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THE GENERALIZED BESSEL MATRIX POLYNOMIALS

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Abstract. In this paper, the generalized Bessel matrix polynomials are introduced, starting from the hypergeometric matrix function. Integral form, Rodrigues's formula and generating matrix function are then developed for the generalized Bessel matrix polynomials. These polynomials appear as finite series solutions of second-order matrix differential equations and orthogonality property for the generalized Bessel matrix polynomials are given. Finally, connections between generalized Bessel matrix polynomials with Laguerre matrix polynomials and Whittaker matrix functions are established.

Keywords: Bessel matrix polynomials; Hypergeometric matrix functions; Matrix functional calculus.

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1. Introduction

The theory of generalized special matrix functions has witnessed a rather significant evolution during the last years. The reasons of interest have a manifold motivation. Restricting ourselves to the applicative field, we note that for some physical problems the use of new classes of special matrix functions provided solutions hardly achievable with conventional analytical and numerical means. Hermite, Chebyshev, Jacobi, Laguerre and

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Gegenbauer matrix polynomials were introduced and studied in [1, 4, 5, 13, 17]. Recently, a new extension of hypergeometric matrix functions and Humbert matrix functions have been introduced as a matrix power series in [15, 18].

The Bessel polynomials were proposed nearly a century ago [3, 16]. Since then, they have been recognized as a unique tool in both pure and applied mathematics. the main object of this paper is study some important properties of the generalized Bessel matrix polynomials which is a matrix extension of Bessel scalar polynomials [2, 6, 7, 8, 9, 19, 20]. The paper is organized as follows: Section 2, provides the definition of the generalized Bessel matrix polynomials, $Y_n(A, B; z)$, for parameter matrices A and B, and integral form of the generalized Bessel matrix polynomials is given. Section 3, Rodrigues's formula and the generating matrix function of Bessel matrix polynomials is established. In Section 4, these polynomials appear as finite series solutions of second-order matrix differential equations and orthogonality property for the Bessel matrix polynomials is given. Section 5, connections between generalized Bessel matrix polynomials with Laguerre matrix polynomials and Whittaker matrix functions are obtained.

Throughout this paper, consider the complex space $\mathbb{C}^{N\times N}$ of complex matrices of common order N. A matrix A is a positive stable matrix in $\mathbb{C}^{N\times N}$ if $Re(\lambda) > 0$ for all $\lambda \in \sigma(A)$ where $\sigma(A)$ is the set of all eigenvalues of A. If $A_0, A_1, ..., A_n$ are elements of $\mathbb{C}^{N\times N}$ and $A_n \neq 0$, then we call

$$P_n(z) = A_n z^n + A_{n-1}z^{n-1} + A_{n-2}z^{n-2} + ... + A_0,$$

a matrix polynomial of degree n in z.

If $f(z)$ and $g(z)$ are holomorphic functions of the complex variable z which are defined in an open set Ω of the complex plane and A is a matrix in $\mathbb{C}^{N\times N}$ such that $\sigma(A) \subset \Omega$, then from the properties of the matrix functional calculus [10], it follows that

$$f(A)g(A) = g(A)f(A). \tag{1.1}$$

Hence, if B in $\mathbb{C}^{N\times N}$ is a matrix for which $\sigma(B) \subset \Omega$ and if $AB = BA$, then

$$f(A)g(B) = g(B)f(A). \tag{1.2}$$

The reciprocal gamma function denoted by $\Gamma^{-1}(z) = \frac{1}{\Gamma(z)}$ is an entire function of the complex variable z. Then for any matrix A in $\mathbb{C}^{N\times N}$, the image of $\Gamma^{-1}(z)$ acting on A denoted by $\Gamma^{-1}(A)$ is a well defined matrix. Furthermore, if

$$A + nI \quad \text{is invertible for all integer } n \geq 0 \tag{1.3}$$

where I is the identity matrix in $\mathbb{C}^{N\times N}$, then $\Gamma(A)$ is invertible, its inverse coincides with $\Gamma^{-1}(A)$ and one gets the formula [12]

$$(A)_n = A(A+I)...(A+(n-1)I) = \Gamma(A+nI)\Gamma^{-1}(A); \quad n \geq 1; (A)_0 = I. \tag{1.4}$$

Jódar and Cortés have proved in [12] that

$$\Gamma(A) = \lim_{n\longrightarrow\infty} (n-1)![(A)_n]^{-1} n^A. \tag{1.5}$$

Let P and Q be two positive stable matrices in $\mathbb{C}^{N\times N}$. The gamma matrix function $\Gamma(P)$ and the beta matrix function $B(P,Q)$ have been defined in [11] as follows

$$\begin{aligned} \Gamma(P) &= \int_0^\infty e^{-t} t^{P-I} dt; \\ t^{P-I} &= \exp\Big((P-I)\ln t\Big) \end{aligned} \tag{1.6}$$

and

$$B(P,Q) = \int_0^1 t^{P-I}(1-t)^{Q-I} dt. \tag{1.7}$$

In [15] Kishka et al are given the definition of hypergeometric matrix series as

$$F(\alpha,\beta;\gamma;X) = \sum_{k=0}^{\infty} \frac{(\alpha)_k(\beta)_(k)[(\gamma)_k]^{-1}}{k!} X^k. \tag{1.8}$$

Laguerre matrix polynomials may be defined in [13] by

$$L_n^A(z) = \sum_{m=0}^{n} \frac{(-1)^m}{m!\,(n-m)!} (A+I)_n \left[(A+I)_m\right]^{-1} z^m. \tag{1.9}$$

2. On generalized Bessel matrix polynomials

Definition 2.1. Let A and B be a matrix in $\mathbb{C}^{N\times N}$ satisfying the spectral condition (1.3). For any natural number $n \geq 0$, the n-th generalized Bessel matrix polynomial $Y_n(A,B;z)$ is defined by

$$(2.1)\quad Y_n(A,B;z) = \sum_{k=0}^{n} \binom{n}{k} (A+(n+k-2)I)^{(k)}(z\,B^{-1})^k;\, (n \in \mathbb{N}_0 := \{0,1,2,..\}).$$

If we replace A by $A+2I$ then the n-th generalized Bessel matrix polynomial $Y_n(A,B;z)$ given by

$$(2.2)\quad Y_n(A,B;z) = \sum_{k=0}^{n} \binom{n}{k} (A+(n+1)I)_k(z\,B^{-1})^k;\ (n \in \mathbb{N}_0 := \{0,1,2,..\}),$$

where $\binom{n}{k}$ is a binomial coefficient and $(A)^{(k)}$ as usual means $(A)(A-I)...(A-(k-1)I)$.

By the explicit formulas for generalized Bessel matrix polynomials $Y_n(A,B;z)$ the first four of these polynomials are therefore given by

$$Y_0(A,B;z) = I,$$

$$Y_1(A,B;z) = I + A(z\,B^{-1}),$$

$$Y_2(A,B;z) = I + 2(A+I)(z\,B^{-1}) + (A+I)(A+2I)(z\,B^{-1})^2$$

and

$$Y_3(A,B;z) = I+3(A+2I)(z\,B^{-1})+3(A+2I)(A+3I)(z\,B^{-1})^2+(A+2I)(A+3I)(A+4I)(z\,B^{-1})^3.$$

From (1.8) we can written

$$\begin{aligned}
&{}_2F_0(-nI, A+(n-1)I; -; -z\,B^{-1})\\
&= \sum_{k=0}^{n} \frac{(-nI)(I-nI)(2I-nI)..(kI-nI-I)(A+(n+1)I)..(A+(n+k-2)I)}{k!}(-z\,B^{-1})^k\\
&= \sum_{k=0}^{n} \frac{(nI)(In-I)(nI-2I)..(nI-kI+I)(A+(n+1)I)..(A+(n+k-2)I)}{k!}(z\,B^{-1})^k\\
&= \sum_{k=0}^{n} \binom{n}{k} (A+(n+k-2)I)^{(k)}(z\,B^{-1})^k,
\end{aligned}$$

where

$$\frac{(-1)^k}{(n-k)!}I = \frac{(-n)_k}{n!}I = \frac{(-nI)_k}{n!};\ \ 0 \le k \le n.$$

This gives the generalized Bessel matrix polynomials

$$(2.3)\ Y_n(A,B;z) = {}_2F_{\mathbf{0}}(-nI, A+(n-1)I; -; -z\ B^{-1}); (n \in \mathbb{N}_0 := \{0,1,2,..\}).$$

To get an integral form for the generalized Bessel matrix polynomials of complex variable. By (2.3), we can write

$$\begin{aligned} Y_n(A,B;z) &= {}_2F_0(-nI, A+(n-1)I; -; -z\ B^{-1}) \\ &= \sum_{k=0}^{\infty} \frac{(-nI)_k \Gamma^{-1}(A+(n-1)I)\Gamma(A+(k+n-1)I)}{k!}\ (z\ B^{-1})^k. \end{aligned} \tag{2.4}$$

According to (1.6), we find that

$$\Gamma(A+(k+n-1)I) = \int_0^{\infty} t^{A+(n+k-2)I}\ e^{-t} dt, \tag{2.5}$$

therefore, we see that

$$Y_n(A,B;z) = \Gamma^{-1}(A+(n-1)I) \int_0^{\infty} t^{A+(n-2)I}\ e^{-t} \sum_{k=0}^{\infty} \frac{(-nI)_k}{k!} (-t\ z\ B^{-1})^k dt, \tag{2.6}$$

thus,

$$\begin{aligned} Y_n(A,B;z) &= \Gamma^{-1}(A+(n-1)I) \int_0^{\infty} t^{A+(n-2)I}\ {}_1F_{\mathbf{0}}(-nI; -; -t\ z\ B^{-1})\ e^{-t} dt \\ &= \Gamma^{-1}(A+(n-1)I) \int_0^{\infty} t^{A+(n-2)I} (I + t\ z\ B^{-1})^n e^{-t} dt. \end{aligned} \tag{2.7}$$

Summarizing, the following result has been established.

Theorem 2.1. *Let A and B be a matrix in $\mathbb{C}^{N\times N}$ satisfying the spectral condition (1.3), let z, t be a complex numbers. Then for any integer $n \ge 0$, expressions (2.7) hold true.*

3. The generating function of generalized Bessel matrix polynomials

It is well known the interest in the applications of the generating function of classical generalized Bessel polynomials, (see [8, 9, 19, 20]). The aim of this section to obtain formula for the generating function of generalized Bessel matrix polynomials.

Lemma 3.1. *Let A and B be a matrix in $\mathbb{C}^{N\times N}$ satisfying the spectral condition (1.3) and let $Y_n(A,B;z)$ be the n-th generalized Bessel matrix polynomial. Then the following formula holds for $n \geq 0$*

$$Y_n(A,B;z) = B^{-n} z^{2I-A} e^{\frac{B}{z}}\ D^n(z^{A+(2n-2)I} e^{-\frac{B}{z}}); \quad D \equiv \frac{d}{dz}. \tag{3.1}$$

Proof. By expanding the right side of (3.1) and identifying it which formula (2.1) for $Y_n(A,B;z)$. We see that

$$\begin{aligned} & B^{-n} z^{2I-A} e^{\frac{B}{z}}\ D^n(z^{A+(2n-2)I} e^{-\frac{B}{z}}) \\ &= B^{-n} z^{2I-A} \sum_{m=0}^{\infty} \frac{B^m}{m! z^m} D^n(\sum_{s=0}^{\infty} (-1)^s \frac{B^s}{s!} z^{A+(2n-2-s)I}) \\ &= B^{-n} z^{2I-A} \sum_{m=0}^{\infty} \frac{B^m}{m! z^m} \sum_{s=0}^{\infty} (-1)^s \frac{B^s}{s!} (A+(2n-2-s)I)^{(n)}, z^{A+(n-2-s)I}, \end{aligned} \tag{3.2}$$

set $m+s=k$, then the double sum can be written as

$$\begin{aligned} & B^{-n} z^{2I-A} e^{\frac{B}{z}}\ D^n(z^{A+(2n-2)I} e^{-\frac{B}{z}}) \\ &= \sum_{k=0}^{\infty} \frac{2^{k-n} z^{n-k}}{k!} \sum_{s=0}^{k} (-1)^s \binom{k}{s} ((A+2(n-1)I)-sI)^{(n)}. \end{aligned} \tag{3.3}$$

Using lemma 2 in [9] and applying the matrix functional calculus [10], one gets

$$\sum_{s=0}^{k} (-1)^s \binom{k}{s} ((A+2(n-1)I)-sI)^{(n)} = \begin{cases} 0, & for\ k>n, \\ n^{(k)}((A+2(n-1)I-kI)^{(n-k)} & for\ k\leq n, \end{cases} \tag{3.4}$$

the final expression in (3.3) reduces to

$$\begin{aligned} & B^{-n} z^{2I-A} e^{\frac{B}{z}}\ D^n(z^{A+(2n-2)I} e^{-\frac{B}{z}}) \\ &= \sum_{k=0}^{n} \frac{B^{k-n} z^{n-k}}{k!}\ n^{(k)} (A+(2n-2-k)I)^{(n-k)} \\ &= \sum_{k=0}^{n} \binom{n}{k} (A+(n+k-2)I)^{(k)} (z\ B^{-1})^k = Y_n(A,B;z). \end{aligned} \tag{3.5}$$

□

The following result formula for the generating function of generalized Bessel matrix polynomials.

Theorem 3.1. *Let A and B be a matrix in $\mathbb{C}^{N\times N}$ satisfying the spectral condition (1.3) and let z, t be complex number. Then the generating function for generalized Bessel matrix polynomial is given by*

$$\begin{aligned}
&[\frac{1}{2}+\frac{1}{2}\sqrt{1-2zt}]^{2I-A}\ (1-2zt)^{-\frac{1}{2}}\ \exp\,[\frac{B\,(1-\sqrt{1-2zt})}{2z}] \\
&=\sum_{n=0}^{\infty}\frac{(B/2)^n\ Y_n(A,B;z)}{n!}\ t^n.
\end{aligned} \tag{3.6}$$

Proof. Lagrange's theorem, c.f [8] states that if $w=z+t\psi(w)$; when w is many valued, that branch is taken which converges to z when $t\to 0$, then

$$f(w)=f(z)+\sum_{n=1}^{\infty}\frac{t^n}{n!}\ \frac{d^{n-1}}{dz^{n-1}}\big[\{\psi(z)\}^n\ \frac{d}{dz}f(z)\big]. \tag{3.7}$$

From (3.7) we easily deduce that

$$\frac{\partial}{\partial z}f(w)=\sum_{n=0}^{\infty}\frac{t^n}{n!}\ D^n\big[\{\psi(n)\}^n.f'(z)\big]. \tag{3.8}$$

Now, let us rediscover the generating function (3.6) by use of Lagrange's theorem for this purpose we take $\psi(z)=z^2$

$$f'(z)=z^{A-2I}\ e^{-\frac{B}{z}}\ \textit{and}\ w=z+(\frac{t}{2})w^2,$$

so that

$$w=\frac{1}{t}(1-(\sqrt{1-2zt})),$$

since we are to have $w=z$, when $t\to 0$. Thus it at once follows from (3.8) that

$$f'(w)\frac{\partial w}{\partial z}=\sum_{n=0}^{\infty}\frac{(t/2)^n}{n!}\ D^n\big[z^{2n}.z^{A-2I}e^{-\frac{B}{z}}\big]. \tag{3.9}$$

Thus, we obtain from (3.1) and (3.9)

$$w^{A-2I}\ e^{-\frac{B}{w}}\ (1-2zt)^{-\frac{1}{2}}=z^{A-2I}\ e^{-\frac{B}{z}}\sum_{n=0}^{\infty}\frac{(B/2)^n\ Y_n(A,B;z)}{n!}\ t^n,$$

□

whence, we derive

$$\begin{aligned}&\left[1-\frac{1}{2}(1-\sqrt{1-2zt})\right]^{2I-A}(1-2zt)^{-\frac{1}{2}}\exp\left[\frac{B\,(1-\sqrt{1-2zt})}{2z}\right]\\&=\sum_{n=0}^{\infty}\frac{(B/2)^n\;Y_n(A,B;z)}{n!}\;t^n.\end{aligned} \tag{3.10}$$

Therefore, the proof of Theorem 3.1 is completed.

4. Orthogonality for generalized Bessel matrix polynomial

Orthogonal matrix polynomials is an emergent field whose development is reaching important results from both the theoretical and practical points of view. Important connections between orthogonal matrix polynomials and matrix differential equations appear in [4, 5, 14].

The operator $\theta = z(\frac{d}{dz})$, already used in the derivation of the many matrix differential equations, c.f [15], is helpful in deriving a matrix differential equation satisfied by (2.1), we obtain

$$\begin{aligned}(\theta I)\;Y_n(A,B;z)=&B^{-1}\sum_{k=0}^{\infty}\frac{(-1)^k k(-nI)_k(A+(n-1)I)_k\;z^k\;(B^{-1})^{k-1}}{k!}\\=&B^{-1}\sum_{k=1}^{\infty}\frac{(-1)^k(-nI)_k(A+(n-1)I)_k z^k\;(B^{-1})^{k-1}}{(k-1)!}.\end{aligned} \tag{4.1}$$

A shift of index yields

$$\begin{aligned}(\theta I)\;Y_n(A,B;z)=&B^{-1}\sum_{k=0}^{\infty}\frac{(-1)^{k+1}(-nI)_{k+1}(A+(n-1)I)_{k+1}z^{k+1}\;(B^{-1})^k}{k!}\\&=-\,(z\;B^{-1})\sum_{k=0}^{\infty}(k-n)I)(A+(n-1+k)I)\;U_k(z)\\&=-\,(z\;B^{-1})[(\theta I-nI)(\theta I+(A+(n-1)I))]\;Y_n(A,B;z),\end{aligned}$$

where, $U_k(z)=\frac{(-1)^k(-nI)_k(A+(n-1)I)_k}{k!}\;(z\;B^{-1})$, thus,

$$\left[(\theta I)+(z\;B^{-1})\big((\theta I-nI)(\theta I+(A+(n-1)I))\big)\right]\;Y_n(A,B;z)=0. \tag{4.2}$$

Equation (4.2) is easily put in the form

$$z^2\;Y_n^{''}(A,B;z)+(Az+B)Y_n^{'}(A,B;z)=nI(A+(n-1)I)Y_n(A,B;z). \tag{4.3}$$

Therefore, the following result has been established.

Theorem 4.1. *For each natural number $n \geq 0$, then the generalized Bessel matrix polynomial $Y_n(A, B; z)$ satisfies the equation (4.3).*

Now, we shall show that the generalized Bessel matrix polynomials form an orthogonal system with path of integration an arbitrary curve surrounding the origin, and with the weight function $\rho(z)$ given by

$$\rho(z) = \frac{1}{2\pi i}\sum_{n=0}^{\infty}\Gamma(A)\Gamma^{-1}(A+(n-1)I)\ (\frac{-B}{z})^n, \tag{4.4}$$

which satisfies the related a matrix nonhomogeneous equation

$$(z^2\rho(z))' = (Az+B)\rho(z) - \frac{1}{2\pi i}[(A-I)(A-2I)]z. \tag{4.5}$$

If equation (4.3) is multiplied by $\rho(z)$, we have

$$(z^2\rho(z)Y_n')' - (z^2\rho(z))'Y_n' + (Az+B)\rho(z)Y_n' = nI(A+(n-1)I)Y_n\rho(z)$$

and using (4.5) we find that

$$(z^2\rho(z)Y_n')' + \frac{[(A-I)(A-B)]z}{2\pi i}Y_n' = nI(A+(n-1)I)Y_n\rho(z). \tag{4.6}$$

If we multiply equation (4.6) by $Y_m(A, B; z)$ and integrate around the unit circle, we get

$$\begin{aligned}&\int_C (z^2\rho(z)Y_n'(z))'Y_m\ dz + \int_C \frac{[(A-I)(A-2I)]z}{2\pi i}Y_n'\ Y_m\ dz\\ &= nI(A+(n-1)I)\int_C \rho(z)Y_nY_m\ dz.\end{aligned} \tag{4.7}$$

By integrating by parts we have

$$nI(A+(n-1)I)\int_C \rho(z)Y_nY_m\ dz = -\int_C z^2\rho(z)Y_n'Y_m'\ dz. \tag{4.8}$$

Interchanging n and m, that is

$$mI(A+(m-1)I)\int_C \rho(z)Y_mY_n\ dz = -\int_C z^2\rho(z)Y_m'Y_n'\ dz \tag{4.9}$$

and subtracting gives

$$[nI(A+(n-1)I) - mI(A+(m-1)I)]\int_C \rho(z)Y_mY_n\ dz = 0. \tag{4.10}$$

Finally, for $n \neq m$ we have

$$\int_C \rho(z) Y_m Y_n \, dz = 0,$$

this shows that the generalized Bessel matrix polynomials are orthogonal with $\rho(z)$ as weight function.

5. A connections between Bessel with Laguerre matrix Polynomials and Whittaker matrix functions

In this section, we shall study relations of the generalized Bessel Polynomials and to other matrix polynomials and matrix functions.

From (1.9), we can written

$$L_n^A(z) = \sum_{m=0}^{n} \frac{(-1)^m}{m!\,(n-m)!}\,(A+I)_n\,[(A+I)_m]^{-1}\,z^m. \tag{5.1}$$

It follows that

$$L_n^{-2nI-A+I}(\frac{B}{z}) = \sum_{m=0}^{n} \frac{(-1)^m B^m}{m!\,(n-m)!}\,(-2nI-A+2I)_n\,[(-2nI-A+2I)_m]^{-1}\,z^{-m}$$

and multiply both sides by $n!\,(-zB^{-1})^n$, one gets

$$\begin{aligned} n!\,(-zB^{-1})^n\,L_n^{-2nI-A+I}(\frac{B}{z}) &= \sum_{m=0}^{n} \frac{(-1)^{m+n} B^{m-n}}{m!\,(n-m)!}\,(-2nI-A+2I)_n n!\,[(-2nI-A+2I)_m]^{-1}\,z^{n-m} \\ &= \sum_{m=0}^{n} \frac{(2nI+A-mI-2I)...(nI+A-I)n!}{m!\,(n-m)!}\,(z\,B^{-1})^{n-m}. \end{aligned}$$

Here by changing the indices $n-m \to k$, we have

$$\sum_{k=0}^{n} \frac{(nI+A+kI-2I)...(nI+A-I)n!}{k!\,(n-k)!}\,(z\,B^{-1})^k = \sum_{k=0}^{n} \frac{(nI+A+kI-2I)_k}{k!\,(n-k)!}(z\,B^{-1})^k.$$

This leads to

$$Y_n(A,B;z) = n!(-zB^{-1})^n L_n^{-2nI-A+I}(\frac{B}{z}); (n \in \mathbb{N}_0 := \{0,1,2,..\}). \tag{5.2}$$

The Whittaker function $W_{\nu,-\nu+1/n+2}(z)$ is defined by

$$\begin{aligned} W_{\nu,-\nu+1/n+2}(z) &= e^{-z/2}\,z^{\nu}\,\sum_{k=0}^{n} \frac{n(n+1-2\nu)(n-1)(n+2-2\nu)...(n-k+1)(n+k-2\nu)}{k!\,z^k} \\ &= e^{-z/2}\,z^{\nu}\,{}_2F_0(-n, n+1-2\nu; -; -\frac{1}{z}), \end{aligned} \tag{5.3}$$

where $\nu \in N_0$, see [9, pp.37]. Hence, if C is a matrix satisfying (1.3), then by the properties of the matrix functional calculus [10] and (2.6), the matrix $W_{C,-C+I/n+2}(z)$ is well defined for z and

$$W_{C,-C+I/n+2}(z) = e^{-z/2}\ z^C \sum_{k=0}^{n} \frac{(-1)^k(-nI)_k\ ((n+1)I-2C)_k}{k!\ z^k} \tag{5.4}$$
$$= e^{-z/2}\ z^C\ {}_2F_0(-nI,(n+1)I-2C;-;-\frac{1}{z}).$$

From (2.1), we can written

$$Y_n(2I-2C,B;\frac{1}{z}) = \sum_{k=0}^{n} \frac{(-1)^k(-nI)_k\ ((n+1)I-2C)_k}{k!} (\frac{B^{-1}}{z})^k$$
$${}_2F_0(-nI,(n+1)I-2C;-;-\frac{B^{-1}}{z}),$$

comparison with (5.4), we obtain

$$Y_n(A,B;z) = e^{B/2z}(z\ B^{-1})^{I-A/2}\ W_{I-A/2,(A-I)/n+2}\ (\frac{B}{z}); (n \in \mathbb{N}_0 := \{0,1,2,..\}). \tag{5.5}$$

Now, we formalize of the results obtained so far in the following theorem:

Theorem 5.1. *With previous notations for hypergeometric matrix functions, Laguerre matrix polynomials and Whittaker matrix functions, the following relations connect the generalized Bessel matrix polynomials with these special matrix functions:*

(i) $Y_n(A,B;z) = {}_2F_0(-nI,A+(n-1)I;-;-z\ B^{-1})$.

(ii) $Y_n(A,B;z) = n!\ (-zB^{-1})^n\ L_n^{-2nI-A+I}(\frac{B}{z})$.

(iii) $Y_n(A,B;z) = e^{B/2z}\ (z\ B^{-1})^{I-A/2}\ W_{I-A/2,(A-I)/n+2}\ (\frac{B}{z})$.

References

[1] R.S. Batahan, A new extension Hermite matrix polynomials and its applications, *Linear Algebra Appl.*, **419** (2006), 82-92.

[2] J.O. Baek and D.W. Lee, A differential equation for multiple Bessel polynomials with raising and Lowering operators, *Bull. Korean Math. Soc.*, **48** (2011), 445-454.

[3] J.L. Burchnall, The Bessel polynomials, *Canad. J. Math.*, **3** (1951), 62-67.

[4] E. Defez and L. Jódar, Chebyshev matrix polynomails and second order matrix differential equations, *Utilitas Math.*, **61** (2002), 107-123.

[5] E. Defez, L. Jódar and A. Law, Jacobi matrix differential equation, polynomial solutions and their properties, *Comput. Math. Applicat.*, **48** (2004), 789-803.

[6] G. Dattoli, H.M. Srivastava, and D. Sacchetti, A note on the Bessel Polynomials, *Math. Maced.*, **1** (2003), 1-7.

[7] E.H. Doha and H.M. Ahmed, Recurrences and explicit formulae for the expansion and connection coefficients in series of Bessel polynomials, *J. Phys. A: Math. Gen.*, **37** (2004), 8045-8063.

[8] M.K. Das, A generating function for the generalized Bessel polynomials, *Amer. Math. Month.*, **74** (1967), 182-183.

[9] E. Grosswald, *Bessel Polynomials, Lecture Notes in Mathematics*, vol. 698, Springer, Berlin, 1978.

[10] G. Golub and C.F. Van Loan, *Matrix Computations*, The Johns Hopkins University Press, Baltimore, MA, (1989).

[11] L. Jódar and J.C. Cortés, Some properties of Gamma and Beta matrix functions, *Appl. Math. Lett.*, **11** (1998), 89-93.

[12] L. Jódar and J.C. Cortés, On the hypergeometric matrix function, *J. Comp. Appl. Math.*, **99** (1998), 205-217.

[13] L. Jódar and J. Sastre, On the Laguerre matrix polynomials, *Utilitas Math.*, **53** (1998), 37-48.

[14] L. Jódar and E. Defez and E. Ponsoda, Orthogonal matrix polynomials with respect to linear matrix moment functionals: Theory and applications, *J. Appr. Theory Appl.*, **12** (1996), 96-115.

[15] Z. M. G. Kishka, A. Shehata and M. Abul-Dahab, A new extension of hypergeomatric matrix functions, *Advan. Appl. Math. Scie.*, **10** (2011), 349-371.

[16] H. L. Krall and O. Frink, A new class of orthogonal polynomials: The Bessel polynomials, *Trans. Amer. Math. Soc.*, **65** (1949), 100-115.

[17] A. Shehata; A new extension of Gegenbauer matrix polynomials and their properties, *Bulletin Internat. Math. Virtual Institute*, **2** (2012), 29-42.

[18] A. Shehata and M. Abul-Dahab, A new extension of Humbert matrix function and their properties, *Advan. Pure. Math.*, **1** (2011),315-321.

[19] H.M. Srivastava, Orthogonality relations and generating functions for the generalized Bessel polynomials, *Appl. Math. Comput.*, **61** (1994) 99-134.

[20] S. Yang and H.M. Srivastava, Some families of generating functions for the Bessel polynomials, *J. Math. Anal. Appl.*, **211** (1997), 314-325.