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SINGLE AND MULTI-OBJECTIVE OPTIMIZATION - A COMPARATIVE ANALYSIS

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Abstract. We compare the order relations used in single and in multi-objective optimization. The single objective optimization is based on a unique complete ordering. In multi-objective case there are infinite inequivalent partial orderings. The implications of these facts are analysed with examples. Some implications on Pareto techniques are also pointed out. A technique will be presented on the use of optimal frontier which synthesises it into a "compiled" guide to control the system successfully. Some applications as examples are presented like economic planning strategies etc.

Keywords: multi-objective optimization; maximal and minimal optimum.

2010 AMS Subject Classification: 49Q10, 49N20.

1. Introduction

The multi-objective optimization is very wide, it is not the purpose of this lecture to review all. A comprehensive book on the different concepts and techniques of multi-objective optimization is written by (Branke, Greco, Sowiski, and Zielniewicz, 2010). See also (18).

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Some aspects of the linear optimization will be discussed. The history of linear optimization goes back to 1827 when Fourier solved the problem of finding solution of a system of linear inequalities. His method was based on elimination of variables and he made a n^3 algorithm to find a feasible point or states that there is no feasible point when there is no feasible solution to the problem. This procedure was forgotten and rediscovered by Dines (1918) and by Motzkin (1936). This algorithm became Fourier-Motzkin algorithm which actually should be Fourier-Dines-Motzkin and it is similar to Gaussian elimination (1800) (17). However, the algorithm is slow compared to interior point techniques. This method remained important even after the development of the simplex method since it is capable of stating the existence or non-existence of a feasible point and also gives all the optimal solutions of a problem in integer linear programming (2).

The linear programming and its solution techniques was first developed by L. Kantorovich in 1939 but it was used for military operations hence it was top secret. Similar approaches were used in British and American armies. It was in 1947 that Dantzig was mandated to write his paper about the formulation of simplex method (14). The simplex method in principle involves the investigation of the existence of feasible points. However, the simplex method is a semi-decisive algorithm since it can fall into infinite loop especially when there is no solution. The complexity of this method is polynomial n^4 however, the worst case scenario is exponential. Non-polynomial algorithms are non-tractable on digital computers which means that they work for comparatively small number of variables only (4), (19).

N. Karmarkar published the interior point algorithm in 1984 (4). While the simplex method moves on the vertices of the set of feasible points, this method is based on the interior points of the set of feasible points. In his conference lecture, N. Karmarkar claimed that his algorithm has a complexity of $\sqrt{n}\, log\left(\frac{n}{\varepsilon}\right)$ which is much better than simplex method. However, Denardo on the cited book states that some of Karmarkars statements are difficult or cannot be reproduced and also there are examples which show that the worst case scenario is similar to that of simplex method. In conclusion, he says that simplex method and interior point algorithms have more or less the same performance and behaviour. Denardos book contains interesting comparisons in industrial solution where the range of number variables goes from $40,000\ to\ 1,000,000$.

The formulation of multi-objective optimization started in economics and in parallel Borel, Cantor, (11), (12) and others developed the mathematical background (10). The first paper in this research area was published in 1881 by Edgeworth, an economist and a mathematician. He formulated the idea of evaluating the economic problem in terms of more than one objective function though he did not present this idea formally. He realized that one objective function cannot cope with the complexity of evaluating the performance of a system of economy (5). Along the same lines, another economist Vilfredo Pareto formulated the idea of evaluating a multi-objective optimization problem in a formal way (16). Karush, Kuhn and Tucker worked out the conditions for optimality. Originally, Karush had these conditions in his unpublished thesis of 1939 but later published as (12) and the reference to Karush is made in (7). What is often neglected in the literature of multi-objective optimization is that the presence of multiple solutions is no solution at all if there is no instructional guide to give directives to the users of such system. The main issue here is what to do with all the generated (maybe infinite) optimal solutions.

A number of techniques have been developed to handle the problem of multi-objective optimization problems. Solution techniques are summarised for mathematics audience in (10), engineers and economists in (Zhou, Qu, Li, Zhao, Suganthan, and Zhang, 2011), (13), (8). For extensive reading on a number of classical solution techniques to solving multi-objective problems see (3).

Figure 1 shows the difference between single and multi-objective optimization. The single objective optimization gives a global optimum, timeless, while the multi-objective optimization gives a local type optimization valid within the positive cone.

2. Solution Techniques

Multi-objective optimization is classified into three major groups of methods according to the phase when the decision maker chooses his preferences; Apriori, Interactive and Posteriori methods (9). In the posteriori or generation method, all the solutions are generated and presented before the decision maker who will either take it or reject it (1). The various methods are listed below.

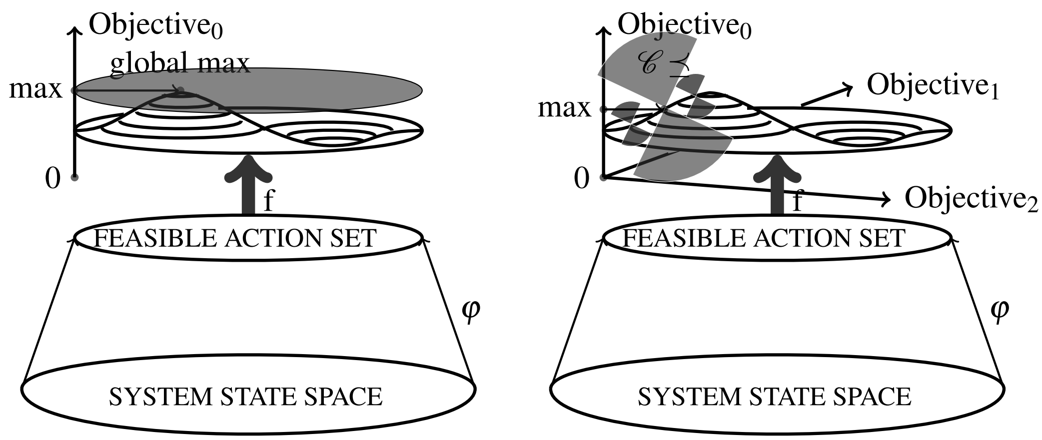

FIGURE 1. Single and multi-objective optimums.

Scalarization methods, Weighting sum method, E-constraint method, Goal programming method, Evolutionary algorithms, Direct search methods, Genetic algorithms,
Pareto optimality.

It took a long time until the concepts of single and multi-objective optimization crystallized. The critical difference comes from the fact that in scalar optimization there is only one order relation - the order relation of numbers while in two or more dimensions there are infinite inequivalent order relations. Hence defining the objective function does not determine the optimization problem.

Pareto's method: Among the first approaches to solve multi-objective optimization problems, Pareto developed his Pareto optimality theory in 1910 and was published in (6). He introduced the dominance relation between pairs of points of R^n as follows: Let $x, y \in R^n$. Then x dominates y if $x_j \geq y_j, \forall\, 1 \leq j \leq n$ and $\exists\, 1 \leq i \leq n$ such that $x_i > y_i$, where $x = (x_1, x_2, \ldots x_n)$ and $y = (y_1, y_2, \ldots y_n)$.

Let the order relation $\preceq$ be defined between the vectors of R^n as follows:

Definition 1. If $x, y \in R^n$ and $x = (x_1, x_2, \ldots x_n)$ and
$y = (y_1, y_2, \ldots y_n)$ then $x \preceq y \Leftrightarrow x_j \geq y_j, \forall\, 1 \leq j \leq n$.

Lemma 1. $\forall\, x, y \in R^n, x$ *dominates* $y \;\Leftrightarrow x \preceq y \;\&\; x \neq y$.

Pareto actually introduced an order relation to find an optimal solution to a multi-objective problem like in one dimension, the ordering of real numbers is used to find an optimal solution.

2.1. **Single and multi-objective optimization - technical aspects.** Optimization requires a means to compare any two entities to find out which one is better for our purpose than the other. In this comparison qualities expressed in terms of scalar or vector values are compared and order relations perform the comparisons.

Comparison of scalar values uses the usual order relation of numbers (single objective optimization) while comparing vectors uses order relation (reflexive, asymmetric and transitive relation) on vectors (multi-objective optimization) carrying the basis of comparison needed by the analyst. We formulate the order relation used in vector spaces.

Definition 2. Let H be a set then a relation $\preceq$ between the elements of H is an order relation on H if it fulfils:
1. $x \preceq x$, $\forall\, x \in H$ (reflexive);
2. $x \preceq y \;\&\; y \preceq x \Rightarrow x = y$, $\forall\, x, y \in H$ (asymmetric);
3. $x \preceq y \;\&\; y \preceq z \Rightarrow x \preceq z$, $\forall\, x, y, z \in H$ (transitive)

When $H = V$ is a vector space then the order relationship is assumed to fulfil some compatibility properties to make handling of expressions with vector operations combined with order relations easier. This part of the paper draws from Jahn's book (10) and related literature.

Definition 3. Let V be a vector space over R and let $\preceq$ be an order relation between the elements of V. The order relation $\preceq$ is compatible with the vector structure of V if
1. $\forall\, x, y, a \in V$, $x \preceq y \Rightarrow x + a \preceq y + a$ (additive translation invariance)
2. $\forall\, x, y \in V \;\&\; a \in R^{+}$, $x \preceq y \Rightarrow x * a \preceq y * a$ (multiplicative translation invariance).

A vector space with an order relation compatible with the vector structure is an ordered vector space and it has some important properties.

2.2. **Implication of the translation invariance.** An order relation compatible with the vector structure is translation invariant therefore it can be checked at the zero of the vector space. Precisely

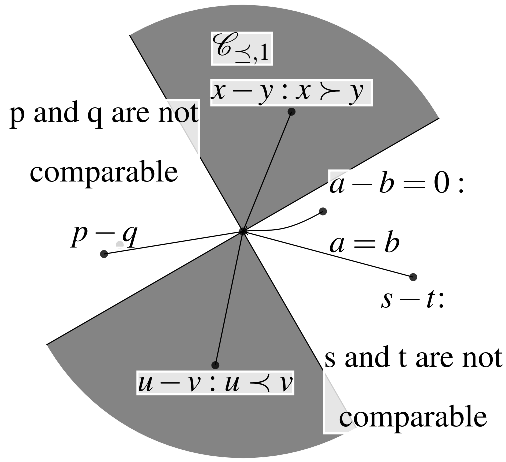

FIGURE 2. Vector comparison: $a \prec b$ or $a = b$ or $b \prec a$ or a and b are not comparable.

Theorem 1. *Let V be a vector space and let $\preceq$ be an order relation on V compatible with the vector structure. If $C_{\preceq} := \{0 \preceq x\}$ is the set of those vectors which are greater than or equal to 0 then from the compatibility conditions 1. and 2. follows that $C_{\preceq}$ is a convex cone with zero as its tip and $\forall\, x, y \in V$, $x \preceq y \Leftrightarrow 0 \preceq y - x \Leftrightarrow y - x \in C_{\preceq}$.*

The convexity of $C_{\preceq}$: Let $a, b \in C_{\preceq} \Rightarrow 0 \preceq a$ & $0 \preceq b$ then $0 \preceq a \Rightarrow b \preceq a + b$ hence $0 \preceq b$ & $b \preceq a + b \Rightarrow 0 \preceq a + b$ by transitivity. Applying this to ca and $(1-c)b$ with $c \in [0,1]$ gives the convexity.

Corollary 1. *In **one dimensional vector space** $C_{\preceq} = [0, \infty)$ or $C_{\preceq} = (-\infty, 0]$. We select the cone that contains the non-negative numbers. Hence there is only one order relation compatible with the vector structure in a one dimensional vector space.*
In one dimensional vector space any two elements $x, y \in V$ are comparable which means $x \preceq y$ or $y \preceq x$ is true.
*In **two or more dimensional vector space** there are infinite inequivalent order relations compatible with the vector structure. In two or more dimensions in addition to $x, y \in V \Rightarrow x \preceq y$ or $y \preceq x$ the elements x, y may not be comparable (see Figure 2, points p, q or s, t are not comparable).*

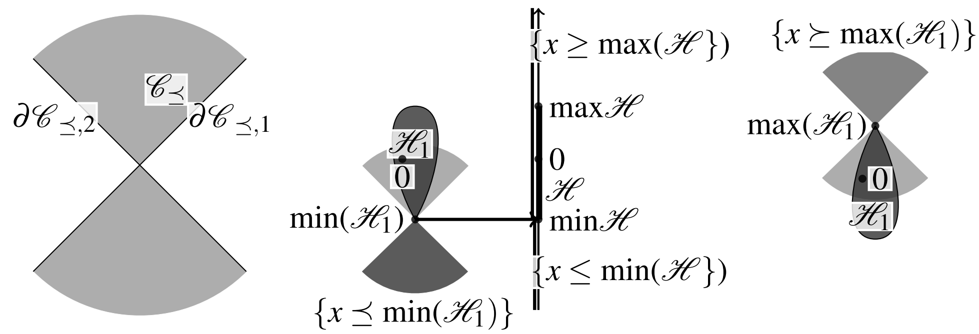

FIGURE 3. Strong multiobjective maximum and minimum like in single objective optimization.

2.3. **Optimization concepts.** The single objective function optimization mostly targets to find the best objective function value among all possible values (minimum or maximum). This value is taken as a control action found by the optimization process. Precisely,

Definition 4. Let $H \subset R$. Then

$\max(H) \in H$ fulfils $x \leq \max(H)$, $\forall\, x \in H$;

$\min(H) \in H$ fulfils $x \geq \min(H)$, $\forall\, x \in H$;

Definition 5. Let $H \subset R^p$ and let $\mathscr{C}_{\preceq}$ be the positive cone of the order relation. Then

$\max_{\mathscr{C}_{\preceq}}(H) \in H$ fulfils $x \preceq \max_{\mathscr{C}_{\preceq}}(H)$, $\forall\, x \in H$;

$\min_{\mathscr{C}_{\preceq}}(H) \in H$ fulfils $\min_{\mathscr{C}_{\preceq}}(H) \preceq x$, $\forall\, x \in H$;

Figure 3 shows the practical appearance of multi-objective maximum/minimum. As maximum or minimum must be the tip of the cone moreover, this maximum/minimum must be comparable with the whole set hence the whole set must be inside the cone. This gives a drop shape to the set and strongly delimits range of usability of the multi-objective optimization.

2.3.1. *Maximal and minimal optimum.* The maximum or minimum solutions to single objective optimization problems give global solutions - a best choice against any other selection. We will now investigate the case when a process under the control of a multi-objective optimal solution drifts out to a state not comparable with the optimal solution in use.

The nature of non-comparable vectors. Let us consider the situation on figure 4 where a turning vehicle's movement is analysed. To cover the centripetal force from the static friction of the tyres with the soil, the centripetal force must not exceed the available maximum static friction

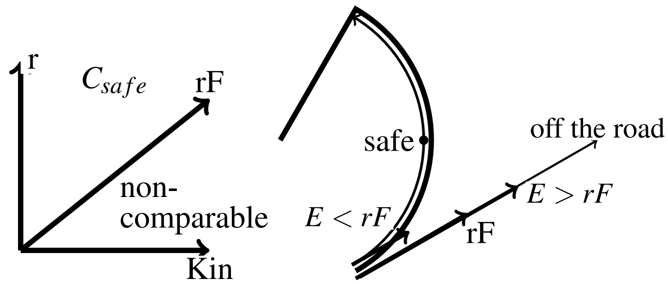

FIGURE 4. Safe turning formulated with the kinetic energy and the turning radius.

F_{fr}. Using the formula for the needed force to turn at a radius r while having v speed gives us $F_{cp} = \frac{mv^2}{r} = \frac{2E_{kin}}{r} < F_{fr}$. This leads to the safety cone: The turning is safe if $E_{kin} < r\frac{F_{fr}}{2} = rF$ (F is a constant defined by the formula). As figure 4 shows, a non-comparable objective vector leads to disaster. The car movement becomes independent from the intentions of the controller. He/she has no effect on the movement of a car in such a situation since the brake, the accelerator and the steering are all based on the friction force which is switched off. Many other examples show the same situation. Therefore, we may conclude:

Conclusion 1. *A selected optimal solution is valid only over the situations with dominated objective vectors. Hence each optimal solution has a domain where it acts. An attempt to use it outside the domain may lead to a disaster.*

Using the conclusion that the optimal solution controls dominated situations only we will introduce maximal and minimal elements in an object space such that such elements are in the domain and dominate all comparable elements in the domain.

Definition 6. Let $H \subset R^p$ and let $\mathscr{C}_{\preceq}$ be the positive cone of the order relation. Then the maximal and minimal elements are

$\max_{w,\mathscr{C}_{\preceq}}(H) \in H$ fulfils $x \preceq \max_{w,\mathscr{C}_{\preceq}}(H)$, $\forall\, x \in H$ comparable with $\max_{w,\mathscr{C}_{\preceq}}(H)$;

$\min_{w,\mathscr{C}_{\preceq}}(H) \in H$ fulfils $\min_{w,\mathscr{C}_{\preceq}}(H) \preceq x$, $\forall\, x \in H$ comparable with $\min_{w,\mathscr{C}_{\preceq}}(H)$;

Remark 1. As Figure 5 shows, $\mathscr{H} \cap \{x \succ \max(\mathscr{H})\} = \emptyset$ which means there is no larger element in $\mathscr{H}$ than the maximal element. Similarly, $\mathscr{H} \cap \{\min(\mathscr{H}) \succ x\} = \emptyset$ which means that there is no smaller element in H than a minimal element. Hence $\mathscr{H}$ may contain elements not

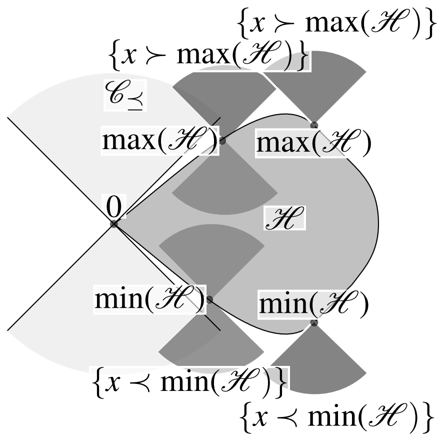

FIGURE 5. Multi-objective maximal and minimal solutions.

comparable with the maximal/minimal elements or elements less than or equal to the maximal element or greater than or equal to the minimal element.

3. OPTIMAL FRONTIER.

We essentially concluded in the previous sections that the maximal or minimal optimality concepts suit more the multi-objective optimization because these concepts require to fulfil the least conditions equivalent to the guarantee that there is no better option in the object space.

These solutions must be boundary points of the objective space since they themselves must be in the objective space and any neighbourhood of them must contain points from outside the objective space (see the reasoning in remark 1). Therefore, the objective space will be assumed to be a closed set. It also follows from the discussion represented on figure 5 that there are many minimal or maximal points to a problem.

Any two different optimal solutions may be either comparable or non-comparable. When comparable, out of two maximal solutions the larger takes care of the smaller one and out of two minimal solutions the smaller one takes care of the other one.

Definition 7. If $C_{\preceq}$ is a positive cone on the objective space $H \subset R^p$ then $\forall\, x, y \in H$, $C_x := x + C_{\preceq}$ & $C_y := y + C_{\preceq}$ are comparable if $x \in C_y \vee y \in C_x$ The relation C_x is comparable to C_y is denoted by $C_x \sim_c C_y$.

Lemma 2. *The relation $\sim_c$ is an equivalence relation in $\{x + C_{\preceq} \,|\, x \in H\}$.*

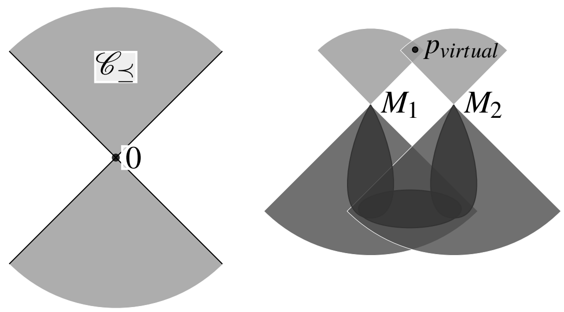

FIGURE 6. Two non-comparable Multi-objective maximums.

If we form the equivalence classes of all solution cones over H and form the union of cones in each of the equivalence class, we get a system of optimal cones equivalent to the original system of optimal solutions which are pairwise non-comparable. Hence comparable pairs of solutions can be eliminated by selecting one out of any two. Therefore, we may assume that any two different optimal solutions are non-comparable. Example of non-comparable solutions is shown on figure 6. Non-comparable maximal or minimal solutions occur when non of the tips of such cones is in the other cone.

Definition 8. Let $H \subset R^p$ be the objective space and let $\mathscr{C}_{\preceq}$ be the positive cone of the order relation. Then the optimal frontier $\mathscr{F}_{opt,max}(H) := \{\max_{w,\mathscr{C}_{\preceq}} | \max_{w,\mathscr{C}_{\preceq}} = u \in \partial H\}$. Similarly, $\mathscr{F}_{opt,min}(H) := \{\min_{w,\mathscr{C}_{\preceq}} | \min_{w,\mathscr{C}_{\preceq}} = u \in \partial H\}$.

The concept originates from Pareto in optimality theory developed in 1910 and was published in (6). His concept of dominating elements (definition 1) actually shows that his multi-objective solutions dominate points other than the maximal/minimal solutions. However, Pareto's order relation is defined by the cone of positive quadrant (in p-dimensional case $C_{Pareto}; = \prod_{j=1}^{n} [0, \infty)_j$ which is a special selection of order relation.

To cover all points of an objective space with dominating cones requires to create optimal strategy for all the border points in the direction. Such family of solutions is the optimal frontier. Figure 7 shows the structure of optimal frontier.

Note that having an infinite system of optimal strategies does not make life easy because one should be used at a time. This leads us to the problem of synthesis which offers an optimal solution to each situation.

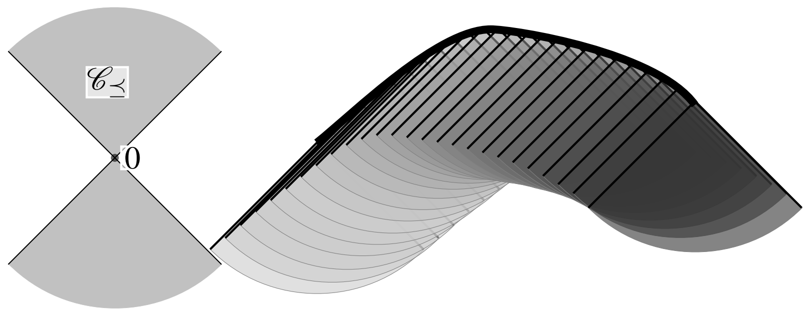

FIGURE 7. Infinite number of multiobjective solutions.

4. SYNTHESIS OF OPTIMAL SOLUTIONS.

4.1. **The switch cone.** When a system is controlled by an optimal strategy this strategy has to be changed when the system movement reaches the boundary point of the domain of the optimal control. If the state point of the system is close to the boundary of the domain of the newly selected strategy then soon after switching one may expect another switching. Hence knowing the position of the current state we should select a new optimal strategy where the current state is relatively far from the boundary of the newly selected optimal behaviour. This requirement for the selection is solved by introducing a switch-cone as shown on figure 8. Figure 9 shows the use of switching cones on the optimal frontier.

4.2. **Strategy on compact subsets of objective space.** Any real system operates in a sub-optimal way and the strategy directs the process towards the optimal operation. There are two reasons for this: The information as basis of decisions is never complete and the operations are always subject to random effects. In addition, having infinite set of strategies, at the border points one has to change strategy for each change of working point which already and impossible requirement from practical point of view. Therefore, we will assume that a system moves within a compact subset of the object space at a distance from the boundary. This can be covered by switching cones hence reaching the boundary of the domain of a strategy, there is a switching cone containing the current state vector hence the system can switch to the new strategy. Figure 10 shows the situation.

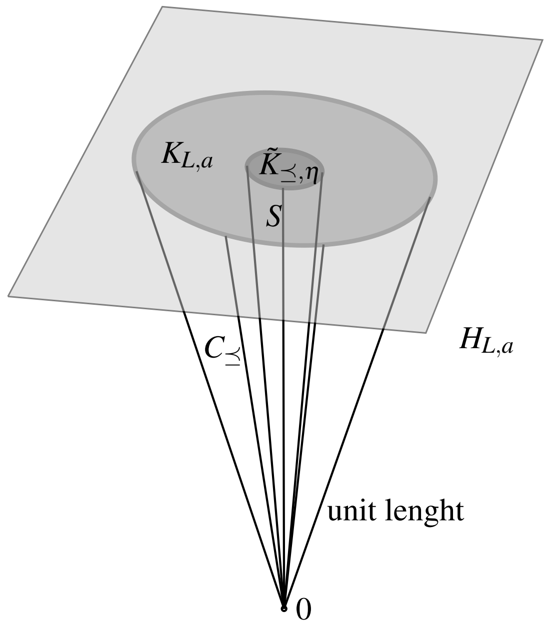

FIGURE 8. The definition of the switchcone.

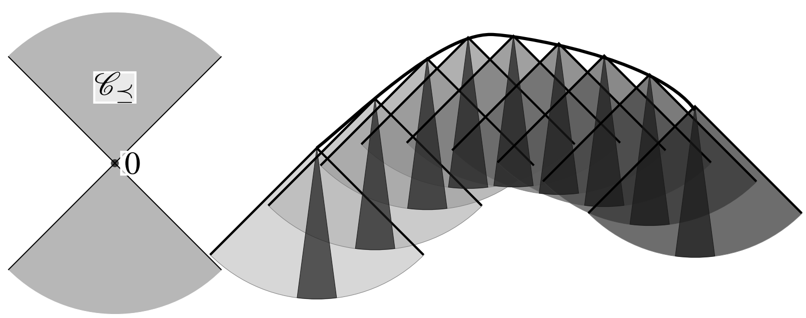

FIGURE 9. Introduction of switching zones to select more lasting strategies.

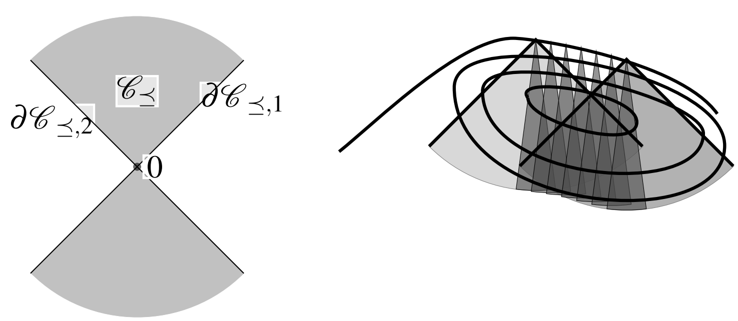

FIGURE 10. Building the objective set from an increasing system of compact sets, we can cover each set with finite system of switching cones. This gives a locally finite system of strategies at each point to select.

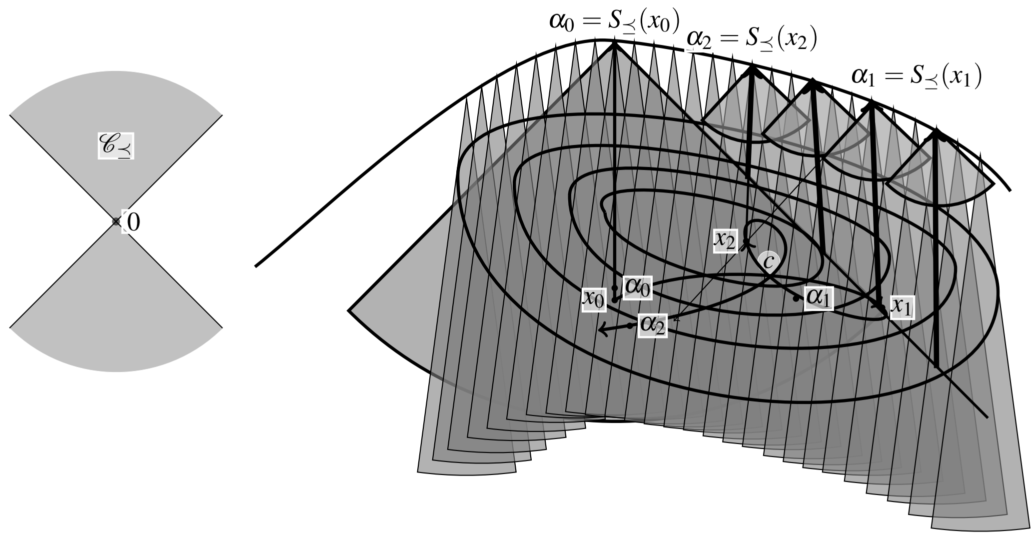

FIGURE 11. The constructed synthesis function and its use.

4.3. **The synthesis and its use.** The interior of a closed set can be obtained as a countable union of ascending sequence of compact sets. If there is a member of such a sequence which is needed for our operations, then there is a finite subset of the frontier points such that the corresponding switching cones cover the selected compact set. These cones can be combined into a finite pairwise disjoint covering of the selected compact set so that reaching the boundary point of the domain of the currently used optimal strategy, there is a unique strategy to continue with. The situation is shown on figure 11. This synthesis facilitates stability and asymptotic analysis of the solution if a dynamical system (ordinary differential equations etc.) model is created to describe the system's operation.

5. SOLUTION METHODS.

An arsenal of methodologies and techniques has been developed to solve multi objective optimization problems. We noted earlier in the statement of problem to this presentation that it is very interesting to generate the local optimal solutions but to successfully give an instructional guide on how to run a multi-objective system is much more complex than generating these optimal solutions.

In this lecture we give a comparison among three methods: Scalarization, Pareto's method and the general multi-objective solutions. Figure 12 gives a comparison among the order rela-

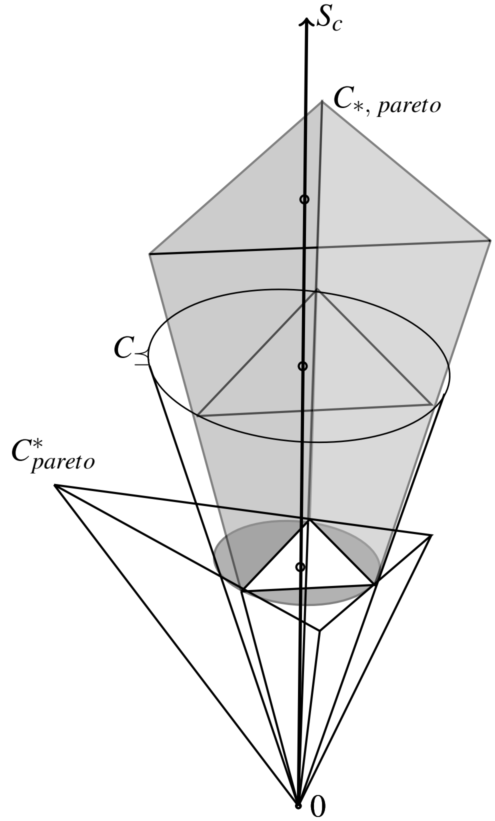

FIGURE 12. The tower shows four ordering cones: The positive cone $\mathscr{C}_{\preceq}$, outer and inserted pareto cones $\mathscr{C}^*_{paret}, \mathscr{C}_{*,paret}$ and the scalar cone S_c.

tion, Pareto and scalariztion methods.

In line with the multi-objective optimization, the existence of an order relation is defined in terms of a convex positive cone $C_{\preceq}$. It is known that a convex cone is intersection of hyperplanes and any finite subsystem of these hyperplanes contains $C_{\preceq}$. If we select n such hyperplanes, their intersection is a cone spanned by a polygon the sides of this cone gives a pareto basis (with axes not parallel with the original basis). Look at the cone $C^{*\,pareto} \supset C_{\preceq}$. Also, $C_{\preceq}$ contains a polygon base cone $C_{*\,pareto} \subset C_{\preceq}$. Finally all contains S_c straight line which is a one dimensional cone.

We state a lemma to show the relationship among the four cones on Figure 12.

Lemma 3. *Let $C_1, C_2 \subset R^p$ be positive cones such that $C_1 \supset C_2$ over an objective space H. Then 1. $max_{C_1}(H) \in H \Leftrightarrow \{max_{C_1}(H) \prec x \in C_1\} = \emptyset \Rightarrow \{max_{C_1}(H) \prec x \in C_2\} = \emptyset$ holds. Hence or otherwise, $max_{C_1}(H)$ is maximal with respect to C_1 implies $max_{C_1}(H)$ is maximal with respect*

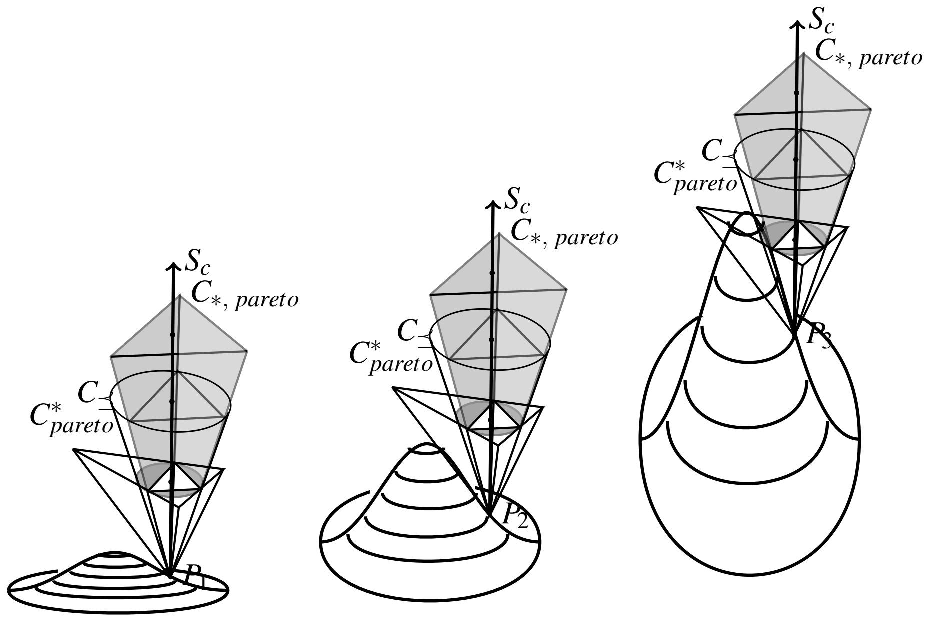

FIGURE 13. The figure shows the relationship between the set of objective values and the existence of the various maximal points: One may notice that S_c optimum always exists while the external pareto, the positive cone and the internal Pareto may not exist.

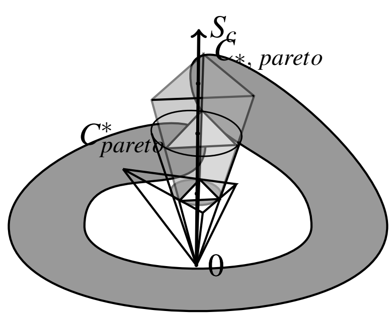

FIGURE 14. In this case there is no multi-objective optimal solution, not even for S_c.

to C_2*;*

2. If $max_{C_2}(H)$ *is maximal and* $\{max_{C_2}(H) \prec x \in C_1\} = \emptyset$ *then* $max_{C_2}(H)$ *is maximal with respect to* C_1*. The same holds for minimal elements in the opposite direction.*

In short, if $C_1 \supset C_2$, then $\forall\ max_{C_1}(H) \Rightarrow max_{C_1}(H)$ is maximal with respect to C_2 also.

Corollary 2. *In the chain of positive cones a maximal element with respect to any positive cone is maximal with respect to S_c. The same holds for minimal elements.*

The corollary states that scalarization gives solution to a maximisation/minimisation problem, but out of the scalarized solutions one has to select the maximal/minimal ones with respect to the selected positive cone.

Figure 13 shows that the four positive cones represent different orderings, and on different objective spaces they give different results. Figure 14 shows that there can be a case such that there is no optimal solution inside an enclosed loop. Maximal solutions will exist if the tip of the cone group goes on the top of the boundary points.

Conclusion 2. *In this paper we pointed out that the significant differences between single and multi-objective optimizations originate from the existence of a unique complete ordering compatible with the vector structure in one dimension against the higher dimensional vector spaces, infinite inequivalent partial orderings compatible with the vector structure. The existence of non-comparable vectors in higher dimension lead to limited concept of maximum/minimum and infinite non-comparable inequivalent optimal solutions to a problem. This leads to the necessity of synthesis techniques for the creation of a practical control guide for the user.*

Remark:

This paper is a summary of some of the results from the Ph.D. thesis (15) of Obikwere, Clare A. supervised by Dr. Effanga, E. O. and Prof. Z. Lipcsey.

Conflict of Interests

The authors declare that there is no conflict of interests.

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