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ALEXANDROFF L-FUZZY TOPOLOGICAL SPACES AND REFLEXIVE L-FUZZY RELATIONS

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Abstract. Galois connection in category theory palys an important role in establish the relationships between different spatial structures. In this paper, we prove that there exist many interesting Galois connections between the category of Alexandroff L-fuzzy topological spaces and the category of reflexive L-fuzzy relations.

Keywords: complete residuated lattice; Alexandroff L-fuzzy topological space; L-fuzzy approximation space; Galois correspondence

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1. Introduction

Hájek [8] introduced a complete residuated lattice which is an algebraic structure for many valued logic. Bělohlávek [2] investigated information systems and decision rules over complete residuated lattices. Hence residuated lattices and their generalizations are the main structures of truth degree used in many-valued logic [4, 28, 33]. Höhle [12] introduced L-fuzzy topologies with algebraic structure L (cqm, quantales, MV-algebra). It has developed in many directions [3, 5, 6, 7, 18].

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Fang and Yue [7] studied the relationship between L-fuzzy closure systems and L-fuzzy topological spaces from a category viewpoint for a complete residuated lattice L. Ramadan [29] studied the relationship between L-fuzzy interior systems and L-fuzzy topological spaces over complete residuated lattices. Yao and Han characterized Alexandroff L-topological spaces by means of L-orderings [39].

The rough set theory was originally proposed by Pawlak [22, 23] as a mathematical approach for handling imprecision and uncertainty in data analysis. In recent years, rough set theory has developed significantly due to its widespread applications. Various generalized rough set models have been established and their properties or structures have been investigated intensively [3, 17, 19, 25, 30, 34, 36, 37, 38, 41]. Radzikowska [26, 27] developed fuzzy rough sets in complete residuated lattice. An interesting and natural research topic in rough set theory is the study of rough set theory via topology. Kortelainen [15] considered the relationship between modified sets, topological spaces, and rough sets based on a preorder. Subsequently, as generalizations of rough sets from the viewpoint of fuzzy sets, Qin and Pei [32] showed that there exists a one-to-one correspondence between the family of all the lower approximation sets based on fuzzy preorder and the set of all fuzzy topologies that satisfy the so-called (TC) axiom. Pei et al. [23] observed that inverse serial relations are the weakest relations that can induce topological spaces, and that different relations based on generalized rough set models will induce different topological spaces. In addition, Hao and Li [10] determined a one-to-one correspondence between the set of all reflexive, transitive L-fuzzy relations and the set of all Alexandroff L-fuzzy topologies. Ma and Hu [20] investigated the topological and lattice structures of L-fuzzy rough sets determined by lower and upper sets. Qiao and Hu [24] studied the relationship between L-fuzzy pretopological spaces [40] and L-fuzzy approximation spaces based on the reflexive L-fuzzy relations from a category viewpoint. Kim [13, 14] investigated the properties of various approximation operators and Alexandroff topologies in complete residuated lattices.

In this paper, we investigate the relationships between the category of Alexandroff L-fuzzy topological spaces and the category of reflexive L-fuzzy approximation spaces. In particular, we obtain some interesting adjunctions between the considered categories.

2. Preliminaries

Throughout this paper, L denotes a complete lattice. The greatest element of L is denoted by $\top$ and the least element of L is denoted by $\bot$. For $A \subseteq L$, we write $\bigvee A$ for the least upper bound of A and $\bigwedge A$ of A for the greatest lower bound of A. Specifically, $\bigvee L = \top$ and $\bigwedge L = \bot$ are respectively the universal upper and the universal lower bounds in L. We assume that $\top \neq \bot$, i.e. L has at least two elements.

Definition 2.1. ([2, 4, 8, 33]) An algebra $(L, \wedge, \vee, \odot, \rightarrow, \bot, \top)$ is called a complete residuated lattice if it satisfies the following conditions:

(1) $(L, \leq, \vee, \wedge, \bot, \top)$ is a complete lattice with the greatest element $\top$ and the least element $\bot$;

(2) $(L, \odot, \top)$ is a commutative monoid;

(3) $x \odot y \leq z$ iff $x \leq y \rightarrow z$ for $x, y, z \in L$.

An operator $^* : L \rightarrow L$ defined by $a^* = a \rightarrow \bot$ is called a strong negation if $a^{**} = a$.

In this paper, we assume that $(L, \leq, \odot)$ is a complete residuated lattice unless otherwise specified.

Some basic properties of the binary operation $\odot$ and residuated operation $\rightarrow$ are collected in the following lemma, and they can be found in many works.

Lemma 2.2. [2, 4, 8, 33] Let L be a complete residuated lattice. Then the following properties hold for each $x, y, z, x_i, y_i \in L$,

(1) If $y \leq z$, $x \odot y \leq x \odot z$, $x \rightarrow y \leq x \rightarrow z$ and $z \rightarrow x \leq y \rightarrow x$.

(2) $x \odot y \leq x \wedge y$.

(3) $x \rightarrow y = \top$ iff $x \leq y$, $x \rightarrow \top = \top$ and $\top \rightarrow x = x$.

(4) $x \odot (\bigvee_{i\in\Gamma} y_i) = \bigvee_{i\in\Gamma}(x \odot y_i)$ and $(\bigvee_{i\in\Gamma} x_i) \odot y = \bigvee_{i\in\Gamma}(x_i \odot y)$.

(5) $x \rightarrow (\bigwedge_{i\in\Gamma} y_i) = \bigwedge_{i\in\Gamma}(x \rightarrow y_i)$ and $(\bigvee_{i\in\Gamma} x_i) \rightarrow y = \bigwedge_{i\in\Gamma}(x_i \rightarrow y)$.

(6) $\bigvee_{i\in\Gamma}(x_i \rightarrow y) \leq (\bigwedge_{i\in\Gamma} x_i) \rightarrow y$ and $\bigvee_{i\in\Gamma}(x \rightarrow y_i) \leq x \rightarrow (\bigvee_{i\in\Gamma} y_i)$.

(7) $y \rightarrow z \leq x \odot y \rightarrow x \odot z$, $y \leq x \rightarrow (x \odot y)$.

(8) $\bigwedge_{i\in\Gamma} x_i \rightarrow \bigwedge_{i\in\Gamma} y_i \geq \bigwedge_{i\in\Gamma}(x_i \rightarrow y_i)$ and $\bigvee_{i\in\Gamma} x_i \rightarrow \bigvee_{i\in\Gamma} y_i \geq \bigwedge_{i\in\Gamma}(x_i \rightarrow y_i)$

(9) $(x \rightarrow y) \odot x \leq y$ and $(x \rightarrow y) \odot (y \rightarrow z) \leq (x \rightarrow z)$.

(10) $x \to y \leq (y \to z) \to (x \to z)$ and $x \to y \leq (z \to x) \to (z \to y)$.

(11) $(x \odot y) \to z = x \to (y \to z) = y \to (x \to z)$ and $x \odot (y \to z) \leq y \to (x \odot z)$.

If the strong negation law is done, then L satisfies moreover

(12) $\bigwedge_{i \in \Gamma} x_i^* = (\bigvee_{i \in \Gamma} x_i)^*$ and $\bigvee_{i \in \Gamma} x_i^* = (\bigwedge_{i \in \Gamma} x_i)^*$.

(13) $x \to y = y^* \to x^*$ and $x \odot y = (x \to y^*)^*$.

An L-subset on a set X is a mapping from X to L, and the family of all L-subsets on X will be denoted by L^X. For $\alpha \in L, \lambda \in L^X$, we denote $(\alpha \to \lambda), (\alpha \odot \lambda), \alpha_X \in L^X$ as $(\alpha \to \lambda)(x) = \alpha \to \lambda(x)$, $(\alpha \odot \lambda)(x) = \alpha \odot \lambda(x)$, $\alpha_X(x) = \alpha$, $\top_x \in L^X$,

$$\top_x(y) = \begin{cases} \top, & \text{if } y = x, \\ \bot, & \text{otherwise,} \end{cases}$$

Definition 2.3. [2] Let X be a set. A mapping $R_X : X \times X \to L$ is called L-fuzzy relation on X. Then R is said to be

(1) reflexive if $R(x,x) = \top$ for all $x \in X$,

(2) symmetric if it satisfies $R(x,y) = R(y,x)$ for all $x, y \in X$,

(3) transitive if $R(x,y) \odot R(y,z) \leq R(x,z)$ for all $x, y, z \in X$.

An L-fuzzy relation on X is called an L-fuzzy preorder if it is reflexive and transitive. And an L-fuzzy equivalence relation if it is reflexive,symmetric and transitive.

There exists an inherent L-order S on L^X defined by

$$S(\lambda, \mu) = \bigwedge_{x \in X} (\lambda(x) \to \mu(x)).$$

The lemma below collects some properties of S used in this paper.

Lemma 2.4. [2, 6, 7] Let $\lambda, \mu, \rho, \nu \in L^X$, and $\alpha \in L$. Then the following properties hold.

(1) $\lambda \leq \mu \Leftrightarrow S(\lambda, \mu) = \top$.

(2) If $\lambda \leq \mu$, then $S(\rho, \lambda) \leq S(\rho, \mu)$ and $S(\lambda, \rho) \geq S(\mu, \rho)$.

(3) $S(\lambda, \mu) \odot S(\nu, \rho) \leq S(\lambda \odot \nu, \mu \odot \rho)$ and $S(\lambda, \alpha \odot \lambda) \geq \alpha$.

(4) $S(\lambda, \mu) \odot S(\mu, \rho) \leq S(\lambda, \rho)$ and $\lambda \odot S(\lambda, \mu) \leq \mu$.

(5) $S(\lambda,\alpha_X\to\mu)=S(\alpha_X\odot\lambda,\mu)=\alpha_X\to S(\lambda,\mu)$ and $S(\mu,\lambda)\to\lambda\geq\mu$.

(6) Let $\varphi:X\to Y$ be an ordinary mapping. Define $\phi^{\to}:L^X\to L^Y$ and $\varphi^{\leftarrow}:L^Y\to L^X$ by $\varphi^{\to}(\lambda)(y)=\bigvee_{\varphi(x)=y}\lambda(x),\ \ \forall\lambda\in L^X,\ y\in Y$ and $\varphi^{\leftarrow}(\mu)(x)=\mu(\varphi(x))=\mu\circ\varphi(x),\ \ \forall\mu\in L^Y$. Then, for $\lambda,\mu\in L^X$ and $\rho,\nu\in L^Y$, we have $S(\lambda,\mu)\leq S(\varphi^{\to}(\lambda),\varphi^{\to}(\mu))$ and $S(\rho,\nu)\leq S(\varphi^{\leftarrow}(\rho),\varphi^{\leftarrow}(\nu))$ and the equalities hold if φ is bijective.

Definition 2.5. (See Adámek et al. [1], Herrlich and Hušek [11]) Suppose that $F:\mathscr{D}\to\mathscr{C}$, $G:\mathscr{C}\to\mathscr{D}$ are concrete functors.

(1) (F,G) is called a Galois correspondence between $\mathscr{C}$ and $\mathscr{D}$ if for each $Y\in\mathscr{C}$, $id_Y:F\circ G(Y)\to Y$ is a $\mathscr{C}$-morphism, and for each $X\in\mathscr{D}$, $id_X:X\to G\circ F(X)$ is a $\mathscr{D}$-morphism.

(2) The categories $\mathscr{C}$ and $\mathscr{D}$ are said to be *isomorphic* if $F\circ G=id_{\mathscr{C}}$ and $G\circ F=id_{\mathscr{D}}$.

Definition 2.6. [13, 14] A map $\mathscr{T}:L^X\to L$ is called an Alexandroff L-fuzzy topology on X if it satisfies the following conditions:

(AT1) $\mathscr{T}(\top_X)=\top$ and $\mathscr{T}(\bot_X)=\top$.

(AT2) $\mathscr{T}(\bigwedge_{i\in\Gamma}\lambda_i)\geq\bigwedge_i\mathscr{T}(\lambda_i)$ and $\mathscr{T}(\bigvee_{i\in\Gamma}\lambda_i)\geq\bigwedge_i\mathscr{T}(\lambda_i)$ for all $\{\lambda_i\}_{i\in\Gamma}\subseteq L^X$.

(AT3) $\mathscr{T}(\alpha_X\odot\lambda)\geq\mathscr{T}(\lambda)$ and $\mathscr{T}(\alpha_X\to\lambda)\geq\mathscr{T}(\lambda)$ for all $\lambda\in L^X$ and $\alpha\in L$.

The pair $(X,\mathscr{T})$ is called an Alexandroff L-fuzzy topological space. A mapping $\varphi:(X,\mathscr{T}_X)\longrightarrow(Y,\mathscr{T}_Y)$ between Alexandroff L-fuzzy topological spaces is called continuous if $\mathscr{T}_X(\varphi^{\leftarrow}(\lambda))\geq\mathscr{T}_Y(\lambda)$ for all $\lambda\in L^Y$. The category of Alexandroff L-fuzzy topological spaces with continuous mappings as morphisms is denoted by **AFTop**.

Definition 2.7. [13, 14] A map $\mathscr{J}:L^X\to L^X$ is called an L-lower approximation operator on X if

(J1) $\mathscr{J}(\top_X)=\top_X$,

(J2) $\mathscr{J}(\lambda)\leq\lambda$ for all $\lambda\in L^X$,

(J3) $\mathscr{J}(\bigwedge_{i\in\Gamma}\lambda_i)=\bigwedge_{i\in\Gamma}\mathscr{J}(\lambda_i)$ for all $\lambda_i\in L^X$, and

(J4) $\mathscr{J}(\alpha\to\lambda)=\alpha\to\mathscr{J}(\lambda)$.

The pair $(X,\mathscr{J})$ is called L-lower approximation space.

Let **LAS** be a category with object $(X, \mathscr{J}_X)$, where $\mathscr{J}_X$ is an L-fuzzy lower approximation operator with a lower approximation mapping $\varphi : (X, \mathscr{J}_X) \to (Y, \mathscr{J}_Y)$ such that $\varphi^{\leftarrow}(\mathscr{J}_Y(\lambda)) \leq \mathscr{J}_X(\varphi^{\leftarrow}(\lambda))$ for each $\lambda \in L^Y$.

Definition 2.8. [13, 14] A map $\mathscr{H} : L^X \to L^X$ is called an L-upper approximation operator on X if

(H1) $\mathscr{H}(\perp_X) = \perp_X$,

(H2) $\mathscr{H}(\lambda) \geq \lambda$ for all $\lambda \in L^X$,

(H3) $\mathscr{H}(\bigvee_{i\in\Gamma} \lambda_i) = \bigvee_{i\in\Gamma} \mathscr{H}(\lambda_i)$ for all $\lambda_i \in L^X$, and

(H4) $\mathscr{H}(\alpha \odot \lambda) = \alpha \odot \mathscr{H}(\lambda)$.

The pair $(X, \mathscr{H})$ is called L-upper approximation space.

Let **UAS** be a category with object $(X, \mathscr{H}_X)$, where $\mathscr{H}_X$ is an L-fuzzy upper approximation operator with an upper approximation mapping $\varphi : (X, \mathscr{H}_X) \to (Y, \mathscr{H}_Y)$ such that $\varphi^{\leftarrow}(\mathscr{H}_Y(\lambda)) \geq \mathscr{H}_X(\varphi^{\leftarrow}(\lambda))$for each $\lambda \in L^Y$,

Let **RFR** be a category with object (X, R_X), where R_X is a reflexive L-fuzzy relation with an order preserving map $\varphi : (X, R_X) \to (Y, R_Y)$ such that $R_X(x,y) \leq R_Y(\varphi(x), \varphi(y))$ for all $x, y \in X$.

Let $\mathscr{H}$ and $\mathscr{J}$ be an L-upper and L-lower approximation on X, respectively. The pair $(\mathscr{J}(\lambda), \mathscr{H}(\lambda))$ is called a *fuzzy rough set for* λ.

Theorem 2.9. [13, 14] (1) Let $(X, \mathscr{J})$ be an L-upper approximation space. Define a map $\mathscr{T}_{\mathscr{H}} : L^X \to L$ by

$$\mathscr{T}_{\mathscr{H}}(\lambda) = S(\mathscr{H}(\lambda), \lambda).$$

Then $\mathscr{T}_{\mathscr{H}}$ is an Alexandroff L-fuzzy topology on X.

(2) Let $(X, \mathscr{J})$ be an L-lower approximation space. Define a map $\mathscr{T}_{\mathscr{J}} : L^X \to L$ by

$$\mathscr{T}_{\mathscr{J}}(\lambda) = S(\lambda, \mathscr{J}(\lambda)).$$

Then $\mathscr{T}_{\mathscr{J}}$ is an Alexandroff L-fuzzy topology on X.

3. Alexandroff L-fuzzy topological spaces and reflexive L-fuzzy relations

We devote this section to the categorical aspect of the relationship between Alexandroff L-fuzzy topological spaces and reflexive L-fuzzy relations.

Theorem 3.1. Let $(X, \mathscr{T}_X)$ be an Alexandroff L-fuzzy topological space. Define a mapping $R_{\mathscr{T}_X} : X \times X \longrightarrow L$ as

$$R_{\mathscr{T}_X}(x,y) = \bigwedge_{\lambda \in L^X} (\mathscr{T}_X(\lambda) \to (\lambda(x) \to \lambda(y))).$$

(1) $R_{\mathscr{T}_X}$ is a reflexive L-fuzzy relation.

(2) Let $\varphi : (X, \mathscr{T}_X) \longrightarrow (Y, \mathscr{T}_Y)$ be continuous mapping between Alexandroff L-fuzzy topological spaces. Then $\varphi : (X, R_{\mathscr{T}_X}) \longrightarrow (Y, R_{\mathscr{T}_Y})$ is an order preserving mapping.

Proof. (1) For any $x \in X$,

$$R_{\mathscr{T}}(x,x) = \bigwedge_{\lambda \in L^X} (\mathscr{T}(\lambda) \to (\lambda(x) \to \lambda(x))) = \top,$$

i.e., $R_{\mathscr{T}}$ is reflexive.

(2) For any $x, z \in X$,

$$\begin{aligned} R_{\mathscr{T}_X}(x,z) &= \bigwedge_{\lambda \in L^X} (\mathscr{T}_X(\lambda) \to (\lambda(x) \to \lambda(z))) \\ &\leq \bigwedge_{\mu \in L^Y} (\mathscr{T}_X(\varphi^{\leftarrow}(\mu)) \to (\varphi^{\leftarrow}(\mu)(x) \to \varphi^{\leftarrow}(\mu)(z))) \\ &\leq \bigwedge_{\mu \in L^Y} (\mathscr{T}_Y(\mu) \to (\mu(\varphi(x)) \to \mu(\varphi(z)))) = R_{\mathscr{T}_Y}(\varphi(x), \varphi(z)). \end{aligned}$$

The above theorem shows that $\Gamma : \mathbf{AFTop} \to \mathbf{RFR}$ is a concrete functor with

$$\Gamma(X, \mathscr{T}_X) = (X, R_{\mathscr{T}_X}),\ \Gamma(\varphi) = \varphi.$$

Theorem 3.2 Let (X, R_X) be a reflexive L-fuzzy relation. Define a mapping $\mathscr{T}_{R_X} : L^X \longrightarrow L$ as

$$\mathscr{T}_{R_X}(\lambda) = \bigwedge_{x,y \in X} (R_X(x,y) \to (\lambda(x) \to \lambda(y))).$$

(1) $\mathscr{T}_{R_X}$ is an Alexandroff L-fuzzy topological space such that $R_{\mathscr{T}_{R_X}} \geq R_X$.

(2) If $\mathscr{T}_X$ is an Alexandroff L-fuzzy topological space, then $\mathscr{T}_{R_{\mathscr{T}_X}} \geq \mathscr{T}_X$.

(3) If $\varphi : (X, R_X) \longrightarrow (Y, R_Y)$ is an order preserving mapping, then $f : (X, \mathscr{T}_{R_X}) \longrightarrow (Y, \mathscr{T}_{R_Y})$ is continuous.

Proof. (1)

$$\begin{aligned}
\mathscr{T}_{R_X}(\bigwedge_{i\in\Gamma} \lambda_i) &= \bigwedge_{x,y\in X} (R_X(x,y) \to (\bigwedge_{i\in\Gamma} \lambda_i(x) \to \bigwedge_{i\in\Gamma} \lambda_i(y))) \\
&\geq \bigwedge_{x,y\in X} (R_X(x,y) \to \bigwedge_{i\in\Gamma}(\lambda_i(x) \to \lambda_i(y))) \\
&= \bigwedge_{i\in\Gamma} \bigwedge_{x,y\in X} (R_X(x,y) \to (\lambda_i(x) \to \lambda_i(y))) \\
&= \bigwedge_{i\in\Gamma} \mathscr{T}_{R_X}.
\end{aligned}$$

$$\begin{aligned}
\mathscr{T}_{R_X}(\bigvee_{i\in\Gamma} \lambda_i) &= \bigwedge_{x,y\in X} (R_X(x,y) \to (\bigvee_{i\in\Gamma} \lambda_i(x) \to \bigvee_{i\in\Gamma} \lambda_i(y))) \\
&\geq \bigwedge_{x,y\in X} (R_X(x,y) \to \bigwedge_{i\in\Gamma}(\lambda_i(x) \to \lambda_i(y))) \\
&= \bigwedge_{i\in\Gamma} \bigwedge_{x,y\in X} (R_X(x,y) \to (\lambda_i(x) \to \lambda_i(y))) \\
&= \bigwedge_{i\in\Gamma} \mathscr{T}_{R_X}(\lambda_i).
\end{aligned}$$

$$\begin{aligned}
\mathscr{T}_{R_X}(\alpha \to \lambda) &= \bigwedge_{x,y\in X} (R_X(x,y) \to ((\alpha \to \lambda(x)) \to (\alpha \to \lambda(y)))) \\
&\geq \bigwedge_{x,y\in X} (R_X(x,y) \to (\lambda(x) \to \lambda(y))) \\
&= \mathscr{T}_{R_X}(\lambda).
\end{aligned}$$

$$\begin{aligned}
\mathscr{T}_{R_X}(\alpha \odot \lambda) &= \bigwedge_{x,y\in X} (R_X(x,y) \to ((\alpha \odot \lambda(x)) \to (\alpha \odot \lambda(y)))) \\
&\geq \bigwedge_{x,y\in X} (R_X(x,y) \to (\lambda(x) \to \lambda(y))) \\
&= \mathscr{T}_{R_X}(\lambda).
\end{aligned}$$

$$\begin{aligned}
R_{\mathscr{T}_{R_X}}(x,y) &= \bigwedge_{\lambda\in L^X}(\mathscr{T}_{R_X}(\lambda)\to(\lambda(x)\to\lambda(y)))\\
&\geq \bigwedge_{\lambda\in L^X}((\bigwedge_{z,w\in X}(R_X(z,w)\to(\lambda(z)\to\lambda(w))\to(\lambda(x)\to\lambda(y)))\\
&\geq \bigwedge_{\lambda\in L^X}((R_X(x,y)\to(\lambda(x)\to\lambda(y)))\to(\lambda(x)\to\lambda(y)))\\
&\geq R_X(x,y).
\end{aligned}$$

(2) For any $\lambda\in L^X$,

$$\begin{aligned}
\mathscr{T}_{R_{\mathscr{T}_X}}(\lambda) &= \bigwedge_{x,y\in X}(R_{\mathscr{T}_X}(x,y)\to(\lambda(x)\to\lambda(y)))\\
&\geq \bigwedge_{x,y\in X}((\mathscr{T}_X(\lambda)\to(\lambda(x)\to\lambda(y))\to(\lambda(x)\to\lambda(y)))\\
&\geq \mathscr{T}_X(\lambda).
\end{aligned}$$

(3) For any $\mu\in L^Y$,

$$\begin{aligned}
T_{R_X}(\varphi^{\leftarrow}(\mu)) &= \bigwedge_{x,y\in X}(R_X(x,y)\to(\varphi^{\leftarrow}(\mu)(x)\to\varphi^{\leftarrow}(\mu)(y)))\\
&\geq \bigwedge_{x,y\in X}(R_Y(\varphi(x),\varphi(y))\to(\varphi^{\leftarrow}(\mu)(x)\to\varphi^{\leftarrow}(\mu)(y)))\\
&\geq \mathscr{T}_{R_X}(\varphi^{\leftarrow}(\mu)).
\end{aligned}$$

The above theorem shows that $\Delta:\mathbf{RER}\to\mathbf{AFTop}$ is a concrete functor with

$$\Delta(X,R_X)=(X,\mathscr{T}_{R_X}),\ \Delta(\varphi)=\varphi.$$

Theorem 3.3. (Δ,Γ) forms a Galois connection between the category **RER** and the category **AFTop**.

Proof. (1) From Theorem 3.2(2), $id_X:(X,\Delta\circ\Gamma(\mathscr{T}))\longrightarrow(X,\mathscr{T})$ is continuous.

(2) From Theorem 3.2(1), $id_X:(X,R_X)\longrightarrow(X,\Gamma\circ\Delta(R_X))$ is an order preserving mapping. Thus (Δ,Γ) forms a Galois connection between the category **RER** and the category **AFTop**.

The following theorem can be obtained in a method similar to reference [14].

Theorem 3.4.[14] Let R be a reflexive L-fuzzy relation. Then

(1) $\mathscr{J}_R : L^X \to L^X$ defined as $\mathscr{J}_R(\lambda) = \bigwedge_{y\in X}(R(x,y) \to \lambda(y))$ is an L-lower approximation operator with $\mathscr{J}_{R_{\mathscr{J}}} = \mathscr{J}$, where $R_{\mathscr{J}}(x,y) = \mathscr{J}^*(\top_y^*)(x)$ and $R_{\mathscr{J}_R} = R$.

(2) $\mathscr{H}_R : L^X \to L^X$ defined as $\mathscr{H}_R(\lambda)(x) = \bigvee_{y\in X}(R(x,y) \odot \lambda(y))$ is an L-upper approximation operator with $\mathscr{H}_{R_{\mathscr{H}}} = \mathscr{H}$, where $R_{\mathscr{H}}(x,y) = \mathscr{H}(\top_y)(x)$ and $R_{\mathscr{H}_R} = R$.

(3) From (1), Θ : **RER** $\to$ **LAS** is a concrete functor with

$$\Theta(X,R_X) = (X, \mathscr{J}_{R_X}),\ \Delta(\varphi) = \varphi.$$

Moreover, Λ : **LAS** $\to$ **RER** is a concrete functor with

$$\Lambda(X, \mathscr{J}_{R_X}) = (X, R_{\mathscr{J}_X}),\ \Lambda(\varphi) = \varphi.$$

Then **RER** and **LAS** are isomorphic.

(4) From (2), Υ : **RER** $\to$ **UAS** is a concrete functor with

$$\Upsilon(X,R_X) = (X, \mathscr{T}_{R_X}),\ \Upsilon(\varphi) = \varphi.$$

Moreover, Φ : **UAS** $\to$ **RER** is a concrete functor with

$$\Phi(X,R_X) = (X, \mathscr{T}_{R_X}),\ \Phi(\varphi) = \varphi.$$

Then **RER** and **UAS** are isomorphic.

Theorem 3.5 Let $(X, \mathscr{T})$ be an Alexandroff L-fuzzy topological space. Define a mapping $\mathscr{J}_{\mathscr{T}_X} : L^X \longrightarrow L^X$ as

$$\mathscr{J}_{\mathscr{T}_X}(\lambda)(y) = \bigwedge_{x\in X}(R_{\mathscr{T}_X}(y,x) \to \lambda(x)).$$

(1) $\mathscr{J}_{\mathscr{T}_X}$ is an L-lower approximation operator such that $\mathscr{T}_{R_{\mathscr{T}_X}} = \mathscr{T}_{\mathscr{J}_{\mathscr{T}_X}}$.

(2) If $\varphi : (X, \mathscr{T}_X) \longrightarrow (Y, \mathscr{T}_Y)$ be a continuous mapping. Then $f : (\mathscr{J}_{\mathscr{T}_X}) \longrightarrow (\mathscr{J}_{\mathscr{T}_X})$ is an lower approximation mapping.

(3) If R is an L-fuzzy reflexive relation, then $R_{\mathscr{J}_{\mathscr{T}_R}} = R_{\mathscr{T}_R}$.

Proof. (1) Since $R_{\mathscr{T}_X}$ is reflexive, by Theorem 3.4(1), $\mathscr{J}_{\mathscr{T}_X}$ is an L-lower approximation operator. For any $\lambda \in L^X$,

$$\begin{aligned}
\mathscr{T}_{\mathscr{J}_{\mathscr{T}_X}}(\lambda) &= \bigwedge_{x\in X}(\lambda(x) \to \mathscr{J}_{\mathscr{T}_X}(\lambda)(x)) \\
&= \bigwedge_{x\in X}(\lambda(x) \to \bigwedge_{y\in X}(R_{\mathscr{T}_X}(y,x) \to \lambda(y))) \\
&= \bigwedge_{x,y\in X}(R_{\mathscr{T}_X}(x,y) \to (\lambda(x) \to \lambda(y))) = \mathscr{T}_{R_{\mathscr{T}_X}}(\lambda).
\end{aligned}$$

(2) For any $\lambda \in L^Y$, by Theorem 3.1(2),

$$\begin{aligned}
\mathscr{J}_{\mathscr{T}_X}(\varphi^{\leftarrow}(\lambda))(x) &= \bigwedge_{z\in X}(R_{\mathscr{T}_X}(x,z) \to \varphi^{\leftarrow}(\lambda)(z)) \\
&\geq \bigwedge_{z\in X}(R_{\mathscr{T}_Y}(\varphi(x),\varphi(z)) \to \lambda(\varphi(z))) \\
&\geq \bigwedge_{y\in Y}(R_{\mathscr{T}_Y}(\varphi(x),y) \to \lambda(y)) = \varphi^{\leftarrow}(\mathscr{J}_{\mathscr{T}_Y}(\lambda))(x).
\end{aligned}$$

(3) For any $x,y \in X$,

$$\begin{aligned}
R_{\mathscr{T}_{\mathscr{J}_R}}(x,y) &= \bigwedge_{\lambda\in L^X}(\mathscr{T}_{\mathscr{J}_R}(\lambda) \to (\lambda(x) \to \lambda(y))) \\
&= \bigwedge_{\lambda\in L^X}(S(\lambda, \mathscr{J}_R(\lambda)) \to (\lambda(x) \to \lambda(y))) \\
&= \bigwedge_{\lambda\in L^X}(\bigwedge_{x\in X}(\lambda(x) \to \mathscr{J}_R(\lambda)(x)) \to (\lambda(x) \to \lambda(y))) \\
&= \bigwedge_{\lambda\in L^X}(\bigwedge_{x,y\in X}(\lambda(x) \to (R(x,y) \to \lambda(y))) \to (\lambda(x) \to \lambda(y))) \\
&= \bigwedge_{\lambda\in L^X}(\bigwedge_{x,y\in X}(R(x,y) \to (\lambda(x) \to \lambda(y))) \to (\lambda(x) \to \lambda(y))) \\
&= R_{\mathscr{T}_R}(x,y).
\end{aligned}$$

Lemma 3.6 (1) If $\mathscr{T}$ is an Alexandrov L-fuzzy topology such that $\mathscr{T}(\lambda) = \mathscr{T}(\lambda^*)$ for each $\lambda \in L^X$, then $R_{\mathscr{T}}$ is symmetric.

(2) If R is symmetric, then $\mathscr{T}_R(\lambda) = \mathscr{T}_R(\lambda^*)$ for each $\lambda \in L^X$.

Proof. (1) If $\mathscr{T}(\lambda)=\mathscr{T}(\lambda)$ for each $\lambda\in L^X$, then

$$\begin{aligned} R_{\mathscr{T}}(x,y) &= \bigwedge_{\lambda\in L^X}(\mathscr{T}(\lambda)\to(\lambda(x)\to\lambda(y))) \\ &= \bigwedge_{\lambda\in L^X}(\mathscr{T}(\lambda)\to(\lambda^*(y)\to\lambda^*(x))) \\ &= \bigwedge_{\lambda\in L^X}(\mathscr{T}(\lambda^*)\to(\lambda^*(y)\to\lambda^*(x))) \\ &= R_{\mathscr{T}}(y,x). \end{aligned}$$

(2)If R is symmetric, then for any $\lambda\in L^X$,

$$\begin{aligned} \mathscr{T}_R(\lambda^*) &= \bigwedge_{x,y\in X}(R(x,y)\to(\lambda^*(x)\to\lambda^*(y))) \\ &= \bigwedge_{x,y\in X}\bigwedge_{y\in X}(R(y,x)\to(\lambda(y)\to\lambda(x))) \\ &= \mathscr{T}_R(\lambda). \end{aligned}$$

Theorem 3.7 Let $(X,\mathscr{T})$ be an Alexandroff L-fuzzy topological space. Define a mapping $\mathscr{H}_{\mathscr{T}_X}:L^X\longrightarrow L^X$ as

$$\mathscr{H}_{\mathscr{T}_X}(\lambda)(y)=\bigvee_{x\in X}(R_{\mathscr{T}_X}(x,y)\odot\lambda(x)).$$

(1) $\mathscr{H}_{\mathscr{T}_X}$ is an L-lower approximation operator such that $\mathscr{T}_{R_{\mathscr{T}_X}}=\mathscr{T}_{\mathscr{H}_{\mathscr{T}_X}}=\mathscr{T}_{\mathscr{J}_{\mathscr{T}_X}}$.

(2) If $\varphi:(X,\mathscr{T}_X)\longrightarrow(Y,\mathscr{T}_Y)$ be a continuous mapping. Then $\varphi:(\mathscr{H}_{\mathscr{T}_X})\longrightarrow(\mathscr{H}_{\mathscr{T}_X})$ is an upper approximation mapping.

(3) If R is a reflexive L-fuzzy relation, then $R_{\mathscr{H}_{\mathscr{T}_R}}=R_{\mathscr{T}_R}$.

Proof. (1) Since $R_{\mathscr{T}_X}$ is reflexive, by Theorem 3.4(2), $\mathscr{H}_{\mathscr{T}_X}$ is an L-upper approximation operator. For any $\lambda\in L^X$,

$$\begin{aligned}
\mathscr{T}_{\mathscr{J}_{\mathscr{T}_X}}(\lambda) &= \bigwedge_{x\in X}(\lambda(x)\to \mathscr{J}_{\mathscr{T}_X}(\lambda)(x))\\
&= \bigwedge_{x\in X}(\lambda(x)\to \bigwedge_{y\in X}(R_{\mathscr{T}_X}(x,y)\to \lambda(y)))\\
&= \bigwedge_{x,y\in X}(R_{\mathscr{T}_X}(x,y)\to(\lambda(x)\to\lambda(y))) = \mathscr{T}_{R_{\mathscr{T}_X}}(\lambda)\\
&= \bigwedge_{x,y\in X}(\bigwedge_{\mu\in L^X}(\mathscr{T}(\mu)\to(\mu(x)\to\mu(y))\to(\lambda(x)\to\lambda(y)))\\
&= \bigwedge_{y\in X}(\bigvee_{x\in X}(R_{\mathscr{T}_X}(x,y)\odot\lambda(x))\to\lambda(y))) = \mathscr{T}_{\mathscr{H}_{\mathscr{T}_X}}(\lambda).
\end{aligned}$$

(2) For any $\lambda\in L^Y$, by Theorem 3.1(2),

$$\begin{aligned}
\mathscr{H}_{R_{\mathscr{T}_X}}(\varphi^{\leftarrow}(\lambda))(x) &= \bigvee_{z\in X}(R_{\mathscr{T}_X}(x,z)\odot\varphi^{\leftarrow}(\lambda)(z))\\
&\leq \bigvee_{z\in X}(R_{\mathscr{T}_Y}(\varphi(x),\varphi(z))\odot\lambda(\varphi(z)))\\
&\leq \bigvee_{y\in Y}(R_{\mathscr{T}_Y}(\varphi(x),y)\odot\lambda(y)) = \varphi^{\leftarrow}(\mathscr{H}_{R_{\mathscr{T}_Y}}(\lambda))(x).
\end{aligned}$$

(3) For any $x,y\in X$,

$$\begin{aligned}
R_{\mathscr{T}_{\mathscr{H}_R}}(x,y) &= \bigwedge_{\lambda\in L^X}(\mathscr{T}_{\mathscr{H}_R}(\lambda)\to(\lambda(x)\to\lambda(y)))\\
&= \bigwedge_{\lambda\in L^X}(S(\mathscr{H}_R(\lambda),\lambda)\to(\lambda(x)\to\lambda(y)))\\
&= \bigwedge_{\lambda\in L^X}(\bigwedge_{s\in X}(\mathscr{H}_R(\lambda)(s)\to\lambda(s))\to(\lambda(x)\to\lambda(y)))\\
&= \bigwedge_{\lambda\in L^X}(\bigwedge_{s,t\in X}(R(t,s)\odot\lambda(t))\to\lambda(s))\to(\lambda(x)\to\lambda(y)))\\
&= \bigwedge_{\lambda\in L^X}(\bigwedge_{s,t\in X}(R(t,s)\to(\lambda(t)\to\lambda(s)))\to(\lambda(x)\to\lambda(y)))\\
&= R_{\mathscr{T}_R}(x,y)
\end{aligned}$$

Theorem 3.8. Let $\varphi : (X, \mathscr{H}_X) \longrightarrow (Y, \mathscr{H}_Y)$ be an upper approximation mapping. Then $\varphi : (X, \mathscr{T}_{\mathscr{H}_X}) \longrightarrow (Y, \mathscr{T}_{\mathscr{H}_Y})$ is continuous.

Proof. For any $\lambda \in L^Y$,

$$\begin{aligned}
\mathscr{T}_{\mathscr{H}_Y}(\lambda) &= S((X, \mathscr{H}_Y(\lambda), \lambda) \leq S(\varphi^{\leftarrow}(\mathscr{H}_Y(\lambda)), \varphi^{\leftarrow}(\lambda)) \\
&\leq S(\mathscr{H}_X(\varphi^{\leftarrow}(\lambda)), \varphi^{\leftarrow}(\lambda)) = \mathscr{T}_{\mathscr{H}_X}(\varphi^{\leftarrow}(\lambda)).
\end{aligned}$$

Example 3.9. Let $X = \{a, b, c\}$ be a set and $(L = [0,1], \leq, \wedge, \odot, 0, 1)$ a complete residuated lattice with $x * y = (x + y - 1) \vee 0$ and $x \to y = (1 - x + y) \wedge 1$. Put $\lambda \in [0,1]^X$ as follows:

$$\lambda(a) = 0.9, \lambda(b) = 0.4, \lambda(c) = 0.6.$$

Define Alexandrov $[0,1]$-fuzzy topology as $\mathscr{T} : [0,1]^X \to [0,1]$ as follows:

$$\mathscr{T}(B) = \begin{cases} 1, & \text{if } B \in \{0_X, 1_Y\}, \\ 0.8, & \text{if } B \in \{\alpha \to \lambda, \alpha \odot \lambda \mid \alpha \in [0,1]\} - \{0_X, 1_Y\}, \\ 0, & \text{otherwise.} \end{cases}$$

We obtain $R_{\mathscr{T}} \in [0,1]^{X \times X}$ as follows:

$$R_{\mathscr{T}}(a,a) = R_{\mathscr{T}}(b,a) = R_{\mathscr{T}}(b,b) = R_{\mathscr{T}}(b,c) = 1,$$

$$R_{\mathscr{T}}(a,b) = 0.7, R_{\mathscr{T}}(a,c) = 0.9,$$

$$R_{\mathscr{T}}(c,a) = R_{\mathscr{T}}(c,b) = R_{\mathscr{T}}(c,c) = 1.$$

We obtain an Alexandrov $[0,1]$-fuzzy topology as $\mathscr{T}^s : [0,1]^X \to [0,1]$ with $\mathscr{T}^s(\lambda) = \mathscr{T}(\lambda^*)$ as follows:

$$\mathscr{T}^s(\mu) = \begin{cases} 1, & \text{if } \mu \in \{0_X, 1_Y\}, \\ 0.8, & \text{if } \mu \in \{\alpha \to \lambda^*, \alpha \odot \lambda^* \mid \alpha \in [0,1]\} - \{0_X, 1_Y\}, \\ 0, & \text{otherwise.} \end{cases}$$

We obtain $R_{\mathscr{T}^s} \in [0,1]^{X \times X}$ such that

$$R_{\mathscr{T}^s}(a,b) = R_{\mathscr{T}}^{-1}(a,b) = R_{\mathscr{T}}(b,a).$$

For $\mu \in [0,1]^X$ with $\mu(a) = 0.4, \mu(b) = 0.6, \mu(c) = 0.3$,

$$\mathscr{T}_{R_{\mathscr{T}}}(\mu) = 0.7 > \mathscr{T}(\mu) = 0.$$

We obtain $[0,1]$-fuzzy lower approximations $\mathscr{J}_{\mathscr{T}}, \mathscr{J}_{\mathscr{T}^s} : [0,1]^X \to [0,1]^X$ as follows:

$$\mathscr{J}_{\mathscr{T}}(\lambda)(x) = \bigwedge_{y \in X} (R_{\mathscr{T}}(x,y) \to \lambda(y))$$

$$\mathscr{J}_{\mathscr{T}^s}(\lambda)(x) = \bigwedge_{y \in X} (R_{\mathscr{T}}(y,x) \to \lambda(y))$$

$$\begin{pmatrix} \mathscr{J}_{\mathscr{T}}(\lambda)(a) \\ \mathscr{J}_{\mathscr{T}}(\lambda)(b) \\ \mathscr{J}_{\mathscr{T}}(\lambda)(c) \end{pmatrix} = \begin{pmatrix} \lambda(a) \wedge (0.7 \to \lambda(b)) \wedge (0.9 \to \lambda(c)) \\ \lambda(a) \wedge \lambda(b) \wedge \lambda(c) \\ \lambda(a) \wedge \lambda(b) \wedge \lambda(c) \end{pmatrix}$$

We obtain $[0,1]$-fuzzy upper approximations $\mathscr{H}_{\mathscr{T}}, \mathscr{H}_{\mathscr{T}^s} : [0,1]^X \to [0,1]^X$ as follows:

$$\mathscr{H}_{\mathscr{T}}(\lambda)(y) = \bigvee_{x \in X} (R_{\mathscr{T}}(x,y) \odot \lambda(x)$$

$$\mathscr{H}_{\mathscr{T}^s}(\lambda)(y) = \bigvee_{x \in X} (R_{\mathscr{T}}(y,x) \odot \lambda(x)$$

$$\begin{pmatrix} \mathscr{H}_{\mathscr{T}}(\lambda)(a) \\ \mathscr{H}_{\mathscr{T}}(\lambda)(b) \\ \mathscr{H}_{\mathscr{T}}(\lambda)(c) \end{pmatrix} = \begin{pmatrix} \lambda(a) \vee \lambda(b) \vee \lambda(c) \\ (0.7 \odot \lambda(a)) \vee \lambda(b) \vee \lambda(c) \\ (0.9 \odot \lambda(a)) \vee \lambda(b) \vee \lambda(c) \end{pmatrix}$$

Conflict of Interests

The authors declare that there is no conflict of interests.

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