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WARPED PRODUCT PSEUDO-SLANT SUBMANIFOLDS OF QUASI-SASAKIAN MANIFOLDS

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Abstract. The object of the present paper is to study warped product and doubly warped product pseudo-slant submanifolds of quasi-Sasakian manifolds. We derive an example of proper pseudo-slant submanifold of almost contact metric manifold. Also, we study the non-existence of warped product and doubly warped product pseudo-slant submanifolds of quasi-Sasakian manifolds.

Keywords: warped product; pseudo-slant submanifold; quasi-sasakian manifold.

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1. Introduction

The idea of warped product manifolds was given by Bishop and Neill [1] and these manifolds were studied by many authors. The warped product manifolds are generalization of Riemannian manifolds. The warped product manifolds play important role in differential geometry as well as in theory of relativity.

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Carriazo [2] introduced the concept of pseudo-slant submanifolds in almost Hermitian manifolds and the contact version of pseudo-slant submanifolds has been defined and studied by Khan and Khan in [3]. Atceken and Hui [4] studied slant and pseudo slant submanifold of $(LCS)_n$ −manifolds where as pseudo slant submanifold of trans-Sasakian manifolds were studied by Khan and Chahal [5].

In 1967, Blair [6] introduced the notion of quasi-Sasakian manifold to unify Sasakian and cosymplectic manifolds. Again in 1977, Kanemaki [7] defined quasi-Sasakian manifolds. Motivated by the studies, the object of the present paper is to study warped product pseudo-slant submanifolds of quasi-Sasakian manifolds. This paper is organized as follows. Section 2 is concerned with preliminaries and we derive an example of proper pseudo-slant submanifold of contact metric manifold. In section 3, we study warped product pseudo-slant submanifolds of quasi-Sasakian manifolds. Section 4 we study doubly warped product pseudo-slant submanifolds of quasi-Sasakian manifolds.

2. Preliminaries

A $(2n+1)$−dimensional Riemannian manifold $(\overline{M},g)$ is called an almost contact metric manifold if the following results hold [6] :

(1) $$\phi^2 = -I + \eta \otimes \xi,$$

(2) $$\eta(\xi) = 1, \qquad \phi(\xi) = 0, \qquad \eta o \phi = 0,$$

(3) $$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y),$$

(4) $$g(X, \phi Y) = -g(\phi X, Y), \quad g(X,\xi) = \eta(X),$$

(5) $$g(\phi X, X) = 0,$$

(6) $$\left(\overline{\nabla}_X \eta\right) Y = g\left(\overline{\nabla}_X \xi, Y\right)$$

for all $X, Y \in T\overline{M}$. An almost contact metric manifold is called contact metric manifold if

$$d\eta(X,Y) = \phi(X,Y) = g(X,\phi Y), \tag{7}$$

ϕ is called the fundamental two form of the manifold. If the characteristic vector field ξ is a killing vector, then the contact manifold is called a K-contact manifold. A contact metric manifold is $K-$contact, if and only if, $\overline{\nabla}_X \xi = -\phi X$, for any X on $\overline{M}$. On the other hand a normal contact metric manifold is known as Sasakian manifold. An almost contact metric manifold is Sasakian, if and only if,

$$\left(\overline{\nabla}_X \phi\right) Y = g(X,Y)\xi - \eta(Y)X, \tag{8}$$

for any vector field X, Y. An almost contact metric structure (ϕ, ξ, η, g) is called quasi-Sasakian if it is normal and its fundamental 2-form ϕ is closed, that is, for every $X, Y \in T(\overline{M})$,

$$[\phi, \phi](X,Y) + d\eta(X,Y)\xi = 0,$$

$$d\phi = 0, \phi(X,Y) = g(X,\phi Y).$$

This was first introduced by Blair [6]. There are many types of quasi-Sasakian structure ranging from the cosymplectic case, $d\eta = 0$ (rank $\eta = 1$), to the Sasakian case, $\eta \wedge (d\eta)^n \neq 0$ (rank $\eta = 2n+1, \phi = d\eta$). The $1-$form η has rank $r' = 2p$ if $(d\eta)^p \neq 0$ and $\eta \wedge (d\eta)^p = 0$ and has rank $r = 2p+1$ if $(d\eta)^p$= 0 and $\eta \wedge (d\eta)^p = 0$. We also say that r' is the rank of the quasi-Sasakian structure. Blair [6] also proved that there are no quasi-Sasakian manifold of even rank. In order to study the properties of quasi-Sasakian manifolds Blair [6] proved some theorems regarding Kaehlerian manifolds and the existence of quasi-Sasakian manifolds. The fundamental vector field ξ of a quasi-Sasakian structure is a killing vector field that is $\pounds_\xi g = 0$. For quasi-Sasakian manifolds Blair [6] proved that

$$\overline{\nabla}_X \xi = -\frac{1}{2}\phi X, \tag{9}$$

for any vector field X.

Let M be a submanifold immersed in $\overline{M}$ with induced metric g. Also let ∇ and $\nabla^{\perp}$ are induced Levi-Civita connections on the tangent bundle TM and $T^{\perp}M$ of M respectively. Then the Gauss and Weingarten formulae are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y) \tag{10}$$

and

$$\overline{\nabla}_X V = -A_V X + \nabla^{\perp}_X V \tag{11}$$

for all $X,Y \in TM$ and $V \in T^{\perp}M$, h and A_V are second fundamental form and Weingarten map associated with V as

$$g(A_V X,Y) = g(h(X,Y),V). \tag{12}$$

For any $X \in T_x M$, we write

$$\phi X = TX + NX, \tag{13}$$

where $TX \in T_x M$ and $NX \in T_x^{\perp}M$. Similarly, for $V \in T_x^{\perp}M$, we have

$$\phi V = tV + nV, \tag{14}$$

where tV is the tangential component and nV is the normal component of ϕV. From (4) and (13), we have

$$g(TX,Y) = -g(X,TY), \tag{15}$$

where $x \in M$, and $X,Y \in T_x M$.

Definition 2.1. A submanifold M of an almost contact metric manifold $\overline{M}$ is said to be a slant submanifold if for each $x \in M$ and $X \in T_x M$, linearly independent to ξ, the angle between

ϕX and T_xM is constant. The constant angle θ, $0 \leq \theta \leq \frac{\pi}{2}$ is called slant angle of M in $\overline{M}$. If $\theta = 0$ the submanifold is invariant submanifold if $\theta = \frac{\pi}{2}$ then submanifold is anti-invariant submanifold if θ lies strictly between 0 and $\frac{\pi}{2}$, i.e $0 < \theta < \frac{\pi}{2}$ then it is called proper slant submanifold.

Definition 2.2. A submanifold M is called pseudo-slant submanifold of an almost contact metric manifold $\overline{M}$, if there exist two orthogonal distribution $D^{\perp}$ and D_{θ} on M such that

(i) $TM = D^{\perp} \oplus D_{\theta} \oplus < \xi >$,

(ii) the distribution $D^{\perp}$ is anti-invariant that is $\phi D^{\perp} \in T^{\perp}M$,

(iii) the distribution D_{θ} is slant with slant angle $\theta \neq \frac{\pi}{2}$.

From the above definition, it is obvious that if $\theta = 0$ and $\theta = \frac{\pi}{2}$, then the pseudo-slant submanifold becomes semi-invariant submanifold and anti-invariant submanifold respectively. On the other hand if we denote the dimensions of D_{θ} and $D^{\perp}$ by d_1 and d_2 respectively then we have the following cases.

(i) if $d_1 = 0$, then M is an anti-invariant submanifold,

(ii) if d_2 and $\theta = 0$, then M is an invariant submanifold,

(iii) if $d_2 = 0$ and $\theta \neq 0$, then M is a proper slant submanifold.

A pseudo slant submanifold is called proper if $d_1, d_2 \neq 0, \theta \neq 0$ and $\theta \neq \frac{\pi}{2}$. Now, we derive an example of proper pseudo-slant submanifold of almost contact metric manifold.

Example. Let $(R^9, \phi, \xi, \eta, g)$ be an almost contact metric manifold with cartesian coordinates $(x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4, t)$ and the almost contact structure

$$\phi\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial x_i}, \phi\left(\frac{\partial}{\partial y_j}\right) = -\frac{\partial}{\partial y_j}, \phi\left(\frac{\partial}{\partial t}\right) = 0,\ i,\ j = 1,2,3,4$$

where $\xi = \frac{\partial}{\partial t}$, $\eta = dt$ and g is the standard Euclidean metric on R^9. Consider a submanifold M of R^9 defined by

$$\chi(x,y,u,v,t) = (x+y, x-y, x, y, x\cos u, x\sin u, x\cos v, x\sin v, t),$$

such that u, v $(u \neq v)$ are non vanishing real valued functions on M. Then the tangent space TM is spanned by the following vector fields

$$e_1 = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial y_1} + \frac{\partial}{\partial x_2} + \cos u \frac{\partial}{\partial x_3} + \sin u \frac{\partial}{\partial y_3} + \cos v \frac{\partial}{\partial x_4} + \sin v \frac{\partial}{\partial y_4},$$

$$e_2 = \frac{\partial}{\partial x_1} - \frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_2},$$

$$e_3 = -x \sin u \frac{\partial}{\partial x_3} + x \cos u \frac{\partial}{\partial y_3},$$
$$e_4 = -x \sin v \frac{\partial}{\partial x_4} + x \cos v \frac{\partial}{\partial y_4},$$

$$e_5 = \frac{\partial}{\partial t}.$$

Now we have

$$\phi e_1 = \frac{\partial}{\partial x_1} - \frac{\partial}{\partial y_1} + \frac{\partial}{\partial x_2} + \cos u \frac{\partial}{\partial x_3} - \sin u \frac{\partial}{\partial y_3} + \cos v \frac{\partial}{\partial x_4} - \sin v \frac{\partial}{\partial y_4},$$

$$\phi e_2 = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial y_1} - \frac{\partial}{\partial y_2},$$

$$\phi e_3 = -x \sin u \frac{\partial}{\partial x_3} - x \cos u \frac{\partial}{\partial y_3},$$
$$\phi e_4 = -x \sin v \frac{\partial}{\partial x_4} - x \cos v \frac{\partial}{\partial y_4},$$

$$\phi e_5 = \phi \left(\frac{\partial}{\partial t} \right) = 0.$$

Then $D^{\perp} = Span\{e_3, e_4\}$ is an anti-nvariant distribution and $D^{\theta} = Span\{e_1, e_2\}$ is a slant distribution with slant angle $\theta = \cos^{-1}\left(\frac{2}{\sqrt{15}}\right)$. Thus M is $5-$dimensional proper pseudo-slant submanifold of R^9.

Bishop and Neill [1] defined the warped product submanifold as follows:

Definition 2.3. Let (N_1, g_1) and (N_2, g_2) be two Riemannian manifolds with Riemannian meteic g_1 and g_2 respectively and f be a positive definite smooth function on N_1. The warped product of N_1 and N_2 is the Riemannian manifold $N_1 \times_f N_2 = (N_1 \times N_2, g)$, where

$$g = g_1 + f^2 g_2. \tag{16}$$

If the vector field U is tangent to $N_1 \times_f N_2$ at (p,q) then

$$g(X,Y) = g_1(\pi_1 * X, \pi_1 * Y) + f^2(p)\, g_2(\pi_2 * X, \pi_2 * Y),$$

where $\pi_i\,(i = 1,2)$ are the cononical projections of $N_1 \times N_2$ onto N_1 and N_2 respectively and $*$ stands for the derivative map.

For warped product manifold we have [8]

Lemma 2.1. *Let $M = N_1 \times_f N_2$ be a warped product manifold. If $X,Y \in TN_1$ and $U,V \in TN_2$ then*

(i) $\nabla_X Y \in TN_1$,

(ii) $\nabla_U X = \nabla_X U = (XInf)\,U$,

(iii) $\nabla_U V = \nabla'_U V - g(U,V)\,\nabla Inf$,

where ∇ and ∇' denote the Levi-Civita connection on N_1 and N_2 respectively.

Doubly warped product manifolds were introduced as a generalization of warped product manifold by Unal [9].

Definition 2.4. A doubly warped product (M,g) is a product manifold of the form $M =_f N_1 \times_b N_2$ with metric $g = f^2 g_1 \oplus b^2 g_2$, where $b : N_1 \to (O,\infty)$ and $f : N_2 \to (O,\infty)$ are smooth maps and g_1, g_2 are the metric on Riemannian manifolds B and F respectively. If either $b = 1$ or $f = 1$, then we obtain a single warped product. If both $b = 1$ and $f = 1$, then we have a warped product. If neither b nor f is constant, then we have a non trivial doubly warped product. If $X \in T(N_1)$ and $Z \in T(N_2)$, then the Levi-Civita connection is

$$\nabla_X Z = Z(Inf)X + X(Inb)Z. \tag{17}$$

3. warped product pseudo-slant submanifolds of quasi-Sasakain manifolds

Let $M = N_1 \times_f N_2$ be a warped product pseudo-slant submanifold of quasi-Sasakain manifold. Such submanifolds are alwayas tangent to the structure vector field ξ. If N_θ and $N_\perp$ are proper slant submanifolds and anti-invariant submanifolds of a quasi-Sasakain manifold $\overline{M}$ then their warped product pseudo-slant submanifolds may be given by one of the following:

(i) $N_\perp \times_f N_\theta$ (ii) $N_\theta \times_f N_\perp$

Now we prove the following results;

Theorem 3.1. *A warped product pseudo-slant submanifold of a quasi-Sasakain manifold $\overline{M}$ of type $M = N_\perp \times_f N_\theta$, where $N_\perp$ is an anti-invariant submanifold and N_θ is a proper slant submanifold of $\overline{M}$ such that ξ is tangent to N_θ, does not exist.*

Proof. From Lemma (2.1), we have

$$\nabla_X Z = \nabla_Z X = (ZInf)X, \tag{18}$$

for any vector fields $X \in N_\theta$ and $Z \in N_\perp$. If $\xi \in N_\theta$, then we have

$$\nabla_Z \xi = (ZInf)\xi. \tag{19}$$

From equation (9), (10) and (19), we have

$$(ZInf) = 0, \tag{20}$$

for all $Z \in N_\perp$. Which shows that f is constant on M hence the proof is complete.

Corollary 3.1. *A warped product pseudo-slant submanifold of* (1) *Sasakian* (2) *cosymplectic manifold $\overline{M}$ of type $M = N_\perp \times_f N_\theta$, where $N_\perp$ is an anti-invariant submanifold and N_θ is a proper slant submanifold of $\overline{M}$ such that ξ is tangent to N_θ, does not exist.*

Theorem 3.2. *A warped product pseudo-slant submanifold of quasi-Sasakian manifold $\overline{M}$ of type $M = N_\theta \times_f N_\perp$ in $\overline{M}$, where N_θ is a proper slant submanifold tangent to ξ and $N_\perp$ is an anti-invariant submanifold of $\overline{M}$, does not exist.*

Proof. For any vector fields $X \in N_\theta$ and $Z \in N_\perp$, from Lemma 2.1, we get the relation (18). Then for any $\xi \in N_\theta$ from (18), we have

$$\nabla_Z \xi = (\xi Inf) Z. \tag{21}$$

From equations (9),(10) and (21), we get

$$\xi Inf = 0,$$

for all $Z \in N_\perp$. Which shows f is constant on M threrfore in this case warped product does not exist.

Corollary 3.2. *A warped product pseudo-slant submanifold of* (1) *Sasakian* (2) *cosymplectic manifold* $\overline{M}$ *of type* $M = N_\theta \times_f N_\perp$*, where* N_θ *is a proper slant submanifold tangent to* ξ *and* $N_\perp$ *is an anti-invariant submanifold of* $\overline{M}$*, does not exist.*

4. Doubly warped product pseudo-slant submanifolds of quasi-Sasakain manifolds

In this section, we will study doubly warped product pseudo-slant submanifolds of quasi-Sasakain manifolds. Let $M =_f N_1 \times_b N_2$ be a doubly warped product pseudo-slant submanifold of quasi-Sasakain manifolds. If N_θ and $N_\perp$ are proper slant submanifolds and anti-invariant submanifolds of quasi-Sasakain manifolds $\overline{M}$ then their doubly warped product pseudo-slant submanifolds may be given by one of the following:

(i) ${}_fN_\perp \times_b N_\theta$ (ii) ${}_fN_\theta +_b N_\perp$

Theorem 4.1. *A doubly warped product pseudo-slant submanifold of a quasi-Sasakain manifold* $\overline{M}$ *of type* $M =_f N_\perp \times_b N_\theta$*, where* $N_\perp$ *is an anti-invariant submanifold and* N_θ *is a proper slant submanifold of* $\overline{M}$ *such that* ξ *is tangent to* N_θ*, does not exist.*

Proof. Let $M =_f N_\perp +_b N_\theta$ be doubly warped product pseudo-slant submanifold of quasi-Sasakain manifold $\overline{M}$, ξ is tangent to N_θ then, for any $Z \in N_\perp$, we get

$$\nabla_Z \xi = Z(Inf)\,\xi + \xi\,(Inb) Z. \tag{22}$$

From equations (9), (10) and (22), we get

$$Z(Inf) = 0, \tag{23}$$

$$\xi\,(Inb) = 0. \tag{24}$$

Which shows that both f and b are constant. So, there does not exist doubly warped product pseudo-slant submanifold of $\overline{M}$ of type $M =_f N_\perp +_b N_\theta$, with ξ tangent to N_θ.

Corollary 4.1. *A doubly warped product pseudo-slant submanifold of* (1) *Sasakian* (2) *cosymplectic manifold* $\overline{M}$ *of type* $M =_f N_\perp \times_b N_\theta$ *, where* $N_\perp$ *is an anti-invariant submanifold and* N_θ *is a proper slant submanifold of* $\overline{M}$ *such that* ξ *is tangent to* N_θ*, does not exist.*

Theorem 4.2. *A doubly warped product pseudo-slant submanifold of a quasi-Sasakain manifold* $\overline{M}$ *of type* $M =_f N_\theta +_b N_\perp$*, where* N_θ *is a proper slant submanifold of* $\overline{M}$ *tangent to* ξ *and* $N_\perp$ *is an anti-invariant submanifold, does not exist.*

Proof. Let $M =_f N_\theta +_b N_\perp$ be pseudo-slant doubly warped product submanifold of quasi-Sasakain manifold, ξ is tangential to N_θ. For any $X \in N_\theta$ and $Z \in N_\perp$, from equations (9), (10) and (22), we get

$$Z(Inf) = 0, \tag{25}$$

$$\xi\,(Inb) = 0. \tag{26}$$

Therefore both f and b are constant. Hence there does not exist doubly warped product pseudo-slant submanifold of $\overline{M}$ of type $M =_f N_\theta +_b N_\perp$ in $\overline{M}$, with ξ tangent to N_θ.

Corollary 4.2. *A doubly warped product pseudo-slant submanifold of* (1) *Sasakian* (2) *cosymplectic manifold* $\overline{M}$ *of type* $M =_f N_\theta +_b N_\perp$ *in* $\overline{M}$*, where* N_θ *is a proper slant submanifold of* $\overline{M}$ *tangent to* ξ *and* $N_\perp$ *is an anti-invariant submanifold, does not exist.*

Application. The warped product manifolds are used in the theory of relativity as well as in physics.

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Conflict of Interests

The authors declare that there is no conflict of interests.

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