Available online at http://scik.org
J. Math. Comput. Sci. 2 (2012), No. 5, 1377-1386
ISSN: 1927-5307

FIXED POINTS OF CYCLIC WEAK CONTRACTIONS IN METRIC SPACES

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Abstract. In this paper the well known notion of a cyclic contraction for a finite family of non-empty subsets of a metric space X and a mapping T of X into X (respectively, into the collection of nonempty subsets of X) has been generalized. Subsequently, the above idea is used to obtain some new fixed point theorems for single and multi-valued mappings. The results obtained herein generalize some recent fixed point theorems.

Keywords: Fixed points, cyclic weak contraction, metric spaces.

2000 AMS Subject Classification: 54H25; 47H10

1. Introduction and Preliminaries

Throughout this paper $\mathbb{N}$ denotes the set of natural numbers and Φ the class of functions $\varphi : [0, \infty) \to [0, \infty)$ satisfying:

(a): φ is continuous and monotone nondecreasing,

(b): $\varphi(t) = 0 \Leftrightarrow t = 0$.

The function $\varphi \in \Phi$ is also known as altering distance function (see, for instance, [1]).

In [2], Dutta and Chaudury obtained the following generalization of the well known Banach contraction principle.

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Received April 29, 2012

Theorem 1.1. *Let (X, d) be a complete metric space and $T : X \to X$ a self-mapping satisfying*

$$\psi(d(Tx, Ty)) \leq \psi(d(x, y)) - \varphi(d(x, y)) \tag{1.1}$$

for all $x, y \in X$, where $\psi, \varphi \in \Phi$. Then T has a unique fixed point.

The mapping T satisfying (1.1) is known as (ψ, φ)-weakly contraction [3].

Notice that when $\psi(t) = t$ and $\varphi(t) = (1-k)t$, we get the well know Banach contraction principle as a special case of Theorem 1.1.

On the other hand Kirk et al. [4] introduced the following notion of cyclic mappings and obtained a fixed point theorem (see Theorem 1.3 below).

Definition 1.2. Let $A_1, A_2, ..., A_p$ be nonempty subsets of a metric space (X, d). A mapping $T : \bigcup_{i=1}^{p} A_i \to \bigcup_{i=1}^{p} A_i$ is called a cyclic mapping (or p-cyclic mapping) if

$$T(A_i) \subset A_{i+1}, \text{ where } A_{p+1} = A_1.$$

Theorem 1.3. *Let $A_1, A_2, ..., A_p$ be nonempty closed subsets of a complete metric space and $T : \bigcup_{i=1}^{p} A_i \to \bigcup_{i=1}^{p} A_i$ a cyclic mapping. Assume that there exists $k \in (0, 1)$ such that*

$$d(Tx, Ty) \leq kd(x, y) \; \forall x \in A_i \text{ and } y \in A_{i+1}.$$

Then T has a unique fixed point.

For a detailed study of cyclic mappings, we refer to [4 -13] and references thereof.

Recently, Karapinar and Sadarangani [12] (see also [11]) combined the ideas of (ψ, φ)-weakly contractions, and cyclic contractions and introduced the notion of cyclic weak (ψ, φ)-contraction as follows:

Definition 1.4. Let $A_1, A_2, ..., A_p$ be nonempty subsets of a metric space (X, d) such that $X = \bigcup_{i=1}^{p} A_i$. A mapping $T : X \to X$ is said to be cyclic weak (ψ, φ)-contraction if

(1): $X = \bigcup_{i=1}^{p} A_i$ is a cyclic representation of X with respect to T;

(2): $\psi(d(Tx,Ty)) \leq \psi(d(x,y)) - \varphi(d(x,y))$ for all $x \in A_i$ and $y \in A_{i+1}$,

where $\psi, \varphi \in \Phi$ and $A_{p+1} = A_1$.

Example 1.5. [11, Example 4]. Let $X = [-1,1]$ with the usual metric, i.e., $d(x,y) = |x-y|$. Let $A_1 = [-1,0] = A_3$ and $A_2 = [0,1] = A_4$. Then $X = \bigcup_{i=1}^{4} A_i = [-1,1]$. Define $T : X \to X$ by

$$Tx = -\frac{x}{3} \text{ for all } x \in X.$$

It is clear that T is a cyclic mapping on X. Further, if $\psi, \varphi : [0,\infty) \to [0,\infty)$ are defined by $\psi(t) = t$ and $\varphi(t) = t/2$, then $\psi, \varphi \in \Phi$ and T is a cyclic weak (ψ,φ)-contraction.

Following theorem is the main result in [12].

Theorem 1.6. *Let (X,d) be a metric space and $A_1, A_2, ..., A_p$ nonempty closed subsets of X such that $X = \bigcup_{i=1}^{p} A_i$. Let $T : X \to X$ be a cyclic weak (ψ,φ)-contraction. Then T has a unique fixed point $z \in \bigcap_{i=1}^{p} A_i$.*

In this paper we obtain two types of generalizations of the above theorem, One, for single valued mappings, and other for multi-valued mappings in a metric space. Our results extend and generalize certain fixed point theorems of [4], [11], [12] and others.

2. Generalized cyclic weak (ψ,φ)-contraction

First we extend Definition 1.2 as follows.

Definition 2.1. Let $A_1, A_2, ..., A_p$ be nonempty subsets of a metric space (X,d). A cyclic mapping $T : \bigcup_{i=1}^{p} A_i \to \bigcup_{i=1}^{p} A_i$ will be called a *Generalized cyclic weak (ψ,φ)-contraction* if

$$\psi(d(Tx,Ty)) \leq \psi(M(x,y)) - \varphi(M(x,y)) \tag{2.1}$$

for all $x \in A_i$ and $y \in A_{i+1}$, where $\psi, \varphi \in \Phi$, $A_{p+1} = A_1$ and

$$M(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\right\}.$$

Remark 2.2. When $M(x,y) = d(x,y)$ in Definition 2.1, we recover Definition 1.4. Hence the class of *generalized cyclic weak (ψ,φ)-contraction* is larger than cyclic weak (ψ,φ)-contraction.

Now we present our first result.

Theorem 2.3. *Let $A_1, A_2, ..., A_p$ be nonempty closed subsets of a complete metric space (X, d) and $T : \bigcup_{i=1}^{p} A_i \to \bigcup_{i=1}^{p} A_i$ be a generalized cyclic weak (ψ, φ)-contraction on X. Then T has a unique fixed point $z \in \bigcap_{i=1}^{p} A_i$.*

Proof. Suppose for some $i \in \{1, 2, .., p\}$ there exists an $x \in A_i$ satisfying (2.1). Since for any $n \in \mathbb{N}$, either n or $n+1$ is even, we have

$$\begin{aligned} \psi(d(T^n x, T^{n+1}x)) &\leq \psi(M(T^{n-1}x, T^n x)) - \varphi(M(T^{n-1}x, T^n x)) \\ &\leq \psi(M(T^{n-1}x, T^n x)). \end{aligned} \tag{2.2}$$

Since ψ is nondecreasing, we have

$$\begin{aligned} d(T^n x, T^{n+1}x) &\leq \max\{d(T^{n-1}x, T^n x), d(T^{n-1}x, T^n x), d(T^n x, T^{n+1}x), \\ &\quad \frac{d(T^{n-1}x, T^{n+1}x) + d(T^{n+1}x, T^{n+1}x)}{2}\} \\ &\leq d(T^{n-1}x, T^n x). \end{aligned}$$

for $n \in \mathbb{N}$. Thus $\{d(T^n x, T^{n+1}x)\}$ is a decreasing sequence of nonnegative real numbers. If $\lim_{n\to\infty} d(T^n x, T^{n+1}x) = 0$ then we are done. Suppose that $\lim_{n\to\infty} d(T^n x, T^{n+1}x) = r$ for some $r > 0$. Making $n \to \infty$ in (2.2) and using the continuity of ψ and φ, we have

$$\psi(r) \leq \psi(r) - \varphi(r) \leq \psi(r),$$

which is a contradiction. Hence

$$\lim_{n\to\infty} d(T^n x, T^{n+1}x) = 0.$$

We show that $\{T^n x\}$ is a Cauchy sequence. Suppose $\{T^n x\}$ is not Cauchy. Then there exists $\mu > 0$ and increasing sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that for all $n \leq m_k < n_k$,

$$d(T^{m_k}x, T^{n_k}x) \geq \mu \text{ and } d(T^{m_k}x, T^{n_k-1}x) < \mu.$$

By the triangle inequality,

$$d(T^{m_k}x, T^{n_k}x) \leq d(T^{m_k}x, T^{n_k-1}x) + d(T^{n_k-1}x, T^{n_k}x).$$

It follows that $\lim_{k\to\infty} d(T^{m_k}x, T^{n_k}x) = \mu$. Now by (2.1), we have

$$\begin{aligned}\psi(d(T^{m_k+1}x, T^{n_k+1}x)) &= \psi(d(TT^{m_k}x, TT^{n_k}x))\\ &\leq \psi(M(T^{m_k}x, T^{n_k}x)) - \varphi(M(T^{m_k}x, T^{n_k}x))\\ &\leq \psi(M(T^{m_k}x, T^{n_k}x)).\end{aligned}$$

Making $k \to \infty$,

$$\psi(\mu) \leq \psi(\mu) - \varphi(\mu) \leq \psi(\mu),$$

a contradiction unless $\mu = 0$. Therefore $\{T^n x\}$ is Cauchy. Since X is complete there exists a point $z \in \bigcup_{i=1}^{p} A_i$ such that $\{T^n x\}$ converges to z. Now for some $i \in \{1, 2, ..., p\}$ there exist sequences $\{T^{2n}x\}$ and $\{T^{2n-1}x\}$ in A_i and A_{i+1} respectively, with $A_{p+1} = A_1$, both converging to z.

Using (2.1), we get

$$\begin{aligned}\psi(d(T^{2n}x, Tz)) &= \psi(d(TT^{2n-1}x, Tz))\\ &\leq \psi(M(T^{2n-1}x, z)) - \varphi(M(T^{2n-1}x, z))\\ &\leq \psi(M(T^{2n-1}x, z)).\end{aligned}$$

Making $k \to \infty$, we get

$$\psi(d(z, Tz)) \leq \psi(d(z, z)) = \psi(0) = 0,$$

and $\psi(d(z, Tz)) = 0$. This implies $d(z, Tz) = 0$ and $z = Tz$. Uniqueness of the fixed point follows easily. □

Corollary 2.4. *Theorem 1.6.*

Proof. It comes from Theorem 2.3, when $X = \bigcup_{i=1}^{p} A_i$ and $M(x, y) = d(x, y)$. □

Corollary 2.5. *Theorem 1.3.*

Proof. It comes from Theorem 2.3, when $M(x, y) = d(x, y)$, $\psi(t) = t$ and $\varphi(t) = (1 - k)t$ where $k \in (0, 1)$. □

Corollary 2.6. [11, Theorem 6]. *Let $A_1, A_2, ..., A_p$ be nonempty closed subsets of a complete metric space (X, d) and $T : \bigcup_{i=1}^{p} A_i \to \bigcup_{i=1}^{p} A_i$ a cyclic mapping such that*

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y))$$

for all $x \in A_i$ and $y \in A_{i+1}$, where $\varphi \in \Phi$, $A_{p+1} = A_1$. Then T has a unique fixed point $z \in \bigcap_{i=1}^{p} A_i$.

Proof. It comes from Theorem 2.3, when $M(x, y) = d(x, y)$, $\psi(t) = t$. □

The following example shows the generality of Theorem 2.3 over Theorems 1.3 and 1.6.

Example 2.7. Let $X = \{1, 2, 3, 4, 5\}$ endowed with the metric d defined by

$$d(1,2) = d(1,3) = d(3,5) = \frac{13}{8},\ d(1,4) = \frac{3}{2}, \qquad d(3,4) = 2.$$
$$d(1,5) = d(2,4) = \frac{7}{4},\ d(2,3) = d(4,5) = 1, \qquad d(2,5) = \frac{15}{8}.$$

Suppose $A_1 = \{1, 2, 3\}$ and $A_2 = \{1, 4, 5\}$ then $A_1 \cup A_2 = X$. Consider a mapping $T : X \to X$ defined by

$$T1 = 1,\ T2 = T3 = 4,\ T4 = 1,\ T5 = 2.$$

We define $\psi(t) = 2t$ and $\varphi(t) = \frac{t}{20}$ for all $t \geq 0$.

Observe that $T(A_1) = \{1, 4\} \subset A_2$ and $T(A_2) = \{1, 2\} \subset A_1$. It can be easily verified that T satisfies all the hypotheses of Theorem 2.3 and $T1 = 1 \in A_1 \cap A_2$. However T does not satisfy Theorems 1.3 and 1.6. For $x = 3,\ y = 5$ we have

$$d(Tx, Ty) = \frac{7}{4} > \frac{13}{8} - \frac{13}{160} = d(x, y) - \varphi(d(x, y)).$$

4. Multi-valued cyclic weak (ψ, φ)-contraction

Throughout this section X denotes a metric space (X, d), $CB(X)$ the collection of all nonempty closed and bounded subsets of X, $C(X)$ the collection of all nonempty compact subsets of X and H the Hausdorff metric induced by d, i.e.,

$$H(A, B) = \max\left\{\sup_{x \in A} d(x, B),\ \sup_{y \in B} d(y, A)\right\},$$

for all $A, B \subseteq CB(X)$, where $d(x, B) = \inf_{y \in B} d(x, y)$.

First we extend Definitions 1.2 and 1.4 for a multi-valued mapping.

Definition 4.1. Let $A_1, A_2, ..., A_p$ be nonempty subsets of a metric space X such that $X = \bigcup_{i=1}^{p} A_i$. A mapping $T : X \to CB(X)$ is said to be a cyclic representation of X with respect to T if

$$Tx \subset A_{i+1} \text{ for all } x \in A_i, \text{ where } A_{p+1} = A_1.$$

Definition 4.2. Let $A_1, A_2, ..., A_p$ be nonempty subsets of a metric space X such that $X = \bigcup_{i=1}^{p} A_i$. A mapping $T : X \to CB(X)$ will be called a *multi-valued cyclic weak (ψ, φ)-contraction* if

(i): $X = \bigcup_{i=1}^{p} A_i$ is a cyclic representation of X with respect to T;
(ii): $\psi(H(Tx, Ty)) \leq \psi(d(x, y)) - \varphi(d(x, y))$ for all $x \in A_i$ and $y \in A_{i+1}$,

where $\psi, \varphi \in \Phi$ and $A_{p+1} = A_1$.

Theorem 4.3. *Let $A_1, A_2, ..., A_p$ be nonempty closed subsets of a complete metric space X such that $X = \bigcup_{i=1}^{p} A_i$. Let $T : X \to C(X)$ be a multi-valued cyclic weak (ψ, φ)-contraction on X. Then T has a fixed point $z \in \bigcap_{i=1}^{p} A_i$.*

Proof. We construct a sequence $\{x_n\}$ in X in the following way. Let $x_0 \in A_1$ and $x_1 \in Tx_0 \subset A_2$. If $H(Tx_0, Tx_1) = 0$ then $x_1 \in Tx_1$ i.e., x_1 is fixed point of T and we are done. Assume that $H(Tx_0, Tx_1) > 0$. There exits a point $x_2 \in Tx_1 \subset A_3$ such that $d(x_1, x_2) \leq H(Tx_0, Tx_1)$. Such a choice is admissible, since Tx_1 is compact (see Nadler Jr. [14, p. 480]). Since Tx_2 is compact, we choose a point $x_3 \in A_4$ such that $d(x_2, x_3) \leq H(Tx_1, Tx_2)$. Again, if $H(Tx_1, Tx_2) = 0$ then $x_2 \in Tx_2$ i.e., x_2 is fixed point of T. For $n > 0$ there exists $i_{n_0} \in \{1, 2, ..., p\}$ such that $x_{n-1} \in A_{i_n}$ and $x_n \in A_{i_{n+1}}$. Continuing in the same manner for $n \in \mathbb{N}$, we get

$$d(x_n, x_{n+1}) \leq H(Tx_{n-1}, Tx_n) \; .$$

Since T is a multi-valued cyclic weak (ψ, φ)-contraction, we have

$$
\begin{aligned}
(4.1) \qquad \psi(d(x_n, x_{n+1})) &\leq \psi(H(Tx_{n-1}, Tx_n)) \leq \psi(d(x_{n-1}, x_n)) - \varphi(d(x_{n-1}, x_n)) \\
&\leq \psi(d(x_{n-1}, x_n)).
\end{aligned}
$$

Since ψ is nondecreasing, we have

$$d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n).$$

for $n \in \mathbb{N}$. Thus $\{d(x_n, x_{n+1})\}$ is a decreasing sequence of nonnegative real numbers. Let $\lim_{n\to\infty} d(x_n, x_{n+1}) = r$ for some $r \geq 0$. Making $n \to \infty$ in (4.1) and using the continuity of ψ and φ, we have

$$\psi(r) \leq \psi(r) - \varphi(r) \leq \psi(r),$$

which is a contradiction unless $r = 0$. Hence

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$$

We show that $\{x_n\}$ is a Cauchy sequence. Suppose $\{x_n\}$ is not Cauchy. Then there exists $\mu > 0$ and increasing sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that for all $n \leq m_k < n_k$,

$$d(x_{m_k}, x_{n_k}) \geq \mu \text{ and } d(x_{m_k}, x_{n_k - 1}) < \mu.$$

By the triangle inequality,

$$d(x_{m_k}, x_{n_k}) \leq d(x_{m_k}, x_{n_k-1}) + d(x_{n_k-1}, x_{n_k}).$$

It follows that, $\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = \mu$. Using (ii), we get

$$
\begin{aligned}
\psi(d(x_{m_k+1}, x_{n_k+1})) &\leq \psi(H(Tx_{m_k}, Tx_{n_k})) \\
&\leq \psi(d(x_{m_k}, x_{n_k})) - \varphi(d(x_{m_k}, x_{n_k})) \\
&\leq \psi(d(x_{m_k}, x_{n_k})).
\end{aligned}
$$

Making $k \to \infty$,

$$\psi(\mu) \leq \psi(\mu) - \varphi(\mu) \leq \psi(\mu),$$

a contradiction unless $\mu = 0$. Therefore $\{x_n\}$ is Cauchy. Since X is complete $\{x_n\}$ has a limit in X. Call it z. By the property that $X = \bigcup_{i=1}^{p} A_i$ is a cyclic representation of X with respect to T, the sequence $\{x_n\}$ has infinite number of terms in each A_i for $i \in 1, 2, ..., p$. Suppose $z \in A_i$, $Tz \in A_{i+1}$ and we choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with $x_{n_k} \in A_{i-1}$ (the existence of this subsequence is guaranteed by the fact that $\{x_n\}$ has infinite number of terms in each A_i for $i \in \{1, 2, ..., p\}$). Again by (ii), we have

$$\begin{aligned} \psi(d(x_{n_k+1}, Tz)) &\leq \psi(H(Tx_{n_k}, Tz)) \\ &\leq \psi(d(x_{n_k}, z)) - \varphi(d(x_{n_k}, z)) \\ &\leq \psi(d(x_{n_k}, z)). \end{aligned}$$

Making $k \to \infty$, we get

$$\psi(d(z, Tz)) \leq \psi(d(z, z)) = \psi(0) = 0,$$

and $\psi(d(z, Tz)) = 0$. This implies $d(z, Tz) = 0$ and $z \in Tz$. □

Corollary 4.4. *Let $A_1, A_2, ..., A_p$ be nonempty closed subsets of a complete metric space X such that $X = \bigcup_{i=1}^{p} A_i$. Let $T : X \to C(X)$ such that*

$$H(Tx, Ty) \leq d(x, y) - \varphi(d(x, y))$$

for all $x \in A_i$ and $y \in A_{i+1}$, where $\varphi \in \Phi$ and $A_{p+1} = A_1$. Then T has a fixed point $z \in \bigcap_{i=1}^{p} A_i$.

Proof. It comes from Theorem 4.3, when $\psi(t) = t$. □

Corollary 4.5. *Let $A_1, A_2, ..., A_p$ be nonempty closed subsets of a complete metric space X such that $X = \bigcup_{i=1}^{p} A_i$. Let $T : X \to C(X)$ such that*

$$H(Tx, Ty) \leq kd(x, y)$$

for all $x \in A_i$ and $y \in A_{i+1}$, where $k \in (0, 1)$ and $A_{p+1} = A_1$. Then T has a fixed point $z \in \bigcap_{i=1}^{p} A_i$.

Proof. It comes from Theorem 4.3, when $\psi(t) = t$ and $\varphi(t) = (1-k)t$, where $k \in (0, 1)$. □

References

[1] M.S. Khan, M. Swleh and S. Sessa, Fixed point theorems by altering distances between the points, Bull. Aust. Math. Soc. 30(1)(1984), 1-9.

[2] P.N. Dutta and B.S. Choudhury, A generalisation of contraction principle in metric spaces, Fixed Point Theory and Applications (2008), doi:10.1155/2008/406368, pages 1-8.

[3] H.K. Nashine and B. Samet, Fixed point results for (ψ, φ)-weakly contractive conditions in partially ordered metric spaces, Nonlinear Anal. 74(2011), 2201-2209.

[4] W. A. Kirk, P. S. Srinivasan and P. Veeramani, Fixed points for mappings satisfying cyclical contractive conditions, Fixed Point Theory 4(2003), 79-89.

[5] A.A. Eldred and P. Veeramani, Existence and convergence of best proximity points, J. Math. Anal. Appl. 323(2006), 1001-1006.

[6] C. Di Bari, T. Suzuki and C. Vetro, Best proximity points for cyclic Meir–Keeler contractions, Nonlinear Anal. 69(2008) 3790-3794.

[7] T. Suzuki, M. Kikkawa and C. Vetro, The existence of best proximity points in metric spaces with the property UC, Nonlinear Anal. 71(2009), 2918-2926.

[8] S. Karpagam and S. Agrawal, Best proximity point theorems for p-cyclic Meir-Keeler contractions, Fixed Point Theory and Applications, (2009), Article ID 197308, 9 pages.

[9] S. Karpagam and S. Agrawal, Best proximity point theorems for cyclic orbital Meir–Keeler contraction maps, Nonlinear Anal. 74(2011), 1040–1046.

[10] K. Neammanee and A. Kaewkhao, Fixed points and Best proximity points for multi-valued mapping satisfying cyclical condition, Int. J. of Math. Sci. Appl. 1(2011), 1-9.

[11] Erdal Karapinar, Fixed point theory for cyclic weak ϕ-contraction, Applied Mathematics Letters 24(2011) 822–825.

[12] Erdal Karapinar and Kishin Sadarangani, Fixed point theory for cyclic $(\phi - \psi)$-contractions, Fixed Point Theory and Applications 2011, 2011:69 doi:10.1186/1687-1812-2011-69.

[13] M. Derafshpour, S. Rezapour, On the existence of best proximity points of cyclic contractions, Advances in Dynamical Systems and Applications 6(2011), 33–40.

[14] S.B. Nadler Jr., Multivalued contraction mappings, Pacific J. Math. 30(1969), 475-488.