Available online at http://scik.org
J. Math. Comput. Sci. 8 (2018), No. 5, 620-629
https://doi.org/10.28919/jmcs/3765
ISSN: 1927-5307

ASYMPTOTIC BEHAVIOUR OF NONOSCILLATING SOLUTIONS OF NEUTRAL DELAY DIFFERENTIAL EQUATIONS OF THE SECOND ORDER WITH IMPULSES

UBON A. ABASIEKWERE[1,*] AND ITA M. ESUABANA[2]

[1]Department of Mathematics and Statistics, University of Uyo, P.M.B. 1017, Uyo, Akwa Ibom State, Nigeria

[2]Department of Mathematics, University of Calabar, P.M.B 1115, Calabar, Cross River State, Nigeria

Abstract: In this paper, we consider a certain class of second order neutral delay differential equations with constant impulsive jumps and obtain some sufficient conditions for the asymptotic decay of all of its non-oscillating solutions. An example is provided to illustrate the main result.

Keywords: neutral; second-order; delay; oscillations; impulsive differential equations; asymptotic.

1. Introduction

Impulsive differential equations are now known to be an outstanding source of models to simulate processes observed in control theory, physics, chemistry, population dynamics, biotechnology, industrial robotics, economics, etc. Lately, Impulsive differential systems have become a very attractive area of research and we refer the reader to the monographs by Lashmikantham *et al*, Samoilenko and Perestyuk ([4], [2]), where properties of their solutions

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*Corresponding author
E-mail address: ubeeservices@yahoo.com
Received June 10, 2018

are studied and extensive bibliographies are given. Recent researches in this area can also be seen in [36-39]. We should note that in spite of the large number of investigations on impulsive differential equations, the theory of the asymptotic behaviour of their solutions has not yet been fully elaborated. In particular, asymptotic behavior of solutions of impulsive delay differential equations have been studied by several authors (e.g. [3], [5], [13–23]). A number of authors have obtained interesting results in studying the asymptotic behaviour of first order neutral impulsive differential equations by two basic methods, by construction of Lyapunov functionals [29–33], and by considering the asymptotic behavior of non-oscillatory and oscillatory solutions respectively [34–35] and the references therein. (see also [6], [12], [24 – 28]). However, there seems to be little or nothing by way of results for the asymptotic behavior of solutions of second order neutral delay impulsive differential equations.

In this paper, we are concerned with the asymptotic behaviour of non-oscillating solutions of a class of second order neutral impulsive differential equations with constant delays and variable coefficients.

It is known that, in ordinary differential equations, the solutions are continuously differentiable sometimes at least once, whereas the impulsive differential equations generally possess non-continuous solutions. Since the continuity properties of the solutions play an important role in the analysis of the behaviour, the techniques used to handle the solutions of impulsive differential equations are fundamentally different, including the definitions of some of the basic terms. In this section, we examine some of these changes.

In effect, the solution $y(t)$ for $t\in[t_0,T)$ of a given impulsive differential equation or its first derivative $y'(t)$ is a piece-wise continuous function with points of discontinuity $t_k\in[t_0,T),\ t_k\neq t,\ \ 0\le k<\infty.$ Consequently, in order to simplify the statements of the assertions later, we introduce the set of functions PC and PC^r which are defined as follows:

Let $r\in N,\ D:=[T,\infty)\subset R$ and let $S:=\{t_k\}_{k\in N}$ be fixed. Except stated otherwise, we will assume that the elements of S are moments of impulsive effects and satisfy the property:

C1.1: $0<t_1<t_2<\cdots$ and $\lim_{k\to\infty} t_k=+\infty$.

We denote by $PC(D,R)$ the set of all functions $\varphi:D\to R$ which are continuous for all $t\in D,\ t\notin S$. They are continuous from the left and have discontinuity of the first kind at the points for which $t\in S$. By $PC^r(D,R)$, we denote the set of functions $\varphi:D\to R$ having derivative $\frac{d^j\varphi}{dt^j}\in PC(D,R),\ 0\le j\le r$ ([1], [4]).

To specify the points of discontinuity of functions belonging to PC and PC^r, we shall sometimes use the symbols $PC(D,R;S)$ and $PC^r(D,R;S),\ r\in N$ ([8], [9], [10]).

i) **Definition 1.1.** The solution $y(t)$ of an impulsive differential equation is said to be finally positive (finally negative) if there exist $T\ge 0$ such that $y(t)$ is defined and is strictly positive (negative) for $t\ge T$ [9];

ii) non-oscillatory, if it is either finally positive or finally negative; and

iii) oscillatory, if it is neither finally positive nor finally negative ([1], [10]).

Remark 1.1. All functional inequalities considered in this paper are assumed to hold finally, that is, they are satisfied for all t large enough.

2. Statement of the Problem

We consider the second order linear neutral delay impulsive differential equation of the form

$$\begin{cases} \left[y(t)+p(t)y(t-\tau)\right]''+q(t)y(t-\sigma)=0, \\ \qquad t\ge t_0,\ t\notin S \\ \Delta\left[y(t_k)+p(t_k)y(t_k-\tau)\right]'+q_k y(t_k-\sigma)=0, \\ \qquad t_k\ge t_0,\ \forall\ t_k\in S, \end{cases} \tag{2.1}$$

where $p(t),q(t)\in PC([t_0,\infty),R)$ and $\tau,\sigma\in R_+$. Let $\varphi(t)\in PC([t_0-\rho,t_0],R)$ be a given function, where $\rho=\max\{\tau,\sigma\}$ and let $y_0\in R$.

Definition 2.1. The function $y(t)\in PC([t_0-\rho,\infty),R)$ is said to be a solution of equation (2.1) if

$$y(t)=\varphi(t),\ t\in[t_0-\rho,t_0],$$

$$\left[y(t)+p(t)y(t-\tau)\right]''\Big|_{t=t_0}=y_0,$$

the function $y(t)+p(t)y(t-\tau)$ is twice piece-wise continuously differentiable for $t\geq t_0$ and $y(t)$ satisfies equation (2.1) for all $t\geq t_0$. Throughout this study, we shall assume the following:

C2.1. $q_k\geq 0\ \forall\ k\in N;$

C2.2. $p(t)\in PC([t_0,\infty),R),\ p_1\leq p(t)\leq p_2$ for $t\in[t_0,\infty)$, where $p_1,p_2\in R;$

C2.3. $q(t)\in PC([t_0,\infty),R),\ q(t)\geq q_1>0$ for $t\in[t_0,\infty).$

Here, our aim is to investigate the asymptotic behaviour of the non-oscillating solutions of equation (2.1). Throughout this discussion, we shall restrict ourselves to the study of impulsive differential equations for which the impulse effects take place at fixed moments $\{t_k\},\ k\in N.$

In this, we demonstrate how well-known mathematical techniques and methods are extended in proving an existence theorem for neutral delay differential equations with constant impulsive jumps. Before proceeding, we will discuss a lemma that will be useful as we advance through this article. This lemma is the impulsive extension of the work by Grammatikopoulos *et al* [7].

Remark 2.1. Without loss of generality, in order to prove the existence of non-oscillatory solutions, we will deal only with positive solutions of equation (2.1).

Lemma 2.1. Assume conditions C2.1—C2.3 satisfied and let $y(t)$ be a finally positive solution of equation (2.1). Set

$$z(t)=y(t)+p(t)y(t-\tau).\qquad (2.2)$$

Then the following statements are true:

a) The functions $z(t)$ and $z'(t)$ are strictly monotone and either

$$\lim_{t\to\infty} z(t)=\lim_{t\to\infty} z'(t)=-\infty \tag{2.3}$$

or

$$\lim_{t\to\infty} z(t)=\lim_{t\to\infty} z'(t)=0;\ z(t)<0 \text{ and } z'(t)>0.$$

(2.4)

In particular, $z(t)$ is finally negative.

b) Assume that $p_1 \geq -1$, then condition (2.4) holds. In particular, $z(t)$ is bounded.

Proof: **(a)** From equation (2.1), we have that

$$\begin{cases} z''(t)=-q(t)y(t-\sigma)\leq -q_1 y(t-\sigma)<0 \\ \Delta z'(t_k)=-q_k y(t_k-\sigma)\leq -q_1 y(t_k-\sigma)<0 \end{cases} \tag{2.5}$$

which implies that $z'(t)$ is a strictly decreasing function of t and so $z(t)$ is a strictly monotone function. From the above observations it follows that either

$$\lim_{t\to\infty} z(t)=\lim_{t\to\infty} z'(t)=-\infty$$

or

$$\lim_{t\to\infty} z'(t)=\ell \quad \text{is finite.} \tag{2.6}$$

Let us assume that condition (2.6) holds. Integrating both sides of equation (2.5) from t_0 to t with t sufficiently large, and letting $t\to\infty$, we obtain

$$\int_{t_0}^{\infty} q_1 y(s-\sigma)ds+\sum_{t_0\leq t_k<\infty} q_1 y(t_k-\sigma)\leq z'(T)-\ell,$$

which implies that $y(t)\in L_1[T,\infty)$ and so $z(t)\in L_1[T,\infty)$. Since $z(t)$ is monotone, it follows that

$$\lim_{t\to\infty} z(t)=0 \tag{2.7}$$

And therefore $\ell=0$. Finally, by equations (2.7) and (2.6) with $\ell=0$ and the decreasing nature of $z'(t)$, we conclude that $z(t)<0$ and $z'(t)>0$.

(b) By contradiction, we assume condition (2.4) was false, then from condition (2.3), it would follow that

$$\lim_{t\to\infty} z(t)=-\infty \,. \tag{2.8}$$

Using the fact that $p_1 \geq -1$ and $z(t)<0$, we obtain

$$y(t)< p(t)y(t-\tau)\leq -p_1 y(t-\tau)\leq y(t-\tau)\,,$$

which implies that $y(t)$ is bounded, contradicting condition (2.8) and proving that condition (2.4) is fulfilled. This, therefore, completes the proof of Lemma 2.1.

We are now ready to prove the main result.

3. Main Results

The following theorem is the impulsive extension of Theorem 3.1.1 of the monograph by Bainov and Mishev [11].

Theorem 3.1: *Assume that*

i) $q_k \geq 0 \;\forall\; k\in N$,

ii) $p(t)\in PC([t_0,\infty),R)$, $p_1 \leq p(t)\leq p_2$ *for* $t\in[t_0,\infty)$, *where* $p_1, p_2 \in R$,

iii) $q(t)\in PC([t_0,\infty),\mathbb{R})$, $q(t)\geq q_1 >0$ *for* $t\in[t_0,\infty)$,

iv) $-1< p_1 \leq p_2 <0$.

Then every non-oscillating solution $y(t)$ *of equation (2.1) tends to zero as* t *tends to infinity.*

Proof.

As the negative of a solution of equation (2.1) is also a solution of the same equation, it suffices to prove the theorem for a finally positive solution $y(t)$ of equation (2.1). Set

$$z(t)=y(t)+p(t)y(t-\tau)\,.$$

Then by Lemma 2.1(b), condition (2.4) must hold. In particular, $z(t)<0$ and hence

$$y(t) < -p(t)y(t-\tau) < y(t-\tau).$$

Therefore, $y(t)$ is a bounded function. Now assume by contradiction that

$$\lim_{x\to\infty} \sup y(t) \equiv s > 0 \tag{2.9}$$

Let $\{t_n\}$ be a sequence of points such that $\lim_{n\to\infty} t_n = \infty$ and $\lim_{n\to\infty} y(t_n) = s$. Then, for sufficiently large n,

$$z(t_n) = y(t_n) + p(t_n)y(t_n - \tau) \geq y(t_n) + p_1 y(t_n - \tau),$$

and so

$$\lim_{n\to\infty} \sup y(t_n - \tau) \geq \frac{s}{-p_1} > s,$$

which contradicts condition (2.9) and therefore completes the proof of Theorem 3.1.

In the following illustration, we see that if condition (iii) of Theorem 3.1 is violated, the result may not be true.

Example 3.1: Consider the neutral delay impulsive differential equation

$$\begin{cases} \left[y(t) + \left(-\frac{1}{2} + (t-1)^{-\frac{1}{2}}\right)y(t-1)\right]'' + \frac{1}{4}(t-2)^{-\frac{1}{2}}\left(t^{-\frac{3}{2}} - \frac{1}{2}(t-1)^{-\frac{3}{2}}\right)\times \\ \qquad \times y(t-2) = 0,\ t \geq 2,\ t \notin S \\ \Delta\left[y(t_k) + \left(-\frac{1}{2} + (t_k-1)^{-\frac{1}{2}}\right)y(t_k-1)\right]' + \frac{1}{4}(t_k-2)^{-\frac{1}{2}}\left(t_k^{-\frac{3}{2}} - \frac{1}{2}(t_k-1)^{-\frac{3}{2}}\right)\times \\ \qquad \times y(t_k-2) = 0,\ t_k \geq 2,\ t_k \in S. \end{cases}$$

Careful observation shows that all conditions of Theorem 3.1 are satisfied, except for condition (iii). Here, notice that the function $y(t) = \sqrt{t}$ is a solution with $\lim_{t\to\infty} y(t) = \infty$.

Conflict of Interests

The authors declare that there is no conflict of interests.

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