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EXACT SOLUTION OF NONLINEAR KLEIN-GORDON EQUATIONS WITH QUADRATIC NONLINEARITY BY MODIFIED ADOMIAN DECOMPOSITION METHOD

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Abstract. In this paper, we used the modified Adomian decomposition method (ADM) to obtain exact solution to Nonlinear Klein-Gordon equation (NK-GE) with quadratic nonlinearity. The paper contains an introduction and the concept of modified ADM for a generalized three-dimensional NK-GE. And, we applied this concept to obtain exact solution to two one-dimensional NK-GE with quadratic nonlinearity. The modified method is based on Taylors series expansion of the source term and implementation on any computer algebra software (Maple, Mathematica, etc) is simple. We discovered that the results of the examples considered are the same as the series solution of those obtained by using any known analytical method. Furthemore, we depicted our findings in three-dimensional surface and contour plots.

Keywords: nonlinear Klein-Gordon equation; modified Adomian decomposition method; partial differential equations.

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1. Introduction

NK-GE are partial differential equations used to model various space-time phenomena in physics and engineering which can be presented in general form as

$$u_{tt}+\alpha\nabla^2 u+\beta u+\gamma g(u)=f(\bar{x},t) \tag{1}$$

$\bar{x}\varepsilon\Omega=[a,b]\subset\mathbb{R}$, $0<t\leqslant T$ with $u(\bar{x},0)=h_1(\bar{x}),u_t(\bar{x},0)=h_2(\bar{x})$
where α, β and γ are constant, $\bar{x}=x+y+z$ and

$$\nabla^2 u=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}$$

is the laplacian of u. Equation (1) and the given initial values is a generalised three-dimensional case of NK-GE. The nonlinear term g(u) could be a polynomial function in u or any other function. In application, the nonlinear term is a nonlinear force. For $g(u)=\sin u$ equation (1) becomes sine-Gordon equation, which could also be sinh-Gordon equation, double sinh-Gordon equation depending on the form of g(u). For a wide class of g(u), NK-GE has several Hamiltonian quantities [11] . As it is mention in [12], [15], [8], [7], [13] and [16], NK-GE has several types of nonlinear terms, which in general, plays a significant role in very many scientific applications. These include solid state physics, nonlinear optics, quantum theory, radiation theory, thermal equilibrium, field theory, harmonic oscillation of clocks in supersonic flow, you name it.

Many powerful methods have been used to investigate this class of partial differential equation equation. Some of which are the symplectic and multi-symplectic finite differnce scheme, Spectral and pseudo-spectral methods etc. Recently, [12] modified the exponentiation and expansion method and applied it to obtain exact solution to coupled Klein-Gordon-Zakharov equation. The solution were expressed in terms of hyperbolic, trigonomeric, exponential and rational functions. Other methods that gives exact solutions to NK-GE are inverse scattering method, the tanh-function method, F-expansion method etc.

Numerical solutions to NK-GE has also been studied. [14] used ADM with different version of Adomian polynomial to obtain numerical solutions to this class of equation. [9] compared solutions by ADM and Talor Matrix method obtained from the Telegraph equations that describes wave propagation in electric signal in cable transmission and Klein-Gordon equation. [16] applied the chebyshev-Tau meshless method. [10] applied the meshless method of lines. [8] applied the radial basis function based on collocation method. [11] used the laplace decomposition method to solve nonlinear partial differential equation.

In this work, we show, for the first time, that ADM [1] studied in [2], [3], [4], [5], [6] can also be use to obtain exact solution to NK-GE. In the next section we present the generalised modified ADM to equation (1), we illustrate with examples how to apply it and we conclude.

2. The Concept of Modified ADM On NK-GE

By ADM equation (1) can be given as

$$Lu(\bar{x},t)+Ru(\bar{x},t)+Nu(\bar{x},t)=f(\bar{x},t) \tag{2}$$

where L is a second order differential operator (in this case $L_{tt}=\frac{\partial^2}{\partial t^2}$), R is the remaining linear operator (in this case $R=\alpha\nabla^2 u+\beta u$). N is a nonlinear differential operator ($N=\gamma g(u)$) and $f(\bar{x},t)$ is the source term. See [2], [3], [4], [5], [6] and the references there in for more details.

Suppose L_{tt}^{-1} of L_{tt} exist the solution of equation (2) is given as

$$u(\bar{x},t)=\sum_{k=0}^{\infty}u_k(\bar{x},t) \tag{3}$$

and

$$Nu(\bar{x},t)=\sum_{k=0}^{\infty}A_k \tag{4}$$

where

$$A_k=A_k(u_0,u_1,u_2,\ldots,u_{k-1},) \tag{5}$$

is the Adomian polynomial given as

$$A_k=\frac{1}{k!}\frac{d^k}{d\lambda^k}[N(\sum_{i=0}^{\infty}\lambda^i u_i)]_{\lambda=0} \tag{6}$$

Applying equations (3), (4) and (5) on equation (2), equation (1) becomes

$$\sum_{k=0}^{\infty} u_k(\bar{x},t) = \Phi + L_{tt}^{-1}[f(\bar{x},t)] + L_{tt}^{-1}[\alpha\nabla^2 u + \beta u] + L_{tt}^{-1}[A_k] \tag{7}$$

where

$$\Phi = h_1(\bar{x}) + t h_2(\bar{x})$$

We modify $f(\bar{x},t)$ by applying Talor series expansion on it. This result to a form

$$f(\bar{x},t) = \sum_{k=0}^{\infty} f_k(\bar{x},t) \tag{8}$$

putting equation (8) in equation (7) and applying the traditional concept ADM [1], we quickly identify

$$u_0 = \omega(\bar{x},t) \tag{9}$$

where $\omega(\bar{x},t)$ represent the term arising from the source term and/or given initial conditions which must satisfy the origal equation (1). And subsequent terms given as

$$u_1 = L_{tt}^{-1}[f_0(\bar{x},t) - \alpha\nabla^2 u_0 - \beta u_0 - A_0(u_0)] \tag{10}$$

$$u_2 = L_{tt}^{-1}[f_1(\bar{x},t) - \alpha\nabla^2 u_1 - \beta u_1 - A_1(u_0,u_1)] \tag{11}$$

$$u_3 = L_{tt}^{-1}[f_2(\bar{x},t) - \alpha\nabla^2 u_2 - \beta u_2 - A_2(u_0,u_1,u_2)] \tag{12}$$

continuing in this order, we have the recurrence relation as

$$u_{k+1} = L_{tt}^{-1}[f_k(\bar{x},t) - \alpha\nabla^2 u_k - \beta u_k - A_k(u_0,u_1,u_2,...,u_{k-1})] \tag{13}$$

Adomian polynomials of frequently occuring nonlinear terms are given explicitely in [4] and [5].

3. Main results

In this section, we apply the concept of modified ADM to NK-GE with quadratic nonlinearity.

Example 1

In this example, we consider the NK-GE (1) with quadratic nonlinearity in one-dimensional space. The constants are $\alpha=-1$, $\beta=0$, $\gamma=1$ in the interval $-2\leqslant x\leqslant 2$. $0<t\leqslant 6$, $g(u)=u^2$ and $f(\bar{x},t)=-x\cos t+x^2\cos^2 t$. The initial conditions are $u(\bar{x},0)=x$, $u_t(\bar{x},0)=0$ and the exact solution, as given in [8] is

$$u(x,t)=x\cos t \tag{14}$$

Applying equations (3)-(13) in this example, we instantly identify that

$$\begin{aligned}
f_0(x,t)&=-x+x^2\\
f_1(x,t)&=(\frac{x}{4}-x^2)t^2\\
f_2(x,t)&=(-\frac{x}{12}+\frac{x^2}{3})t^4\\
f_3(x,t)&=(\frac{x}{720}-\frac{2x^2}{45})t^6
\end{aligned}$$

and so on. The A_k of u^2 are clearly given in [4] and [6]. Consequently, the following are obtain as

$$\begin{aligned}
u_0(x,t)&=x\\
u_1(x,t)&=-\frac{1}{2!}xt^2\\
u_2(x,t)&=\frac{1}{4!}xt^4\\
u_3(x,t)&=-\frac{1}{6!}xt^6
\end{aligned}$$

and so on. And

$$\begin{aligned}
u(x,t)&=\sum_{n=0}^{\infty}u_n(x,t)\\
&=x(1-\frac{1}{2!}t^2+\frac{1}{4!}t^4-\frac{1}{6!}t^6+\ldots)\\
&=x\cos t
\end{aligned}$$

which is the exact solution of example 1 as shown in equation (14). The physical behavior in form of 3D and contour plots of exact solution and Modified ADM solution are shown in Figures 1 - 4. And, the slight variation, if at all, in the plots is as a result of considering a series of order 20.

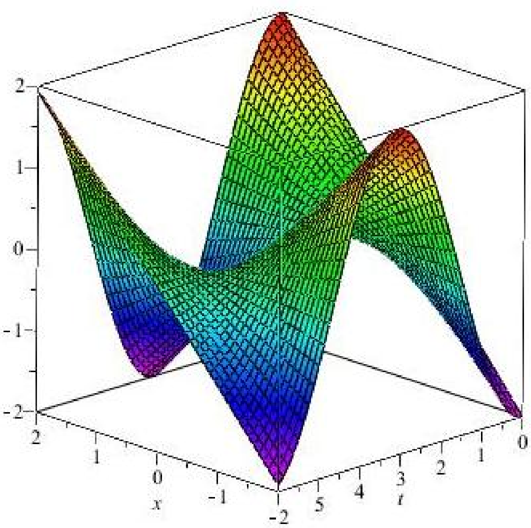

Figure 1: 3D Physical behavior of exact solution

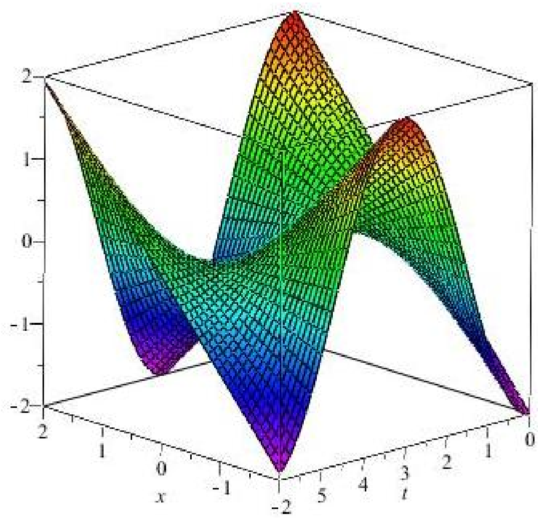

Figure 2: 3D Physical behavior of Modified ADM solution

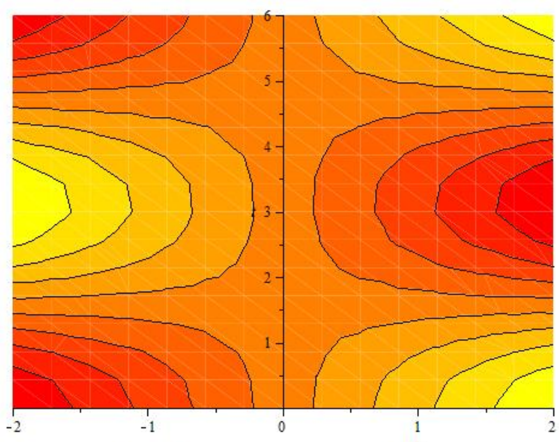

Figure 3: Contour plot of exact solution

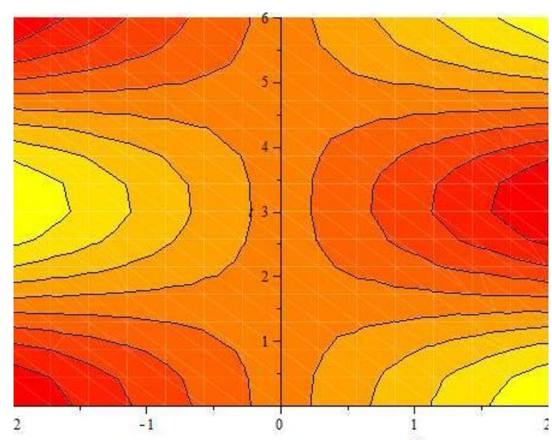

Figure 4: Contour plot of Modified ADM solution

Example 2

Also in this example, we consider the NK-GE (1) with quadratic nonlinearity still in one-dimensional space. The constants are $\alpha = -1$, $\beta = \dfrac{\pi^2}{4}$, $\gamma = 1$ in the interval $-3 \leqslant x \leqslant 3$. $0 < t \leqslant 4$, $g(u) = u^2$ and $f(\bar{x},t) = x^2 \sin^2 \dfrac{\pi t}{2}$. The initial conditions are $u(\bar{x},0) = 0$, $u_t(\bar{x},0) = \dfrac{\pi t}{2}$. The exact solution, as stated in [14], is

$$u(x,t) = x \sin \frac{\pi t}{2} \tag{15}$$

Also applying equations (3)-(13) in the example, we have

$$f_n(x,t) = \frac{(-1)^n(\pi t)^{2n+2}x^2}{2(2n+2)!}$$

and the A_k are as given in [4] and [6]. Consequently,

$$\begin{aligned}
u_0(x,t) &= \frac{\pi t}{2}x \\
u_1(x,t) &= -\frac{1}{48}(\pi t)^2 x \\
u_2(x,t) &= \frac{1}{3840}(\pi t)^5 x \\
u_3(x,t) &= -\frac{1}{645120}(\pi t)^7 x
\end{aligned}$$

and so on. And

$$\begin{aligned}
u(x,t) &= \sum_{n=0}^{\infty} u_n(x,t) \\
&= \frac{\pi t}{2}x - \frac{1}{48}(\pi t)^2 x + \frac{1}{3840}(\pi t)^5 x - \frac{1}{645120}(\pi t)^7 x + \ldots \\
&= x \sin\frac{\pi t}{2}
\end{aligned}$$

which is the exact solution of example 2 as given in equation (15). Figures 5 - 8 shows the contour plots and 3D physical behavior of the exact and modified ADM solutions. The not-too obvious variation in the plots between the two solutions is as a result of taking only a series of order 20 in the modified ADM solution

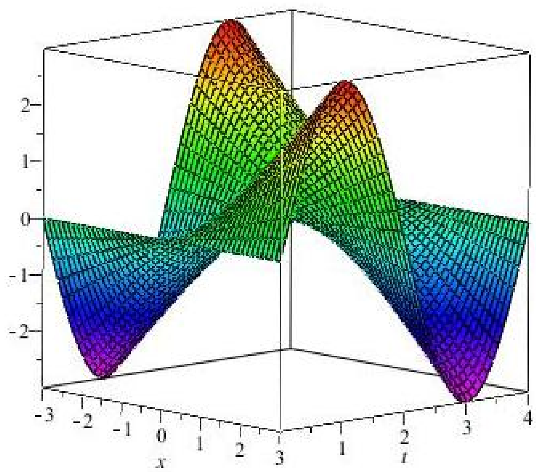

Figure 5: 3D Physical behavior of exact solution

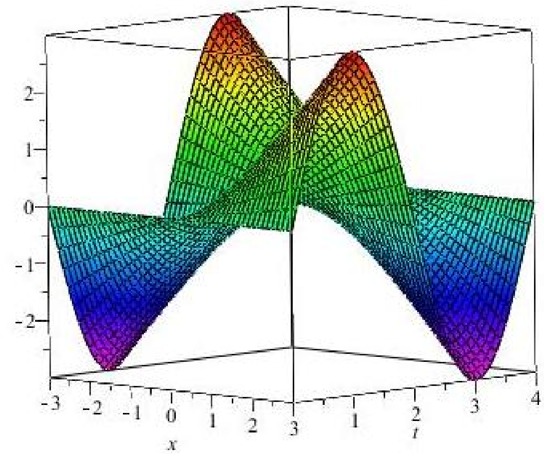

Figure 6: 3D Physical behavior of Modified ADM solution

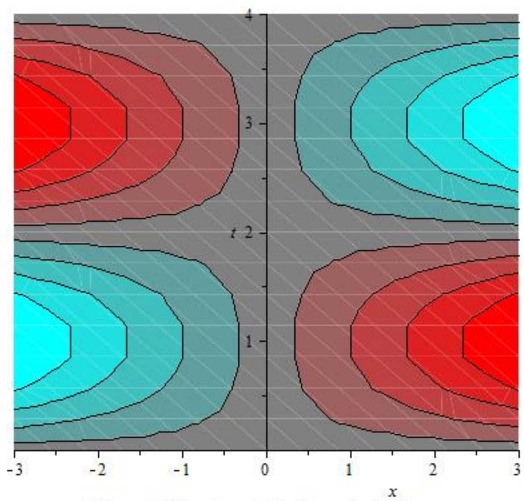

Figure 7: Contour plot of exact solution

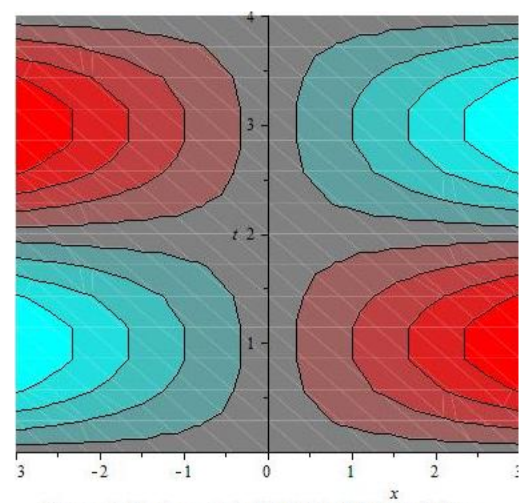

Figure 8: Contour plot of Modified ADM solution

Conclusion

An important conclusion that can be drawn from this work is that the exact solution of NK-GE is easily obtained by taking the Taylors' series expansion of the source term before applying the traditional ADM. This step accelerated the recurrence relation to convergence into the exact solution. The standard ADM when applied to this class of equation gave a numerical result subject to some error in some cases. In other cases, the results were divergent as the terms of the series solution kept growing unboundedly with no possible convergence.

Conflict of Interests

The authors declare that there is no conflict of interests.

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