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CONVERGENCE OF DERIVATIVE OF (0, 2) INTERPOLATORY POLYNOMIAL

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Abstract: In this paper, we consider the non-uniformly distributed zeros on the unit circle of n^{th} Legendre polynomial. Here, we are interested to establish the convergence theorem for the derivative of (0,2) interpolatory polynomial on the above said nodes.

Keywords: weight function; interpolatory polynomials; explicit representation; convergence.

1. Introduction

J. Suranyi and P.Turán [8] was first, who initiated the problem of Lacunary interpolation on zeros of

$$\prod_n(x) = (1 - x^2)P'_{n-1}(x) \ ,$$

where, $P_{n-1}(x)$ is the Legendre polynomial of degree $(n-1)$.The problem of (0,2) interpolation on the roots of unity was first studied by O. Kiš [7]. He obtained its regularity, fundamental

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polynomials and established a convergence theorem for the same. Later on several Mathematicians have considered the Lacunary interpolation on the unit circle. After that author[1] (with K. K. Mathur) [1] considered the weighted $(0,2)^*$-interpolation on the set of nodes obtained by projecting vertically the zeros of $(1-x^2)P_n(x)$ on the unit circle and established a convergence theorem for that interpolatory polynomial. Later on author[1] (with M. Shukla) [3] considered (0,2)-interpolation on the nodes, which are obtained by projecting vertically the zeros of $(1-x^2)P_n^{(\alpha,\beta)}(x)$ on the unit circle, where $P_n^{(\alpha,\beta)}(x)$ stands for Jacobi polynomial, obtained the explicit forms and establish a convergence theorem for the same. Recently, authors [2] considered weighted (0,2)-interpolation on the nodes, which are obtained by projecting vertically the zeros of the $(1-x^2)P_n'(x)$, onto the unit circle, established a convergence theorem for the same.

In 1990, K. Balázs [5] proved the simultaneous convergence of the derivatives of Langrange interpolating polynomials by giving an estimate for $\left|f^{(i)}(x)-L_n^{(i)}(f,x)\right|$ $(i=0,1,2\ldots\ldots\ldots.)$ by the aid of the Lebesgue constant of Lagrange interpolation. In 1993, K. Balázs and T. Kilgore [6] considered the approximation of derivatives by interpolation. Also author[1] (with M. Shukla) [4], considered the convergence of the derivative of the Hermite interpolatory polynomial.

These have motivated us to consider the convergence of the derivative of the Lacunary interpolatory polynomial on the unit circle and established a convergence theorem. In section 2 we give some preliminaries and in section 3, we describe the problem. In section 4, we give the explicit formulae of the interpolatory polynomials. In section 5 and 6, estimation and convergence of interpolatory polynomials are given respectively.

2. Preliminaries

There are some well-known results in this section, which we shall use.

The differential equation satisfied by $P_n(x)$ is

(2.1) $(1-x^2)P_n''(x) - 2xP_n'(x) + n(n+1)P_n(x) = 0$

(2.2) $W(z) = \prod_{k=1}^{2n}(z-z_k) = K_n P_n\left(\frac{1+z^2}{2z}\right)z^n$

(2.3) $R(z) = (z^2-1)W(z)$

We shall require the fundamental polynomials of Lagrange interpolation based on the zeros of $W(z)$ and $R(z)$ are respectively given as:

(2.4) $L_{1k}(z) = \frac{W(z)}{(z-z_k)W'(z_k)}$, $k = 1(1)2n$

(2.5) $L_k(z) = \frac{R(z)}{(z-z_k)\,R'(z_k)}$, $k = 0(1)2n+1$

We will also use the following results

For $k = 1(1)2n,$

(2.6) $\begin{cases} W'(z_k) = -\frac{1}{2}(1-z_k^2)\,P_n'(x_k)z_k^{n-2} \\ W''(z_k) = -K_n\{n(1-z_k^2) + z_k^2\}\,P_n'(x_k)z_k^{n-3} \end{cases}$

For $k = 0(1)2n+1,$

(2.7) $\begin{cases} R'(z_k) = (z_k^2-1)W'(z_k) \\ R''(z_k) = \frac{2\{z_k^2 - n(1-z_k^2)\}}{z_k}W'(z_k) \end{cases}$

We will also use the following well known inequalities:

For, $-1 < x < 1$

(2.8) $(1-x^2)^{1/4}|P_n(x)| \le \sqrt{\frac{2}{\pi}}n^{-1/2},$

(2.9) $(1-x^2)^{3/4}|P_n'(x)| \le \sqrt{2}\,n^{1/2},$

(2.10) $|P_n(x)| \le 1,$

Let $x_k = cos\theta_k$ $(k = 1,2,\dots\dots,n)$ are the zeros of n^{th} Legendre polynomial $P_n(x)$, with

$1 > x_1 > x_2 > \cdots \ldots\ldots\ldots\ldots x_n > -1,$ then

(2.11) $(1-x_k^2)^{-1} \sim \left(\frac{k}{n}\right)^{-2}$

(2.12) $|P_n(x_k)| \sim k^{-1/2}$

(2.13) $|P_n'(x_k)| \sim k^{-3/2}\ n^2$

For more details, one can see [9].

3. The Problem:

Let, $Z_n = \{z_k : k = 0(1)2n+1\}$ satisfying:

(3.1) $Z_n = \begin{cases} z_0 = 1, \quad z_{2n+1} = -1, \\ z_k = \cos\theta_k + i\sin\theta_k,\ z_{n+k} = \overline{z_k},\ k = 1(1)n, \end{cases}$

are non-uniformly distributed zeros of unit circle, obtained by projecting vertically the zeros of $(1-x^2)P_n(x)$, where, $P_n(x)$ stands for n^{th} Legendre polynomial having the zeros $\{x_k = \cos\theta_k : k = 1(1)n\}$,such that $1 > x_1 > x_2 > \cdots \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots > x_n > -1.$

Here we consider the interpolatory polynomial $Q_n(z)$ of degree $\leq 4n+1$ satisfying the following conditions:

(3.2) $\begin{cases} Q_n(z_k) = \alpha_k, & k = 0(1)2n+1, \\ [p(z)Q_n(z)]''_{z=z_k} = \beta_k, & k = 1(1)2n \end{cases}$

where, α_k and β_k are arbitrary complex numbers and $p(z) = (z^2-1)^{1/2}$, $z \in C$. We establish a convergence theorem of the derivative of $\boldsymbol{Q_n(z)}$ on the above said nodes.

4. Explicit Representation of Interpolatory Polynomials:

Let $Q_n(z)$ satisfying (3.2) [1]:

(4.1) $Q_n(z) = \sum_{k=0}^{2n+1} \alpha_k A_k(z) \quad + \sum_{k=1}^{2n} \beta_k B_k(z)$

where, $A_k(z)$ and $B_k(z)$ are fundamental polynomials of first and second kind respectively, each of degree at most $4n+1$ satisfying the conditions :

For $k = 0(1)2n+1$

(4.2) $$\begin{cases} A_k(z_j) = \delta_{jk}, & j = 0(1)2n+1 \\ \left[(z^2-1)^{1/2} A_k(z)\right]''_{z=z_j} = 0, & j = 1(1)2n \end{cases}$$

For $k = 1(1)2n$

(4.3) $$\begin{cases} B_k(z_j) = 0, & j = 0(1)2n+1 \\ \left[(z^2-1)^{1/2} B_k(z)\right]''_{z=z_j} = \delta_{jk}, & j = 1(1)2n \end{cases}$$

Theorem 4.1 [1, Theorem 2]: *For* $k = 1(1)2n$ *, we have*

(4.4) $$B_k(z) = z^{-n} W(z)\{a_k J_k(z) + a_{0k} J_{10}(z) + a_{1k} J_{11}(z)\}$$

where,

(4.5) $$J_k(z) = \int_0^z t^{n-1} L_k(t)\,dt$$

(4.6) $$J_{1j}(z) = \int_0^z t^{n+j-1} W(t)\,dt, \quad j = 0,1$$

(4.7) $$\begin{cases} a_{0k} = \frac{a_k}{2 J_{10}(1)} \{J_k(1) + J_k(-1)\} \\ a_{1k} = \frac{a_k}{2 J_{11}(1)} \{J_k(1) + J_k(-1)\} \end{cases}$$

(4.8) $$a_k = \frac{z_k}{2(z_k^2-1)^{1/2} W'(z_k)}$$

Theorem 4.2 [1, Theorem 3]: *For* $k = 0 \text{ and } 2n+1,$

(4.9) $$A_k(z) = \frac{z^{-n} W(z)}{K_n}\{b_{0k} J_{10}(z) + b_{1k} J_{11}(z)\}$$

For $k = 1(1)2n,$

(4.10) $$A_k(z) = L_k(z) L_{1k}(z) + b_k B_k(z) + z^{-n}\frac{W(z)}{W'(z_k)}\{S_k(z) + b_{0k} J_{10}(z) + b_{1k} J_{11}(z)\}$$

where,

(4.11) $b_k = L_k''(z_k) - L_k'^2(z_k) - L_{1k}''(z_k) - 2L_k'(z_k)L_{1k}'(z_k) - 2z_k(z_k^2 - 1)^{-1}\{L_k'(z_k) - L_{1k}'(z_k)\} + (z_k^2 - 1)^{-2}$

(4.12) $S_k(z) = \int_0^z t^n \frac{L_k'(z_k)L_k(t) - L_k'(t)}{(t - z_k)}\, dt$

For $k = 1(1)\,2n,$

(4.13) $\begin{cases} b_{0k} = \frac{1}{2\,J_{10}(1)}\{S_k(1) + S_k(-1)\} \\ b_{1k} = \frac{(-1)^k}{2\,J_{11}(1)}\{S_k(1) - S_k(-1)\} \end{cases}$

For $k = 0 \; and \; 2n + 1,$

(4.14) $\begin{cases} b_{0k} = \frac{1}{2\,J_{10}(1)} \\ b_{1k} = \frac{(-1)^k}{2\,J_{10}(1)} \end{cases}$

5. Estimation of Fundamental Polynomials:

Lemma 1: *Let* $L_{1k}(z)$ *be given by (2.4). Then*

(5.1) $\max_{|z|=1} \sum_{k=1}^{2n-1} |L_{1k}'(z)| \leq cn \sum_{k=1}^{2n-1} \frac{1}{k^{1/2}}$,

where, c *is a constant independent of* n *and* z.

Proof: Let $z = x + iy$ and $|z| = 1$,

$$\sum_{k=1}^{2n} |L_{1k}(z)| \leq \sum_{k=1}^{2n} \left| \frac{W(z)}{(z - z_k)W'(z_k)} \right|$$

$$\leq c \sum_{k=1}^{2n} \frac{|P_n(x)|(1 - xx_k)^{\frac{1}{2}}}{(x - x_k)(1 - x_k^2)^{1/2} |P_n'(x_k)|}$$

using (2.10) - (2.11) and (2.13), we get the result.

Lemma 2: *Let* $B_k'(z)$ *obtained by differentiating (4.4).Then*

(5.2) $\sum_{k=1}^{2n} |p(z)B_k'(z)| \leq cn^{-1} \log n$

where, c *is a constant independent of* n *and* z.

Proof: Differentiating (4.4), we get

$$\sum_{k=1}^{2n}|p(z)\,B_k'(z)| = \sum_{k=1}^{2n}\left[\begin{matrix}|p(z)W(z)|\{|a_k||J_k'(z)| + |a_{0k}||J_{10}'(z)| + |a_{1k}||J_{11}'(z)|\} + \{n|p(z)W(z)| + |p(z)J_k'(z)|\} \\ +\{|a_k||J_k(z)| + |a_{0k}||J_{10}(z)| + |a_{1k}||J_{11}(z)|\}\end{matrix}\right]$$

using, (5.1) and (2.10)- (2.13), we get the result.

Lemma 3: *Let $A_k'(z)$ be obtained by differentiating (4.9) ($k = 0, 2n+1$) and (4.10) ($k = 1(1)\,2n$).*

Then

(5.3) $\sum_{k=0}^{2n+1}|p(z)\,A_k'(z)| \le cn\,\log n$,

where, c is a constant independent of n and z.

Proof: One can find (5.3), owing to conditions (2.8)-(2.13)

6. Convergence:

In this section, we prove the main theorem.

Theorem: *Let $f(z)$ be continuous for $|z| \le 1$ and analytic for $|z| < 1$. Let the arbitrary β_k's be such that:*

(6.1) $|\beta_k| = O(n\,\omega_2(f, n^{-1})), k = 1(1)\,2n$

Then $\{Q_n'(z)\}$ defined by

(6.2) $[p(z)\,Q_n'(z)] = \sum_{k=0}^{2n+1} f(z_k)[p(z)\,A_k'(z)] + \sum_{k=1}^{2n}\beta_k[p(z)\,B_k'(z)]$

satisfies the relation,

(6.3) $|p(z)\{Q_n'(z) - f(z)\}| = O(n\omega_2(f, n^{-1})\log n)$,

where, $\omega_2(f, n^{-1})$ be the second modulus of continuity of $f(z)$.

Remark: Let $f(z)$ be continuous for $|z| \le 1$ and analytic for $|z| < 1$, and $f' \in Lip\alpha, \alpha > 0$, then the sequence $\{Q_n'(z)\}$ converges uniformly to $f(z)$ in $|z| \le 1$, which follows from (6.3) as :

(6.4) $\omega_2(f, n^{-1}) = O(n^{-1-\alpha})$, To prove the theorem (6.1), we shall need the followings:

Let $f(z)$ be continuous for $|z| \leq 1$ and analytic for $|z| < 1$.Then there exist a polynomial $F_n(z)$ of degree $\leq 4n+1$, satisfying, Jackson's inequality.

(6.5) $|f(z) - F_n(z)| \leq c\omega_2(f, n^{-1}),\ z = e^{i\theta}(0 \leq \theta < 2\pi)$

And also an inequality due to O. Kiš[7].

(6.6) $|F_n^{(m)}(z)| \leq cn^m\,\omega_2(f, n^{-1}),\ \ m\epsilon I^+.$

Proof: Since $Q_n'(z)$ be a uniquely determined polynomial of degree $\leq 4n+1$ and the polynomial $F_n(z)$ satisfying (6.5) and (6.6) can be expressed as:

$$F_n(z) = \sum_{k=0}^{2n+1} F_n(z_k)A_k'(z) + \sum_{k=1}^{2n} F_n'(z_k)B_k'(z)$$

Then,

$$|p(z)\{Q_n'(z) - f(z)\}| \leq |p(z)\{Q_n'(z) - F_n(z)\}| + |p(z)\{F_n(z) - f(z)\}|$$

$$\leq \textstyle\sum_{k=0}^{2n+1}|f(z_k) - F_n(z_k)|\,|p(z)A_k'(z)|$$

$$+\sum_{k=1}^{2n}\{|\beta_k| + |F_n'(z_k)|\}\,|p(z)B_k'(z)|$$

$$+|p(z)\{F_n(z) - f(z)\}|$$

using, (6.1)-(6.2), (6.4),-(6.5), (5.3) - (5.4), we get (6.3).

Conclusion: Convergence of (0,2) Interpolation on non-uniformly distributed zeros on the unit circle by derivative of polynomial is strong than the convergence of the derivative of (0,2) Interpolation on non-uniformly distributed zeros on the unit circle by the same polynomial.

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