Available online at http://scik.org
J. Math. Comput. Sci. 9 (2019), No. 1, 33-45
https://doi.org/10.28919/jmcs/3713
ISSN: 1927-5307

RAINBOW NUMBER OF MATCHINGS IN HALIN GRAPHS

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Abstract. The *rainbow number* $rb(G,H)$ for the graph H in G is defined to be the minimum integer k such that any k-edge-coloring of G contains a rainbow H. As one of the most important structures in graphs, the rainbow number of matchings has drawn much attention and has been extensively studied. In this paper, we determine the rainbow number of some small matchings in Halin graphs.

Keywords: rainbow number; rainbow matching; Halin graph.

2010 AMS Subject Classification: 05C55, 05C70, 05D10.

1. Introduction

An edge-colored graph is called a *rainbow* graph if the colors on its edges are distinct. The anti-Ramsey number $AR(G,H)$ is defined to be the maximum number of colors in an edge coloring of G without any rainbow H. The anti-Ramsey number was introduced by Erdős et al. [2] in 1973 and and always, the anti-Ramsey number plus is called the rainbow number of a

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Received March 29, 2018

graph. The anti-Ramsey numbers of many graphs have been determined, see two comprehensive surveys [3, 11].

The anit-Ramsey number for matchings in complete graphs was determined in [1, 4, 14] independently. During the last ten years, the researchers began to consider the anti-Ramsey problem in more host graphs other than complete graphs, see [12, 6, 7, 13, 5, 9, 8, 10].

In this paper, we consider the rainbow number of matchings in Halin graphs. A Halin graph is a type of planar graph, constructed by connecting the leaves of a tree into a cycle. The tree must have at least four vertices, none of which has exactly two neighbors. It should be drawn in the plane so none of its edges cross (this is called planar embedding), and the cycle connects the leaves in their clockwise ordering in this embedding. Thus, the cycle forms the outer face of the Halin graph, with the tree inside it.

In 1971, Halin introduced the Halin graphs as a class of minimally 3-vertex-connected graphs: for every edge in the graph, the removal of that edge reduces the connectivity of the graph. These graphs gained in significance with the discovery that many algorithmic problems that were computationally infeasible for arbitrary planar graphs could be solved efficiently on them.

Let c be an edge-coloring of the graph G. Denote by $c(G)$ the set colors appearing on the edges of G. For an edge $e \in E(G)$, denote by $c(e)$ the color assigned to the edge e.

2. Main results

Denote by $\mathscr{HL}_n$ the family of Halin graphs of order n. In this section, we give lower and upper bounds on $rb(\mathscr{HL}_n, kK_2)$ for all $k \geq 3$ and $n \geq 2k$. Clearly, if HL_n is a Halin graph of order $n \geq 4$, then $\delta(HL_n) \geq 3$. First we give two definitions.

Definition 2.1. A star is a tree with exactly one internal vertex. Applying the Halin graph construction to a star produces a wheel graph. Definition W_p is a wheel graph with p leaves in its tree.

Definition 2.2. A maximal outerplanar graph is a planar graph that is not a spanning subgraph of another outerplanar graph. Definition M_n is a maximal outerplanar graph of order n.

Lemma 2.3. (**Degree-Sum Formula**) For a graph $G=(V,E)$,

$$\sum_{v\in V(G)} d(v)=2|E(G)|.$$

Lemma 2.4. Let HL_n be a Halin graph, $\lceil \frac{3n}{2}\rceil \leq |E(HL_n)| \leq 2n-2$.

Proof. First we prove the upper bound of the edge of HL_n. HL_n is formed by embedding a tree T having no degree-2 vertices in the plane and connecting its leaves by a cycle C that crosses none of its edges. Since HL_n has n vertices, we get $|E(T)| \leq n-1$. Since there are at most only $n-1$ leaves in T, we get $|E(C)| \leq n-1$. So $|E(HL_n)| \leq 2n-2$.

Next we will prove the lower bound of the edge of HL_n. Since $\delta(HL_n)\geq 3$, we get $\sum\limits_{v\in V(HL_n)} d(v) \geq 3n$ for all $v\in HL_n$. According to the lemma , we can get $|E(HL_n)| \geq \lceil \frac{3n}{2}\rceil$.

Hence $\lceil \frac{3n}{2}\rceil \leq |E(HL_n)| \leq 2n-2$. The proof is complete. □

Lemma 2.5. $rb(\mathscr{HL}_n,2K_2)=\begin{cases} 4, & n=4; \\ 2, & n\geq 5. \end{cases}$.

Proof. Let HL_n be a Halin graph of order n. First we consider the case $n=4$. The edges of HL_4 can be partitioned into E_1,E_2,E_3, where both E_1, E_2 and E_3 are matching of size 2. We color the edges in E_i by the color i for $i=1,2,3$. Clearly, there is not any rainbow matching of size 2. On the other hand, if we color the edges of HL_4 by 4 colors, then at least one of E_1, E_2 and E_3 is rainbow. This proves that $rb(\mathscr{M}_n,2K_2)=4$.

When $n\geq 5$, let HL_n be a Halin graph of order n. We color the edges of HL_n by color 1 and color 2. Let $w\in V(HL_n)$ and the edges connected with w contains two colors. Let the neighbors of w is a set $\{v_1,v_2,...,v_d\}$ and $d\geq 2$. Without loss of generality, we let $c(wv_1)\neq c(wv_2)$. Since $n\geq 5$, there must be two disjoint edges e_1,e_2 that do not belong to $E=\{wv_i|1\leq i\leq d\}$, and e_1,e_2 are connected with v_1,v_2, respectively. Suppose that HL_n does not contain any rainbow $2K_2$, then $c(e_1)=c(wv_2)$ and $c(e_2)=c(wv_1)$. Since $c(wv_1)\neq c(wv_2)$, we get $c(e_1)\neq c(e_2)$. Since e_1,e_2 are disjointed, we get $\{e_1,e_2\}$ is a rainbow $2K_2$, a contradiction.

The proof is complete. □

Now we will show the exact values of $rb(\mathscr{HL}_n, 3K_2)$ for all $n \geq 6$. First we give two lemmas.

Lemma 2.6. Let G be an edge colored graph of order $n \geq 6$ which contains a rainbow 4-cycle, say $v_1v_2v_3v_4v_1$. If there is an edge in $G - \{v_1, v_2, v_3, v_4\}$, then G contains a rainbow $3K_2$.

A graph G is called factor-critical if $G - v$ contains a perfect matching for every vertex $v \in V(G)$.

Lemma 2.7. [15] Given a graph $G = (V, E)$ of order n, let d be the size of a maximum matching of G. Then there exists a subset $S \subset V$ such that $d = \frac{1}{2}(n - (o(G-S) - |S|))$, where $o(G-S)$ is the number of odd components in $G-S$. Moreover, each odd component of $G-S$ is factor-critical.

Theorem 2.8. For all $n \geq 6$, $rb(\mathscr{M}_n, 3K_2) = n+1$.

Proof. We have proved the lower bound in the previous section and here we only consider the upper bound case. Let HL_n be a Halin graph with n vertices. Let c be a $(n+1)$-edge-coloring of HL_n. Clearly, HL_n contains a rainbow $2K_2$. Suppose that HL_n does not contain any rainbow $3K_2$. Now let $G \subset HL_n$ be a rainbow spanning subgraph of size $n+1$ which contains a $2K_2$.

Since the size of the maximum matching of G is 2, by Lemma , there exists a subset $S \subset V(G)$ such that $o(G-S) - |S| = n-4$. Let $|S| = s$, $o(G-S) = q$ and denote the odd components of $G-S$ be $A_1, A_2, ..., A_q$. Let $|V(A_i)| = a_i$ for $1 \leq i \leq q$ and $a_1 \geq a_2 \geq ... \geq a_q \geq 1$. Let $C(G) = V(G-S) \setminus \{\bigcup_{i=1}^{q} V(A_i)\}$.

Since $q = s+n-4$ and $s+q \leq n$, then $0 \leq s \leq 2$. We distinguish the following three cases to finish the proof of the theorem.

Case 1. $s = 0$.

In this case, $q = n-4$. If $a_1 \leq 3$, then $|E(G)| \leq 6 < n+1$, a contradiction. Then $a_1 = 5$ and $a_2 = a_3 = ... = a_q = 1$. When $n \geq 8$, $|E(G)| \leq 2 \times 5 - 2 = 8 < n+1$, a contradiction.

When $n = 7$, suppose that $|E(G)| \geq 8$, we get $G[V(A_1)] \cong W_4$. Then, there are one non-leaf vertex and four leaf vertices in $V(A_1)$. This four leaf vertices will form a cycle. For $n \geq 6$, the remaining vertices in the graph HL_n can only be connected with the non-leaf vertices in A_1. This contradicts that $\delta(HL_n) \geq 3$. Then $|E(G)| < 8 = n+1$, a contradiction. So $n = 6$.

When $n = 6$, suppose that $|E(G)| \geq 8$, we get $G[V(A_1)] \cong W_4$. Contradictions can be seen form the above. Suppose that $|E(G)| = 7$, then $G[V(A_1)] \cong M_5$. Hence $G[V(A_1)]$ contains a rainbow C_4. Since HL_n is a connected plane graph, there must be an edge between $V(A_1) \setminus V(C_4)$ and $V(A_2)$ in graph HL_n. By lemma , we get HL_n contains a rainbow $3K_2$, a contradiction. Then $|E(G)| < 7 = n+1$, a contradiction.

Case 2. $s = 1$.

In this case, $q = s+n-4 = n-3$. If $|C(G)| = 2$, then $a_1 = 1$. Then $|E(G)| \leq 1+n-1 = n < n+1$, a contradiction. So $|C(G)| = 0$. Hence $a_1 = 3$ and $a_2 = a_3 = ... = a_q = 1$.

Since A_1 is factor-critical, $A_1 \cong C_3$. Then, there is only one non-leaf vertex in $V(A_1)$. So $|E_G(V(A_1), S)| \leq 1$. We get $|E(G)| \leq 3+(n-4)+1 = n < n+1$, a contradiction.

Case 3. $s = 2$.

In this case, $q = s+n-4 = n-2$, then $|C(G)| = 0$ and $a_1 = a_2 = ... = a_q = 1$. Let $S = \{w_1, w_2\}$, $V(A_i) = \{v_i\}(i = 1, 2, ..., n-2)$ and $U = \{v_1, v_2, ..., v_{n-2}\}$.

Suppose that $w_1w_2 \notin E(G)$. Since $|E(G)| = n+1$, there are $(n+1)-(n-2) = 3$ vertices in U which have 2 degrees in graph G. Without loss of generality, we let $d_G(v_1) = d_G(v_2) = d_G(v_3) = 2$ and $U_1 = \{v_4, v_5, ..., v_{n-2}\}$, $U_2 = \{v_1, v_2, v_3\}$.

Suppose that w_1, w_2 are non-leaf vertices. Suppose that there is a leaf vertex in U_2, then one leaf vertex connects two non-leaf vertices. This contradicts that HL_n is a Halin graph. So all vertices of U_2 are non-leaf vertices. Then tow vertices of U_2 and w_1, w_2 will form a 4-cycle, that is to say, non-leaf vertices form a 4-cycle. This contradicts that the tree T of HL_n has no cycle. Hence, there is only one non-leaf vertices in S.

Without loss of generality, we assume that w_1 is a non-leaf vertices. There is one vertex of U_2 lie in the inner area of a 4-cycle. Without loss of generality, we let v_3 lie in the inner area of cycle $v_1w_1v_2w_2v_1$. Since w_2 is a leaf vertex, there is only one non-leaf vertex in U_2. Suppose that v_3 is a non-leaf vertex, then v_1, v_2 are leaf vertices and v_1, v_2 are not connected with v_3. We can get $d_{HL_n}(v_3) = 2$, This contradicts that $\delta(HL_n) = 3$. Then v_3 is a leaf vertex. Since there is one non-leaf vertex in U_2, without loss of generality, we let v_1 is a non-leaf vertex. Since one leaf vertex only connects one non-leaf vertex and $w_1v_3 \in E(HL_n)$, we get $v_1v_3 \notin E(HL_n)$. Since

w_2, v_2 are two leaf vertices, $v_2v_3 \notin E(HL_n)$, otherwise leaf vertex w_2, v_2, v_3 will form a C_3. We can get $d_{HL_n}(v_3) = 2$, this contradicts that $\delta(HL_n) = 3$. Then $w_1w_2 \in E(G)$.

Since $|E(G)| = n+1$, we choose two vertices v_1, v_2 from U such that $d_G(v_1) = d_G(v_2) = 2$ and $d_G(v_i) = 1(3 \leq i \leq n-2)$. Let $U_3 = \{v_3, v_4, ..., v_{n-2}\}$ and $U_4 = \{v_1, v_2\}$, we get $|E_{HL_n}(S, U_4)| = 4$ and $|E_{HL_n}(S, U_3)| = n-4$. Let $c(w_1v_1) = 1$, $c(w_1v_2) = 2$, $c(w_2v_1) = 3$, $c(w_2v_2) = 4$, $c(w_1w_2) = 5$. Without loss of generality, we assume that $w_2v_3 \in E(G)$. Let $c(w_2v_3) = 6$, then w_2 is a non-leaf vertex. Since $G[S \cup U_4]$ contains a rainbow 4-cycle, we get $E(HL_n[U_3]) = \emptyset$, otherwise HL_n contains a rainbow $3K_2$.

Suppose that $E(HL_n[U_4]) \neq \emptyset$, then $v_1v_2 \in E(HL_n)$. Suppose that v_1, v_2 are non-leaf vertices, then non-leaf vertices v_1, v_2, w_2 will form a cycle. This contradicts that the tree T of HL_n has no cycle. Suppose that there is only one leaf vertex in $\{v_1, v_2\}$. Without loss of generality, we assume that v_1 is a non-leaf vertex, then v_2 is a leaf vertex. Since leaf vertex v_2 connects two non-leaf vertices v_1, w_2, this contradicts that HL_n is a Halin graph. Suppose that v_1, v_2 are leaf vertices, we get w_1 is a leaf vertex, otherwise leaf vertices v_1, v_2 connects two non-leaf vertices w_1, w_2. Then leaf vertices v_1, v_2, w_1 will form a cycle. Since $n \geq 6$, this contradicts that HL_n is a Halin graph. Then $E(HL_n[U_4]) = \emptyset$.

We get $|E_{HL_n}(U_3, U_4)| \geq \lceil \frac{3n}{2} \rceil - 1 - 4 - (n-4) = \lceil \frac{3n}{2} \rceil - n - 1$. So, when $n \geq 6$, we have $|E_{HL_n}((U_3, U_4)| \geq 1$. Let $|E_{HL_n}(\{v_3\}, U_4)| \geq |E_{HL_n}(\{v_4\}, U_4)| \geq ... \geq |E_{HL_n}(\{v_{n-2}\}, U_4)|$, then $|E_{HL_n}(\{v_3\}, U_4)| \geq 1$. Without loss of generality, we assume that $v_1v_3 \in E(HL_n)$.

Without loss of generality, we assume that $w_2v_4 \in E(G)$ and let $c(w_2v_4) = 7$, then $c(v_1v_3) \in \{2, 7\}$. Now we suppose that there is a vertex u in $\{v_4, v_5, ..., v_{n-2}\}$ such that $v_2u \in E(M_n)$, then $c(v_2u) \in \{1, 6\}$. We get $\{w_1w_2, v_1v_3, v_2u\}$ is a rainbow $3K_2$ in HL_n, a contradiction. So $v_2v_i \notin E(HL_n)(i = 4, ..., n-2)$, then $v_1v_i \in E(HL_n)(i = 3, 4, ..., n-2)$. Hence, v_1 is a non-leaf vertex. Suppose that v_3 is a leaf vertex, then leaf vertex v_3 connects two non-leaf vertices v_1, w_2, this contradicts that HL_n is a Halin graph. Then v_3 is a non-leaf vertex. We get non-leaf vertices $\{v_1, v_3, w_2\}$ form a cycle. This contradicts that the tree T of HL_n has no cycle.

The proof is complete. $\square$

Now, we will show that the exact value of $rb(\mathscr{M}_n, 4K_2)$ for all $n \geq 8$. First we give a lemma.

Lemma 2.9. Let G be an edge-colored graph of order $n \geq 8$ which contains a rainbow 6-cycle, say $v_1v_2v_3v_4v_5v_6v_1$. If there is an edge in $G - \{v_1, v_2, v_3, v_4, v_5, v_6\}$, then G contains a rainbow $4K_2$.

Theorem 2.10. For all $n \geq 8$, $rb(\mathscr{HL}_n, 4K_2) = n+3$.

Proof. We have proved the lower bound in the previous section and here we only consider the upper bound case. Let HL_n be a Halin graph with n vertices. Let c be a $(n+3)$-edge-coloring of HL_n. Clearly, HL_n contains a rainbow $3K_2$. Suppose that HL_n does not contain any rainbow $4K_2$. Now let $G \subset HL_n$ be a rainbow spanning subgraph of size $n+3$ which contains a $3K_2$.

Since the size of the maximum matching of G is 3, by Lemma , there exists a subset $S \subset V(G)$ such that $o(G-S) - |S| = n-6$. Let $|S| = s$, $o(G-S) = q$ and denote the odd components of $G - S$ be $A_1, A_2, ..., A_q$. Let $|V(A_i)| = a_i$ for $1 \leq i \leq q$ and $a_1 \geq a_2 \geq ... \geq a_q \geq 1$. Let $C(G) = V(G-S) \setminus \{\bigcup_{i=1}^{q} V(A_i)\}$.

Since $q = s+n-6$ and $s+q \leq n$, then $0 \leq s \leq 3$. We distinguish the following four cases to finish the proof of the theorem.

Case 1. $s = 0$.

In this case, $q = n-6$. If $a_1 \leq 3$, then $|E(G)| \leq 10 < n+3 (n \geq 8)$, a contradiction. So $a_1 = 5$, $a_2 = 3$ and $a_3 = a_4 = ... = a_q = 1$. When $n \geq 9$, then $|E(G)| \leq 2 \cdot 5 - 2 + 3 = 11 < n+3$, a contradiction. When $n = 8$, suppose that $|E(G)| \geq 11$, then $G[V(A_1)] \cong W_4$ and $G[V(A_2)] \cong C_3$. So there are four leaf vertices in $V(A_1)$ and the four leaf vertices form a cycle. Since $n \geq 8$, this contradicts that HL_n is a Halin graph. Then, $|E(G)| < 11 = n+3$ for all $n \geq 8$, a contradiction. Hence, $a_1 = 7$ and $a_2 = a_3 = ... = a_q = 1$.

When $n \geq 10$, we get $|E(G)| \leq 2 \cdot 7 - 2 = 12 < n+3$, a contradiction. Then $n \leq 9$. When $n = 9$, suppose that $|E(G)| \geq 12$, then $G[V(A_1)] \cong W_6$. So there are six leaf vertices in $V(A_1)$ and the six leaf vertices will form a cycle. since $n = 9$, the remaining vertices in the graph HL_9 can only be connected to the non-leaf vertices in the A_1, which contradicts $\delta(HL_n) = 3$. Then, $|E(G)| < 12 = n+3$, a contradiction. Hence, $n = 8$. Suppose that $|E(G)| \geq 11$, then $G[V(A_1)] \cong M_7$. We get $G[V(A_1)]$ contains a rainbow C_6. Since HL_n is a connected plane graph, there must be an edge between $V(A_1) \setminus V(C_6)$ and $V(A_2)$ in HL_n. By Lemma , HL_n contains a rainbow $4K_2$, a contradiction. Then $|E(G)| < 11 = n+3$, a contradiction.

Case 2. $s = 1$.

In this case, $q = s+n-6 = n-5$. If $|C(G)| = 4$, then $a_1 = a_2 = a_3 = ... = a_q = 1$. Suppose that $G[C(G)] \cong W_3$, there are three leaf vertices in $C(G)$ and this three leaf vertices form a cycle, this contradicts that HL_n is a Halin graph. Then, $G[C(G)] \cong M_4$. There is only one non-leaf vertices in $C(G)$, we get $|E_G(C(G),S)| \leq 1$. Then $|E(G)| \leq 2\cdot 4-3+1+n-5 = n+1 < n+3$, a contradiction. Hence $|C(G)| = 2$, and $a_1 = 3$, $a_2 = a_3 = ... = a_q = 1$. Since A_1 is factor-critical, $A_1 \cong C_3$. There is only one non-leaf vertex in $V(A_1)$, we get $|E_G(V(A_1),S)| \leq 1$. Hence, $|E(G)| \leq 1+3+(n-6)+2+1 = n+1 < n+3$, a contradiction. So $|C(G)| = 0$.

If $a_1 = 5$, then $a_2 = a_3 = ... = a_q = 1$. Suppose that $G[V(A_1)] \cong W_4$, then there are four leaf vertices in $V(A_1)$ and the four leaf vertices form a cycle. Since $n \geq 8$, this contradicts that HL_n is a Halin graph. Then $G[V(A_1)] \cong M_5$. There is only one non-leaf vertex in $V(A_1)$, we get $|E_G(V(A_1),S)| \leq 1$. Hence, $|E(G)| \leq (2\cdot 5-3)+1+(n-6) = n+2 < n+3$, a contradiction.

Case 3. $s = 2$.

In this case, $q = s+n-6 = n-4$. Let $S = \{w_1, w_2\}$. If $|C(G)| = 2$, then $a_1 = a_2 = ... = a_q = 1$.

Suppose that $w_1w_2 \notin E(G)$. Since $|E(G)| = n+3$, there are $n+3-1-(n-4)-2 = 4$ vertices in $V(G)\setminus S$ which are adjacent to both w_1 and w_2. Let this four vertices be v_1, v_2, v_3, v_4 and $U_1 = \{v_1, v_2, v_3, v_4\}$. We get $d_{HL_n}(w_1) \geq 4$, $d_{HL_n}(w_2) \geq 4$, then w_1, w_2 are non-leaf vertices. Suppose there is a vertex in U_1 that is a leaf vertex, then there is a leaf vertex in U_1 which is connected to two non-leaf vertices. This contradicts that HL_n is a Halin graph. So all of the vertices in U_1 are non-leaf vertices. And any two points in U_1 and non-leaf vertex w_1, w_2 form a C_4, this contradicts that the tree T of HL_n has no cycle. Hence $w_1w_2 \in E(G)$.

Since $|E(G)| = n+3$, there are $n+3-1-1-(n-2) = 3$ vertices in $V(G)\setminus S$ which are adjacent to both w_1 and w_2. We get $d_{HL_n}(w_1) \geq 4$, $d_{HL_n}(w_2) \geq 4$. Then we get the contradiction form above. Hence $|C(G)| = 0$. So $a_1 = 3$ and $a_2 = a_3 = ... = a_q = 1$. Since A_1 is factor-critical, $A_1 \cong C_3$. Then, there is only one leaf vertex in $V(A_1)$.

Suppose that there is only one leaf vertex in S and let w_1 be the leaf point in S, then $|E_G(V(A_1),w_1)| \leq 2$ and $|E_G(V(A_1),w_2)| \leq 1$. Hence $|E_G(V(A_1),S)| \leq 3$. Since $|E_G(V(G)\setminus V(A_1),w_1)| \leq 1$, we get $|E(G)| \leq 3+3+1+(n-5) = n+2 < n+3$, a contradiction. Then w_1, w_2 are non-leaf vertices and we get $|E_G(V(A_1),S)| \leq 1$.

Suppose that $w_1w_2 \notin E(G)$, then there are $n+3-3-1-(n-5)=4$ vertices in $V(G)\setminus S$ which are adjacent to both w_1 and w_2. We get $d_{HL_n}(w_1)\geq 4$ and $d_{HL_n}(w_2)\geq 4$, we can get contradictions from above. Hence, $w_1w_2 \in E(G)$. Then there are $(n+3)-3-1-1-(n-5)=3$ vertices in $V(G)\setminus S$ which are adjacent to both w_1 and w_2. We get $d_{HL_n}(w_1)\geq 4$ and $d_{HL_n}(w_2)\geq 4$, we can get contradictions from above.

Case 4. $s=3$.

In this case, $q=s+n-6=n-3$, then $|C(G)|=\emptyset$ and $a_i=1$ for all $1\leq i\leq n-3$. Let $S=\{w_1,w_2,w_3\}$, $V(A_i)=\{v_i\}$ for all $1\leq i\leq n-3$ and $U=\{v_1,v_2,...,v_{n-3}\}$. Suppose that $G[S]\cong C_3$, then there is only one non-leaf vertex in S. So $|E_G(U,S)|\leq 2+2+(n-5)=n-1$. Then $|E(G)|\leq (n-1)+3=n+2<n+3$, a contradiction. Hence $G[S]\cong P_3$ and let $P_3=w_1w_2w_3$.

Suppose that there are 3 vertices of U in graph G have the degree of 2. We choose two vertices v_1,v_2 form U and such that $d_G(v_i)=3(i=1,2)$ and $d_G(v_i)=1(i=3,...,n-3)$, then w_2 is a non-leaf vertex. Since $d_G(v_i)=1(i=3,...,n-3)$, we get that there is one non-leaf vertex in $\{w_1,w_3\}$. Without loss of generality, we assume that w_1 is a non-leaf vertex. Since $G[S\cup\{v_1,v_2\}]\cong W_4$, we get v_1,v_2 are two non-leaf vertices. Then, v_1,v_2 and non-leaf vertex w_1,w_2 form a cycle. This contradicts that the tree T of HL_n has no cycle.

Suppose that there are 3 vertices of U in graph G have the degree of 1. Since HL_n is a Halin graph and $|E(G)|=n+3$, then the degree of 2 vertices in the U is 2 in graph G. Without loss of generality, we assume that w_1v_1,w_1v_2,w_2v_1,w_2v_2,w_2v_3,w_3v_3,$w_3v_2\in E(G)$, then w_2 is a non-leaf vertex. Let $U_1=\{v_1,v_2,v_3\}$ and $U_2=\{v_4,v_5,...,v_{n-3}\}$.

Suppose that there is not only one non-leaf vertex w_2 in S. Without loss of generality, we assume that w_1 is a non-leaf vertex in S. Suppose that v_1 is a leaf vertex, then one leaf vertex v_1 connects two non-leaf vertices w_1,w_2. This contradicts that HL_n is a Halin graph. Then v_1 is a non-leaf vertex. We get v_1 and two non-leaf vertices w_1,w_2 will form a cycle. This contradicts that the tree T of HL_n has no cycle. So there is only one non-leaf vertex w_2 in S. Then $|E_{HL_n}(S,U_2)|\leq n-6$.

Suppose that there is a non-leaf vertex in U_1, then leaf vertex w_1 or w_3 connects two non-leaf vertices w_1, w_2. This contradicts that HL_n is a Halin graph. Then all of vertices in U_1 are leaf vertices. Hence $|E_{HL_n}(S, U_1)| \leq 7$.

Suppose that $E(HL_n[U_2]) \neq \emptyset$. Since $G[S \cup U_1]$ contains a rainbow 6-cycle, by Lemma , we get HL_n contains a rainbow $4K_2$, a contradiction. Then $E(HL_n[U_2]) = \emptyset$.

Suppose that $|E(HL_n[U_1])| \geq 1$. Suppose that $v_1v_2 \in E(HL_n)$ or $v_3v_2 \in E(HL_n)$. Since all of vertices in U_1 are leaf vertices, we get leaf vertices v_1, v_2, w_1 or leaf vertices v_2, v_3, w_3 form a C_3. Since $n \geq 8$, this contradicts that HL_n is a Halin graph. So $v_1v_2 \notin E(HL_n)$ and $v_3v_2 \notin E(HL_n)$, that is to say, $v_1v_3 \in E(HL_n)$. Hence $G[S \cup U_1] \cong W_5$, then all of vertices in U_1 and leaf vertices w_1, w_3 form a cycle. Since $n \geq 8$, the remaining vertices in the graph HL_n can only be connected to the non-leaf vertex w_2. This contradicts that $\delta(HL_n) = 3$. So $|E(HL_n[U_1])| = 0$.

Since $|E_{HL_n}(U_1, U_2)| \geq \lceil \frac{3n}{2} \rceil - 7 - 2 - (n-6) = \lceil \frac{3n}{2} \rceil - n - 3$, when $n \geq 8$, we have $|E_{HL_n}(U_1, U_2)| \geq 1$. Let $|E_{HL_n}(\{v_4\}, U_1)| \geq |E_{HL_n}(\{v_5\}, U_1)| \geq \ldots \geq |E_{HL_n}(\{v_{n-3}\}, U_1)|$, then $|E_{HL_n}(\{v_4\}, U_1)| \geq 1$. Without loss of generality, we let $v_1v_4 \in E(HL_n)$. Since $\delta(HL_n) = 3$ and $|E(HL_n[U_2])| = 0$, $v_4v_3 \in E(HL_n)$, otherwise $d_{HL_n}(v_4) = 2$. So, $G[S \cup U_1 \cup v_4] \cong W_6$, then all of vertices in U_1 and w_1, w_3, v_4 form a cycle. Since $n \geq 8$, the remaining vertices in the graph HL_n can only be connected to the non-leaf vertex w_2. This contradicts that $\delta(HL_n) = 3$. So there are no vertex of U in graph G has degree of 3.

Since HL_n is a Halin graph and $|E(G)| = n+3$, there are four vertex of U which have 2-degree in graph G. Without loss of generality, we assume that w_1v_1, w_1v_2, w_2v_1, w_2v_2, w_2v_3, w_2v_4, w_3v_3, $w_3v_4 \in E(G)$. Then w_2 is a non-leaf vertex. Let $U_1' = \{v_1, v_2, v_3, v_4\}$ and such that $d_G(v_i) = 2 (i = 1,2,3,4)$. Let $U_2' = \{v_5, v_6, \ldots, v_{n-3}\}$ and $d_G(v_i) = 1 (i = 5, \ldots, n-3)$.

Suppose that there is not only one non-leaf vertex w_2 in S. Without loss of generality, we assume that w_1 is a non-leaf vertex in S. Suppose that v_1 is a leaf vertex, then v_1 connects non-leaf vertex w_1, w_2. This contradicts that HL_n is a Halin graph, then v_1 is a non-leaf vertex. We get v_1 and two non-leaf vertices w_1, w_2 will form a cycle. This contradicts that the tree T of HL_n has no cycle. Then there is only one non-leaf vertex w_2 in S.

Suppose that there is a non-leaf vertex in U_1', then leaf vertex w_1 or w_3 connects two non-leaf vertices. This contradicts that HL_n is a Halin graph. Then all of vertices in U_1' are leaf vertices. Hence $|E_{HL_n}(S,U_1')| \leq 8$ and $|E_{HL_n}(S,U_2')| \leq n-7$.

Now suppose that $E(HL_n[U_2']) \neq \emptyset$, then we choose $e \in E(HL_n[U_2'])$. Since $\{w_1v_2, w_3v_3, w_2v_4\}$ is a rainbow $3K_2$ in HL_n, we can get $c(e) \in \{c(w_1v_1), c(w_3v_3), c(w_2v_4)\}$, otherwise HL_n contains a rainbow $4K_2$. So $\{e, w_1v_2, w_2v_3, w_3v_4\}$ is a rainbow $4K_2$ in HL_n, a contradiction. Hence $E(HL_n[U_2']) = \emptyset$.

Suppose that $|E(HL_n[U_1'])| \geq 2$. Suppose that $v_1v_2 \in E(HL_n)$, then leaf vertices v_1, v_2, w_1 will form a C_3. since $n \geq 8$, this contradicts that HL_n is a Halin graph, we get $v_1v_2 \notin E(HL_n)$. The same reason can be obtained $v_3v_4 \notin E(HL_n)$. Since $|E(HL_n[U_1'])| \geq 2$, we let $v_1v_3 \in E(HL_n)$ and $v_2v_4 \in E(HL_n)$. Then all of vertices of U_1' and w_1, w_3 form a cycle. Since $n \geq 8$, the remaining vertices in the graph HL_n can only be connected to the non-leaf vertex w_2. This contradicts that $\delta(HL_n) = 3$. So $|E(HL_n[U_1'])| \leq 1$.

Since $|E_{HL_n}(U_1', U_2')| \geq \lceil \frac{3n}{2} \rceil - 8 - 2 - (n-7) - 1 = \lceil \frac{3n}{2} \rceil - n - 4$. Then, when $n \geq 9$, we get $|E_{HL_n}(U_1', U_2')| \geq 1$. Let $|E_{HL_n}(\{v_5\}, U_1')| \geq |E_{HL_n}(\{v_6\}, U_1')| \geq ... \geq |E_{HL_n}(\{v_{n-2}\}, U_1')|$, we get $|E_{HL_n}(\{v_5\}, U_1')| \geq 1$. Without loss of generality, we let $v_1v_5 \in E(HL_n)$ and we have $w_2v_5 \in E(G)$. Since $\{w_1v_2, w_3v_3, w_2v_4\}$ and $\{w_1v_2, w_2v_3, w_3v_4\}$ are two rainbow $3K_2$ in HL_n, we get $c(v_1v_5) = c(w_1v_2)$, otherwise HL_n contains a rainbow $4K_2$.

Suppose that $|E(HL_n[U_1'])| = 1$. Without loss of generality, we let $v_2v_4 \in E(HL_n)$. Since $\{w_1w_2, v_1v_5, w_3v_3\}$ and $\{w_1v_1, w_2v_5, w_3v_3\}$are two rainbow $3K_2$ in HL_n, we get $c(v_2v_4) = c(w_3v_3)$, otherwise HL_n contains a rainbow $4K_2$.

Suppose that there is a vertex x in U_2' such that $v_3x \in E(HL_n)$. Since $\{v_1v_5, w_1w_2, w_3v_4\}$ is a rainbow $3K_2$ in HL_n, we can get that $c(v_3x) \in \{c(v_1v_5), c(w_1w_2), c(w_3v_4)\}$, otherwise HL_n contains a rainbow $4K_2$. Then $\{v_3x, v_1w_1, v_2v_4, w_2w_3\}$ is a rainbow $4K_2$ in HL_n, a contradiction. Hence $v_3v_i \notin E(HL_n)(i = 6, ..., n-2)$, then $v_1v_i \in E(HL_n)(i = 5, 6, ..., n-2)$. So v_1 is a non-leaf vertex. Suppose that v_5 is a leaf vertex, then one leaf vertex connects two non-leaf vertices v_1, w_2. This contradicts thatHL_n is a Halin graph. Then v_5 is a non-leaf vertex. The non-leaf vertex v_1, v_5, w_2 will form a cycle, this contradicts that the tree T of HL_n has no cycle. Hence $|E(HL_n[U_1'])| = 0$. So we get $v_2v_4 \notin E(HL_n)$.

Since v_2 is a leaf vertex, then $d_{HL_n}(v_2) = 3$. Since $|E(HL_n[U_1'])| = 0$, there exists a vertex y in U_2' such that $v_2y \in E(HL_n)$. Since $\{v_1v_5, w_1w_2, w_3v_4\}$ is a rainbow $3K_2$ in HL_n, we get $c(v_2y) \in \{c(v_1v_5), c(w_1w_2), c(w_3v_4)\}$, otherwise HL_n contains a rainbow $4K_2$. Hence, $\{v_2y, v_1w_1, w_2v_4, v_3w_3\}$ is a rainbow $4K_2$ in HL_n, a contradiction. Then $n = 8$.

When $n = 8$, then $|E(G)| = 11$ and $w_2v_5 \in E(G)$. since all of vertices in U_1' are leaf vertices and are connected to the non-leaf vertex w_2, we get v_5 is leaf vertex, otherwise one leaf vertex connects two non-leaf vertices w_2, v_5. Then $HL_8 \cong W_7$, so $|E(HL_8)| = 14$. Without loss of generality, we let $v_1v_5, v_3v_5, v_2v_4 \in E(HL_8)$.

Since $\{w_1v_2, w_3v_3, w_2v_4\}$ and $\{w_1v_2, w_2v_3, w_3v_4\}$ are two rainbow $3K_2$ in HL_n, we get $c(v_1v_5) = c(w_1v_2)$, otherwise HL_n contains a rainbow $4K_2$. Since $\{w_1w_2, v_1v_5, w_3v_3\}$ and $\{w_1v_1, w_2v_5, w_3v_3\}$ are two rainbow $3K_2$ in HL_n, we get $c(v_2v_4) = c(w_3v_3)$, otherwise HL_n contains a rainbow $4K_2$. Since $\{w_1v_1, v_2v_4, w_2w_3\}$ is a rainbow $3K_2$ in HL_n, we get $c(v_3v_5) \in \{c(w_1v_1), c(v_2v_4), c(w_2w_3)\}$. Hence, $\{v_3v_5, v_1w_2, w_1v_2, v_4w_3\}$ is a rainbow $4K_2$ in HL_n, a contradiction.

The proof is complete. $\square$

Conflict of Interests

The authors declare that there is no conflict of interests.

REFERENCES

[1] H. Chen, X.L. Li, J.H. Tu, Complete solution for the rainbow numbers of matchings, Discrete Math. 309(2009), 3370-3380.

[2] P. Erdős, M. Simonovits, V.T. Sós, Anti-Ramsey theorems, Colloq. Math. Soc. Janos Bolyai. Vol.10, Infinite and Finite Sets, Keszthely (Hungary), 1973, pp. 657-665.

[3] S. Fujita, C. Magnant, K. Ozeki, Rainbow Generalizations of Ramsey Theory: A Survey, Graphs Combin. 26(2010), 1-30.

[4] R. Haas, M. Young, The anti-Ramsey number of perfect matching, Discrete Math. 312(5)(2012), 933-937.

[5] S. Jendrol, I. Schiermeyer, J.H. Tu, Rainbow numbers for matchings in plane triangulations, Discrete Math. 331(28)(2014), 158-164.

[6] Z.M. Jin, Anti-Ramsey numbers for matchings in 3-regular bipartite graphs, Appl. Math. Comput. 292(2017), 114-119.

[7] Z.M. Jin, Oothan Nweit, K.J. Wang, Y.L. Wang, Anti-Ramsey numbers for matchings in regular bipartite graphs, Discrete Math. Algorithms Appl. 9(2)(2017), Article ID 1750019.

[8] Z.M. Jin, Y.F. Sun, S.H.F. Yan, Y.P. Zang, Extremal coloring for the anti-Ramsey problem of matchings in complete graphs, J. Comb. Optim. 34(2017), 1012-1028.

[9] Z.M. Jin, K.C. Ye, Y.F. Sun, H. Chen, Rainbow matchings in edge-colored complete split graphs. Eur. J. Comb. 70(2018), 297-316.

[10] Z.M. Jin, Y.P. Zang, Anti-Ramsey coloring for matchings in complete bipartite graphs, J. Comb. Optim. 33(2017), 1-12.

[11] M. Kano, X.L. Li, Monochromatic and heterochromatic subgraphs in edge-colored graphs - a survey, Graphs Combin. 24(2008), 237-263.

[12] X.L. Li, J.H. Tu, Z.M. Jin, Bipartite rainbow numbers of matchings, Discrete Math. 309(2009), 2575-2578.

[13] X.L. Li, Z.X. Xu, The rainbow number of matchings in regular bipartite graphs, Appl. Math. Lett. 22(2009), 1525-1528.

[14] I. Schiermeyer, Rainbow numbers for matchings and complete graphs, Discrete Math. 286(2004), 157-162.

[15] L. Lovász, M.D. Plummer, Matching Theory, North-Holland, Amsterdam, New York, Oxford, Tokyo, 1986.