B in Equation (5) by arbitrary functions $u_1(x)$ and $u_2(x)$, respectively. We want to determine a particular solution on the nonhomogeneous equation $a_2y'' + a_1y' + a_0y = G(x)$ of the form

$$y_p = u_1(x)y_1(x) + u_2(x)y_2(x) \tag{6}$$

Differentiating Equation (6) we get

$$y_p' = u_1'y_1 + u_2'y_2 + u_1y_1' + u_2y_2' \tag{7}$$

Since u_1 and u_2 are arbitrary functions, we impose two conditions on them so as to simplify our calculations. One condition is that y_p is a solution of the differential equation and we choose the other condition. From Equation (7), let's impose the condition that

(8) $$u_1'y_1 + u_2'y_2 = 0$$

Thus we have

(9) $$y_p' = u_1y_1' + u_2y_2'$$

Then we differentiate Equation (9) and get

(10) $$y_p'' = u_1'y_1' + u_2'y_2' + u_1y_1'' + u_2y_2''$$

Substituting Equations (6), (9) and (10) in Equation (2), we get

$$a_2(u_1'y_1' + u_2'y_2' + u_1y_1'' + u_2y_2'') + a_1(u_1y_1' + u_2y_2') + a_0(u_1y_1 + u_2y_2) = G \quad \text{or}$$

(11) $$u_1(a_2y_1'' + a_1y_1' + a_0y_1) + u_2(a_2y_2'' + a_1y_2' + a_0y_2) + a_2(u_1'y_1' + u_2'y_2') = G$$

Since y_1 and y_2 are solutions of the homogeneous equation, so

$$a_2y_1'' + a_1y_1' + a_0y_1 = 0 \quad \text{and} \quad a_2y_2'' + a_1y_2' + a_0y_2 = 0$$

and Equation (11) becomes

(12) $$a_2(u_1'y_1' + u_2'y_2') = G$$

Equations (8) and (12) form a system of two equations in the unknown functions u_1' and u_2'. After solving this system using different methods such as Elimination, Substitution, and Cramer's Rule or the Wronskian determinant we may be able to integrate to find u_1 and u_2 and then the particular solution is given by Equation (6).

Solving third order equations with method of variation of parameters. The complementary function for 3rd order homogeneous equation is

(13) $$y_c = Ay_1(x) + By_2(x) + Cy_3(x)$$

where y_1, y_2 and y_3 are linearly independent solutions. We replace the constants (parameters) A, B and C in Equation (13) by arbitrary functions $u_1(x), u_2(x)$ and $u_3(x)$. We want to determine a particular solution on the nonhomogeneous equation $a_3y''' + a_2y'' + a_1y' + a_0y = G(x)$ of the form

$$y_p = u_1(x)y_1(x) + u_2(x)y_2(x) + u_3(x)y_3(x) \tag{14}$$

Differentiating Equation (14) we get

$$y_p' = u_1y_1' + u_1'y_1 + u_2y_2' + u_2'y_2 + u_3y_3' + u_3'y_3 \tag{15}$$

Since u_1, u_2 and u_3 are arbitrary functions, we impose three conditions on them so as to simplify our calculations. One condition is that y_p is a solution of the differential equation and we choose the other condition. From Equation (15), let's impose the first condition that

$$u_1'y_1 + u_2'y_2 + u_3'y_3 = 0 \tag{16}$$

Thus we have

$$y_p' = u_1y_1' + u_2y_2' + u_3y_3' \tag{17}$$

Then we differentiate Equation (17) and get

$$y_p'' = u_1y_1'' + u_1'y_1' + u_2y_2'' + u_2'y_2' + u_3y_3'' + u_3'y_3' \tag{18}$$

Let's impose the second condition that

$$u_1'y_1' + u_2'y_2' + u_3'y_3' = 0 \tag{19}$$

Thus we have

$$y_p'' = u_1y_1'' + u_2y_2'' + u_3y_3'' \tag{20}$$

Differentiating Equation (20)

$$y_p''' = u_1y_1''' + u_1'y_1'' + u_2y_2''' + u_2'y_2'' + u_3'y_3'' + u_3y_3''' \tag{21}$$

Substituting Equations (14), (17) and (21) in nonhomogeneous equation gives

$a_3(u_1y_1''' + u_1'y_1'' + u_2y_2''' + u_2'y_2'' + u_3'y_3'' + u_3y_3''') + a_2(u_1y_1'' + u_2y_2'' + u_3y_3'') + a_0(u_1y_1' + u_2y_2' + u_3y_3') = G$ or

(22)

$$u_1(a_3y_1''' + a_2y_1'' + a_1y_1' + a_0y) + u_2(a_3y_2''' + a_2y_2'' + a_1y_2' + a_0y_2) + a_3(u_1'y_1'' + u_2'y_2'' + u_3'y_3'') = G$$

Since y_1, y_2 and y_3 are solutions of the homogeneous equation, so

$$a_3y_1''' + a_2y_1'' + a_1y_1' + a_0y_1 = 0, \quad a_3y_2''' + a_2y_2'' + a_1y_2' + a_0y_2 = 0 \quad \text{and} \quad a_3y_3''' + a_2y_3'' + a_1y_3' + a_0y_3 = 0$$

Thus Equation (22) becomes

$$a_3(u_1'y_1'' + u_2'y_2'' + u_3'y_3'') = G \tag{23}$$

Equations (16), (19) and (23) form a system of three equations in the unknown functions u_1', u_2' and u_3'. After solving this system using the Wronskian determinant we may be able to integrate to find u_1, u_2 and u_3 and then the particular solution is given by Equation (14).

Definition

A general solution of (4) on an open interval I is a solution of (4) on I of the form

$$y_c = c_1y_1 + c_2y_2 + c_3y_3 + \cdots + c_ny_n, \ (c_1, c_2, \cdots, c_n \text{ are arbitrary})$$

where $y_1, y_2, \cdots, y_n$ is a basis (or fundamental system) of solutions of (4) on I.
From the definition, by variation of parameters we have

$$y_p = u_1y_1 + u_2y_2 + u_3y_3 + \cdots + u_ny_n, \ (u_1, u_2, \cdots, u_n \text{ are functions to be determined})$$

The **Wronskian** W of n solutions $y_1, y_2, \cdots, y_n$ is defined as the nth-order determinant

$$W(y_1, y_2, \cdots, y_n) = \begin{vmatrix} y_1 & y_2 & y_3 & \cdots & y_n \\ y_1' & y_2' & y_3' & \cdots & y_n' \\ . & . & . & \cdots & . \\ y_1^{(n-1)} & y_2^{(n-1)} & y_3^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

The particular solution y_p for the nonhomogeneous ODE (1) is given by

$$\begin{aligned} y_p(x) &= \sum_{k=1}^{n} y_k(x) \int \frac{W_k(x)}{W(x)} G(x)\mathrm{d}x \\ &= y_1(x) \int \frac{W_1(x)}{W(x)} G(x)\mathrm{d}x + y_2(x) \int \frac{W_2(x)}{W(x)} G(x)\mathrm{d}x + \cdots + y_n(x) \int \frac{W_n(x)}{W(x)} G(x)\mathrm{d}x \end{aligned}$$

3. Results and Discussion

Examples

(1) Find the particular solution: $y'' - y' = x^2e^x$

Solution

$y'' - y' = 0$

The solution of the homogeneous equation is $y_c = A + Be^x$

Using the method of undetermined coefficient to determine the particular solution

The trial solution is $y_p = (ax^2 + bx + c)e^x$ we modify by multiplying by x

$y_p = (ax^3 + bx^2 + cx)e^x$ finding the first and second derivatives and then substituting in the differential equation

$y_p' = (ax^3 + bx^2 + cx)e^x + (3ax^2 + 2bx + c)e^x,$

$y_p'' = (ax^3 + bx^2 + cx)e^x + (6ax^2 + 4bx + 2c)e^x + (6ax + 2b)e^x,$

$(ax^3 + bx^2 + cx)e^x + (6ax^2 + 4bx + 2c)e^x + (6ax + 2b)e^x - (ax^3 + bx^2 + cx)e^x - (3ax^2 + 2bx + c)e^x = x^2e^x,$

$3ax^2 + (6a + 2b)x + 2b + c = x^2,$

By comparing coefficients gives $a = 1/3,\ b = -1$, and $c = 2$.

Therefore the particular solution is

$$y_p = (\frac{1}{3}x^3 - x^2 + 2x)e^x.$$

Using the method of variation of parameters to determine the particular solution

The particular solution is $y_p = u_1 + u_2e^x$,

The system of equations is

$$u_1' + u_2'e^x = 0$$
$$u_2'e^x = x^2e^x$$

Solving the system for u_1' and u_2' and integrating

$$u_2' = x^2 \Rightarrow u_2 = \frac{1}{3}x^3,$$

$$u_1' + x^2e^x = 0 \Rightarrow u_1 = (-x^2 + 2x - 2)e^x$$

$$y_p = (-x^2 + 2x - 2)e^x + \frac{1}{3}x^3e^x = (\frac{1}{3}x^3 - x^2 + 2x - 2)e^x$$

(2) Find the particular solution: $y'' - 2y' + 2y = e^x \cos x$.

Solution

The solution of the homogeneous equation is

$$y_c = Ae^x \cos x + Be^x \sin x$$

Using the method of undetermined coefficients, the particular solution is modified

$$y_p = axe^x \cos x + bxe^x \sin x$$

Finding the first and second derivatives

$y_p' = axe^x \cos x + ae^x \cos x - axe^x \sin x + bxe^x \sin x + be^x \sin x + bxe^x \cos x$

$y_p'' = axe^x \cos x + ae^x \cos x - axe^x \sin x - ae^x \sin x + ae^x \cos x - axe^x \sin x - ae^x \sin x - axe^x \cos x + bxe^x \sin x + be^x \sin x + bxe^x \cos x + be^x \cos x + be^x \sin x + bxe^x \cos x + be^x \cos x - bxe^x \cos x$

Substituting in the differential equation gives

$$-2ae^x \sin x + 2be^x \cos x = e^x \cos x$$

Comparing coefficients gives $a = 0$ and $b = 1/2$.

The particular solution is

$$y_p = \frac{1}{2}xe^x \sin x$$

Using the method of variation of parameters

$$y_p = u_1e^x \cos x + u_2e^x \sin x$$

We have a system of equations

$$u_1'e^x\cos x + u_2'e^x\sin x = 0$$
$$u_1'(-e^x\sin x + e^x\cos x) + u_2'(e^x\sin x + e^x\cos x) = e^x\cos x$$

Solving the system for u_1' and u_2' by using Wronskian and integrating

$$u_1' = -\cos x\sin x \;\Rightarrow u_1 = -\frac{1}{2}\sin^2 x,$$
$$u_2' = \frac{1}{2}(1+\cos 2x) \;\Rightarrow u_2 = \frac{1}{2}x + \frac{1}{4}\sin 2x$$
$$y_p = -\frac{1}{2}e^x\sin^2 x\cos x + \frac{1}{2}xe^x\sin x + \frac{1}{4}e^x\sin x\sin 2x = \frac{1}{2}xe^x\sin x$$

(3) Find the particular solution: $x^2y'' - 3xy' + 4y = x^2\ln x,\ x > 0.$

Solution

The equation is transformed to

$$y'' - 4y' + 4y = te^{2t}$$

The homogeneous equation is

$$y'' - 4y' + 4y = 0$$

The solution of the homogeneous equation is $y_c = (At+B)e^{2t}$

Using the method of undetermined coefficient to determine the particular solution.

The trial solution is $y_p = (at+b)e^{2t}$ we modify by multiplying by t. This trial fails thus we multiply by t^2,

$y_p = (at^3 + bt^2)e^{2t}$ finding the first and second derivatives and then substituting in the differential equation

$y_p' = (2at^3 + 2bt^2)e^{2t} + (3at^2 + 2bt)e^{2t},$

$y_p'' = (4at^3 + 4bt^2)e^{2t} + (6at^2 + 4bt)e^{2t} + (6at^2 + 4bt)e^{2t} + (6at + 2b)e^{2t},$

$(6at + 2b)e^{2t} = te^{2t},$

By comparing coefficients gives $a = 1/6$ and $b = 0$. Thus

$$y_p = \frac{1}{6}t^3e^{2t}.$$

Therefore the particular solution is

$$y_p = \frac{1}{6}x^2(\ln x)^3.$$

Using the method of variation of parameters to determine the particular solution

The particular solution is $y_p = u_1 te^{2t} + u_2 e^{2t}$,

The system of equations is

$$u_1' te^{2t} + u_2' e^{2t} = 0$$
$$2u_1' te^{2t} + u_1' e^{2t} + 2u_2' e^{2t} = te^{2t}$$

Solving the system for u_1' and u_2' and integrating

$$u_1' = t \Rightarrow u_1 = \frac{1}{2}t^2,$$
$$u_2' = -t^2 \Rightarrow u_2 = -\frac{1}{3}t^3$$
$$y_p = \frac{1}{2}t^3e^{2t} - \frac{1}{3}t^3e^{2t} = \frac{1}{6}t^3e^{2t}$$

Therefore the particular solution is

$$y_p = \frac{1}{6}x^2(\ln x)^3$$

4. Conclusions

We have seen that the method of variation of parameters is easy to implement, saves time and space during computational. Thus it is the best method to use to solve an ODE when the trial solution is part of the complementary function.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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