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A CLASS OF CAYLEY DIGRAPH STRUCTURES INDUCED BY LOOPS

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Abstract. In this paper, we generalize the results in [8] to produce a new classes of Cayley digraph structures induced by loops.

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1. Introduction

A *binary relation* on a set V is a subset E of $V \times V$. A *digraph* is a pair (V, E) where V is a non empty set (called vertex set) and E is a binary relation on V. The elements of E are called edges. Let V be a non empty set and let $E_1, E_2, \ldots, E_n$ be mutually disjoint binary relations on V. Then the $(n+1)$-tuple $G = (V; E_1, E_2, \ldots, E_n)$ is called a digraph structure[8]. The elements of V are called vertices and the elements of E_i are called E_i-edges. The following definition were introduced in [8].

A digraph structure $(V; E_1, E_2, \ldots, E_n)$ is called (i)$E_1E_2\cdots E_n$-*trivial* if $E_i = \emptyset$ for all i, and E_i- *trivial* if $E_i = \emptyset$ (ii)$E_1E_2\cdots E_n$- *reflexive* if for all $x \in G$, $(x, x) \in E_i$ for some i, and E_i- *reflexive* if for all $x \in V$, $(x, x) \in E_i$(iii) $E_1E_2\cdots E_n$- *symmetric* if $E_i = E_i^{-1}$ for

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all i, and E_i- *symmetric* if $E_i = E_i^{-1}$(iv) $E_1E_2\cdots E_n$- *anti symmetric*, if $(x, y) \in E_i$ and $(y, x) \in E_i$ implies $x = y$ for all i, and E_i- anti symmetric if $(x, y) \in E_i$ and $(y, x) \in E_i$ implies $x = y$ (v) $E_1E_2\cdots E_n$- *transitive* if for every i and j, $E_i \circ E_j \subseteq E_k$ for some k, and E_i transitive if $E_i \circ E_i \subseteq E_i$ (vi) an $E_1E_2\cdots E_n$- *hasse diagram* if for every positive integer $n \geq 2$ and every $v_0, v_1, \ldots, v_n$ of V, $(v_i, v_{i+1}) \in \cup E_i$ for all $i = 0, 1, 2, \ldots, n-1$, implies $(v_0, v_n) \notin E_i$ for all i, and E_i- *hasse diagram* if for every positive integer $n \geq 2$ and every $v_0, v_1, \ldots, v_n$ of V, $(v_i, v_{i+1}) \in E_i$ for all $i = 0, 1, 2, \ldots, n-1$, implies $(v_0, v_n) \notin E_i$, (viii)$E_1E_2\cdots E_n$- *complete* if $\cup E_i = V \times V$, and E_i *complete* if $E_i = V \times V$.

A digraph structure $(V; E_1, E_2, \ldots, E_n)$ is called (i) an $E_1E_2\cdots E_n$- quasi ordered set if it is both $E_1E_2\cdots E_n$- reflexive and $E_1E_2\cdots E_n$ -transitive (ii)an $E_1E_2\cdots E_n$ - partially ordered set if it is $E_1E_2\cdots E_n$- anti symmetric and $E_1E_2\cdots E_n$- quasi ordered set. Similarly, we can define E_i quasi ordered set and E_i partially ordered set as in the case of ordinary relations.

An $E_1E_2\cdots E_n$- walk of length k in a digraph structure is an alternating sequence $W = v_0, e_0, v_1, \ldots, e_{k-1}, v_k$, where $e_i = (v_i, v_{i+1}) \in \cup E_i$. An $E_1E_2\cdots E_n$ -walk W is called a $E_1E_2\cdots E_n$- path if all the internal vertices are distinct. We use notation $(v_0, v_1, v_2, \ldots, v_n)$ for the $E_1E_2\cdots E_n$ - path W. As in digraphs, we define E_i- walk and E_i- path. For example, an E_i- path between two vertices u and v consists of only E_i- edges.

A digraph structure $(V; E_1, E_2, \ldots, E_n)$ is called (i) $E_1E_2\cdots E_n$- *connected* if there exits at least one $E_1E_2\cdots E_n$- path from v to u for all $u, v \in V$, (ii)$E_1E_2\cdots E_n$- quasi connected if for every pair of vertices x, y there is a vertex z such that there is an $E_1E_2\cdots E_n$-path from z to x and an $E_1E_2\cdots E_n$-path from z to y, (iii) $E_1E_2\cdots E_n$- *locally connected* iff for every pair of vertices $u, v \in V$ there is an $E_1E_2\cdots E_n$ - path from v to u whenever there is an $E_1E_2\cdots E_n$ - path from u to v and (iv) $E_1E_2\cdots E_n$- *semi connected* for every pair of vertices u, v, there is an $E_1E_2\cdots E_n$- path from u to v or an $E_1E_2\cdots E_n$- path from v to u.

A digraph structure $(V; E_1, E_2, \ldots, E_n)$ is called E_i *-connected* if there exits at least one E_i path from v to u for all $u, v \in V$. Similarly we can define E_i quasi connected, E_i

-locally connected and E_i - semi connected digraph structures.

The $E_1E_2\cdots E_n$ - distance between two vertices x and y in a digraph structure G is the length of the shortest $E_1E_2\cdots E_n$- path between x and y, denoted by $d_{1,2,3,\ldots,n}(x,y)$. Let $G=(V;E_1,E_2,\ldots,E_n)$ be a finite $E_1E_2\cdots E_n$- connected digraph structure. Then the $E_1E_2\cdots E_n$ diameter of G is defined as $d(G)=\max_{x,y\in G}\{d_{1,2,3,\ldots,n}(x,y)\}$. Similarly we can define E_i distance and E_i diameter as in digraphs.

Two digraph structures $(V_1;E_1,E_2,\ldots,E_n)$ and $(V_2;R_1,R_2,\ldots,R_m)$ are said to be isomorphic if (i) $m=n$ and (ii) there exits a bijective function $f\colon V_1\to V_2$ such that $(x,y)\in E_i \Leftrightarrow (f(x),f(y))\in R_i$. This concept of isomorphism is a generalization of isomorphism between two digraphs. An isomorphism of a digraph structure onto itself is called an automorphism. A digraph structure $(V;E_1,E_2,\ldots,E_n)$ is said to be vertex-transitive if, given any two vertices a and b of V, there is some digraph automorphism $f:V\to V$ such that $f(a)=b$. Let $(V;E_1,E_2,\ldots,E_n)$ be a digraph structure and let $v\in V$. Then the $E_1E_2\cdots E_n$ out-degree of u is $|\{v\in V:(u,v)\in\cup E_i\}|$ and $E_1E_2\cdots E_n$ in-degree of u is $|\{v\in V:(v,u)\in\cup E_i\}|$. Similarly we can define the E_i out- degree and E_i in- degree as in the case of digraphs.

Let $(V_1;E_1,E_2,\ldots,E_n)$ be a digraph structure. A vertex $v\in G$ is called an $E_1E_2\cdots E_n$ -source if for every vertex $x\in G$, there is an $E_1E_2\cdots E_n$ - path from v to x. Similarly a vertex $u\in G$ is called an $E_1E_2\cdots E_n$- sink if for very vertex $y\in G$ there is an $E_1E_2\cdots E_n$- path from y to u. As in digraphs, we define E_i - source and E_i - sink. Let $(V_1;E_1,E_2,\ldots,E_n)$ be a digraph structure and let $v\in G$. Then the $E_1E_2\cdots E_n$ reachable set $R_{1,2,3,\cdots,n}(u)$ is $\{x\in G:$ there is an $E_1E_2\cdots E_n$- path from u to $x\}$. Similarly, the $E_1E_2\cdots E_n$- antecedent set $Q_{1,2,\ldots,n}(u)$ is defined as

$$Q_{1,2,\ldots,n}(u)=\{x\in G: \text{ there is an } E_1E_2\cdots E_n\text{- path from } x \text{ to } u\}.$$

As in the case of digraphs, we can define the E_i- reachable set and E_i-antecedent set of a vertex.

A non empty set G, together with a mapping $*:G\times G\longrightarrow G$ is called a *groupoid.* The mapping $*$ is called a *binary operation* on the set G. If $a,b\in G$, we use the symbol

ab to denote $*(a,b)$. A groupoid $(G,*)$ is called a *quasigroup*, if for every $a, b \in G$, the equations, $ax = b$ and $ya = b$ are uniquely solvable in G [6]. This implies both left and right cancelation laws. A quasigroup with an identity element is called a *loop*. Observe that a loop is a weaker algebraic structure than a group.

A subset A of a loop G is said to be a *right associative subset of* G($\mathcal{R}$ associative), if for every $x, y \in G$, $(xy)A = x(yA)$. This means, if $x, y \in G$ and $a \in A$, then $(xy)a = x(ya')$ for some $a' \in A$. Observe that the $\mathcal{R}$ associative law not only allows to interchange the positions of parenthesis, the two elements that are on the left should be in G and they will be same on both sides, the rightmost element in the left hand side is in A and is changed to another element $a' \in A$ as the right most element in the right side [12].

Here we have the following result:

Theorem 1.1.([9]) *Let A and B be $\mathcal{R}$ associative subsets of a loop G. Then AB is also $\mathcal{R}$ associative.*

3. Cayley digraph structures induced by loops

In [11] the authors introduced a class of Cayley digraph structures induced by groups. In this paper, we introduce a class of Cayley digraph structures induced by loops. These class of Cayley digraphs structures can be viewed as a generalization of those obtained in [11]. Further, many graph properties are studied in terms of algebraic properties.
We start with the following definition:

Definition 2.1. *Let G be a loop and $S_1, S_2, \ldots, S_n$ be mutually disjoint $\mathcal{R}$ associative subsets of G. Then Cayley digraph structure of G with respect to $S_1, S_2, \ldots, S_n$ is defined as the digraph structure $X = (G; E_1, E_2, \ldots, E_n)$, where*

$$E_i = \{(x, y) : z \in S_i\}$$

where z denotes the solution of the equation $y = xz$.

The sets $S_1, S_2, \ldots, S_n$ are called connection sets of X. The Cayley digraph structure of G with respect to $S_1, S_2, \ldots, S_n$ is denoted by $\mathrm{Cay}(G; S_1, S_2, \ldots, S_n)$.

In this paper we may use the following notations:

(1) Let $S_1, S_2, \ldots, S_n$ be subsets of a loop G, then we may define the product $S_1, S_2, \ldots, S_n$ as follows:

$$S_1S_2 \ldots S_n = \{(((s_1s_2)s_3)\ldots)s_n : s_i \in S_i, i = 1, 2, \ldots, n\}.$$

If $S_1 = S_2 = \cdots = S_n = S$, we denote the above product as S^n.

(2) Let A_k be the union of set of all k products of the form $S_{i_1}S_{i_2}\cdots S_{i_k}$ from the set $\{S_1, S_2, \ldots, S_n\}$. Then $\bigcup_k A_k$. is denoted by $[S]$.

(3) Let D be a subset of G. We define $D_\ell = \{z_\ell : z_\ell z = 1 \text{ for some } z \in D\}$, where 1 is the identity element in G.

(4) Let A be a subset of a loop G, then the semi group generated by A is denoted by $< A >$.

Theorem 2.2. *If G is a loop and let $S_1, S_2, \ldots, S_n$ are mutually disjoint $\mathcal{R}$ associative subsets of G, then the Cayley digraph structure Cay$(G; S_1, S_2, \ldots, S_n)$ is vertex transitive.*

Proof. Let a and b be any two arbitrary elements in G. Define a mapping $\varphi : G \to G$ by

$$\varphi(x) = (b/a)x \text{ for all } x \in G.$$

where (b/a) denotes the solution of the equation $b = za$. This mapping defines a permutation of the vertices of $\mathrm{Cay}(G; S_1, S_2, \ldots, S_n)$. It is also an automorphism. Let $x, y \in G$ such that $y = xz$. Note that

$$(x, y) \in E_i \Leftrightarrow z \in S_i \text{ for some } i.$$

The equation $y = xz$ can be written as

$$\begin{aligned}(b/a)y &= (b/a)(xz)\\ &= ((b/a)x)z' \quad \text{for some } z' \in S_i\end{aligned}$$

The above equation tells us that $((b/a)x, (b/a)y) \in E_i$. That is, $(\varphi(x), \varphi(y)) \in E_i$. Similarly, assume that $(\varphi(x), \varphi(y)) \in E_i$. Then $(b/a)y = ((b/a)x)z$ for some $z \in S_i$. This implies that $(b/a)y = (b/a)(xz')$ for some $z' \in S_i$. By left cancellation law, we obtain $y = xz'$. This tells us that $(x, y) \in E_i$. Also we note that $\varphi(a) = (b/a)a = b$. Hence $\mathrm{Cay}(G; S_1, S_2, \ldots, S_n)$ is vertex transitive.

Proposition 2.3 *Cay*$(G; S_1, S_2, \ldots, S_n)$ *is an* $E_1E_2 \cdots E_n$ *-trivial digraph structure if and only if* $S_i = \emptyset$ *for all* i.

Proof. By definition, $\mathrm{Cay}(G; S_1, S_2, \ldots, S_n)$ is $E_1E_2 \cdots E_n$ - trivial if and only if $E_i = \emptyset$ for all i. This implies that $S_i = \emptyset$ for all i.

Proposition 2.4 *Cay*$(G; S_1, S_2, \ldots, S_n)$ *is an* E_i*- trivial digraph structure if and only if* $S_i = \emptyset$.

Proposition 2.5 *Cay*$(G; S_1, S_2, \ldots, S_n)$ *is* $E_1E_2 \cdots E_n$*- reflexive if and only if* $1 \in S_i$ *for some* i.

Proof. Assume that $\mathrm{Cay}(G; S_1, S_2, \ldots, S_n)$ is an $E_1E_2 \cdots E_n$- reflexive digraph structure. Then for every $x \in G$, $(x, x) \in E_i$ for some i. This implies that the equation $x = xz$ has a unique solution in S_i for some i. That is, $1 \in S_i$ for some i.

Conversely, assume that $1 \in S_i$ for some i. This implies for each $x \in G$, $(x, x) \in E_i$ for some i. That is, $(x, x) \in \cup E_i$ for all $x \in G$.

Proposition 2.6 *Cay*$(G; S_1, S_2, \ldots, S_n)$ *is* $E_1E_2 \cdots E_n$*- symmetric if and only if* $S_i = S_{i_\ell}$ *for all* i.

Proof. First, assume that $\mathrm{Cay}(G; S_1, S_2, \ldots, S_n)$ is an $E_1E_2 \cdots E_n$ -symmetric digraph structure. Let $a \in S_i$. Then $(1, a) \in E_i$. Since $\mathrm{Cay}(G; S_1, S_2, \ldots, S_n)$ is symmetric $(a, 1) \in E_i$. This implies that the equation $1 = at$ has a solution in S_i. That is $a \in S_{i_\ell}$. Hence $S_i \subseteq S_{i_\ell}$. Similarly, we can prove that $S_{i_\ell} \subseteq S_i$.

Conversely, assume that $S_i = S_{i_\ell}$ for all i. Suppose that $(x, y) \in E_i$. Then the equation $y = xz$ has a solution in S_i. That is $z \in S_i$. Consider the equation $x = yt$. This equation can be written as:

$$xz = (yt)z$$
$$i.e., \quad y = y(tz') \text{ for some } z' \in S_i$$
$$i.e., \quad y1 = y(tz')$$
$$i.e., \quad 1 = tz' \text{ (by left cancelation law).}$$

The above equation tells us that $t \in S_{i_\ell}$. Since $S_i = S_{i_\ell}$, it follows that $t \in S_i$. Hence the equation $x = yt$ has a solution in S_i. That is $(y, x) \in E_i$.

Proposition 2.7 *Cay*$(G; S_1, S_2, \ldots, S_n)$ *is an* $E_1E_2 \cdots E_n$ *- transitive if and only if for every* i, j, $S_iS_j \subseteq S_k$ *for some* k.

Proof. First, assume that Cay$(G; S_1, S_2, \ldots, S_n)$ is $E_1E_2 \cdots E_n$ - transitive. Let $x \in S_iS_j$. Then $x = z_1z_2$ for some $z_1 \in S_i$ and $z_2 \in S_j$. This implies that $(1, z_1) \in E_i$ and $(z_1, z_1z_2) \in E_j$. Since Cay$(G, S_1, S_2, \ldots, S_n)$ is transitive $(1, z_1z_2) \in E_k$ for some k. That is $z_1z_2 \in S_k$. Hence $S_iS_j \subseteq S_k$ for some k.

Conversely assume that for each i, j, $S_iS_j \subseteq S_k$ for some k. Let x, y and $z \in G$ such that $y = xt_1$ and $z = yt_2$. If $(x, y) \in E_i$ and $(y, z) \in E_j$, then $t_1 \in S_i$ and $t_2 \in S_j$. Note that the equation $z = yt_2$ can be written as:

$$z = (xt_1)t_2$$
$$= x(t_1t_2') \text{ for some } t_2' \in S_j$$
$$= xt_3 \text{ where } t_3 = t_1t_2'$$

Note that $t_3 \in S_iS_j$. Since $S_iS_j \subseteq S_k$, $t_3 \in S_k$. That the equation $z = xt$ has a solution t_3 in S_k. Hence Cay$(G; S_1, S_2, \ldots, S_n)$ is transitive.

Proposition 2.8 *Cay*$(G; S_1, S_2, \ldots, S_n)$ *is* $E_1E_2 \cdots E_n$*- complete if and only if* $G = \cup S_i$.

Proof. Suppose Cay$(G; S_1, S_2, \ldots, S_n)$ is $E_1E_2 \cdots E_n$- complete. Then for every $x \in G$, we have $(1, x) \in \cup E_i$. This implies that $x \in S_i$ for some i. This implies that $G = \cup S_i$. Conversely, assume that $G = \cup S_i$. Let x and y be two arbitrary elements in G such that

$y = xz$. Then $z \in G$. This implies that $z \in S_i$ for some i. That is, $(1, z) \in \cup E_i$. That is $(x, xz) = (x, y) \in \cup E_i$. This shows that Cay$(G; S_1, S_2, \ldots, S_n)$ is $E_1E_2 \cdots E_n$- complete.

Proposition 2.9 *Cay$(G; S_1, S_2, \ldots, S_n)$ is E_i- complete if and only if $G = S_i$.*

Proposition 2.10 *Cay$(G; S_1, S_2, \ldots, S_n)$ is $E_1E_2 \cdots E_n$- connected if and only if $G = [S]$.*

Proof. Suppose Cay$(G; S_1, S_2, \ldots, S_n)$ is $E_1E_2 \cdots E_n$- connected and let $x \in G$. Let $(1, y_1, y_2, \ldots, y_k, x)$ be a $E_1E_2 \cdots E_n$- path leading from 1 to x. Then we have, $y_1 = z_1, y_2 = y_1z_2, \ldots, y_k = y_{k-1}z_k, x = y_kz_{k+1}$ for some $z_j \in S_{i_j}, j = 1, 2, \cdots, k+1$. Note that the equation $x = y_kz_{k+1}$ can be written as

$$
\begin{aligned}
x &= (y_{k-1}z_k)z_{k+1} \\
&= ((y_{k-2}z_{k-1})y_{k-1}z_k)z_{k+1} \\
&= (z_1z_2) \cdots z_{k+1}
\end{aligned}
$$

The last equation tells us that $x \in S_{i_1}S_{i_2} \cdots S_{i_{k+1}}$ This implies that $x \in A$ for some $A \in [S]$. Since x is arbitrary, $G = [S]$.

Conversely, assume that $G = [S]$. Let x and y be any arbitrary elements in G. Let $y = xz$. Then $z \in G$ Then $z \in S_iS_j \cdots S_k$ for some $i, j, \ldots$ and k. This implies that $z = s_is_j \ldots s_k$ for some $i, j \ldots$ and k. Then clearly, $(1, s_i, s_is_j, \ldots, s_is_j \ldots s_k)$ is an $E_1E_2 \cdots E_n$ -path from 1 to z. That is, $(x, xs_i, xs_is_j, \ldots, xs_is_j \ldots s_k)$ is a $E_1E_2 \cdots E_n$- path from x to y. Hence Cay$(G; S_1, S_2, \ldots, S_n)$ is connected.

Proposition 2.11 *Cay$(G; S_1, S_2, \ldots, S_n)$ is E_i- connected if and only if $G =< S_i >$, where $< S_i >$ is the semi group generated by the set S_i.*

Proposition 2.12 *Cay$(G; S_1, S_2, \ldots, S_n)$ is $E_1E_2 \cdots E_n$- quasi connected if and only if $G = [S]_\ell[S]$.*

Proof. First, assume that Cay$(G; S_1, S_2, \ldots, S_n)$ is quasi connected. Let x be any arbitrary element in G. Then there exits a vertex $y \in G$ such that there is a path from y to 1, say, $(y, y_1, y_2, \cdots, y_n, 1)$ and a path from y to x, say,$(y, x_1, x_2, \ldots, x_m, x)$. Then we have the following system of equations:

$$
\begin{aligned}
y_1 &= yz_1 \text{ for some } z_1 \in S_{i_1}\\
y_2 &= y_1z_2 \text{ for some } z_2 \in S_{i_2}\\
y_3 &= y_2z_3 \text{ for some } z_3 \in S_{i_3}\\
&\vdots\\
1 &= y_nz_{n+1} \text{ for some } z_{n+1} \in S_{i_{n+1}}.
\end{aligned} \tag{1}
$$

and

$$
\begin{aligned}
x_1 &= yt_1 \text{ for some } z_1 \in S_{i_1}\\
x_2 &= x_1t_2 \text{ for some } z_2 \in S_{i_2}\\
x_3 &= x_2t_3 \text{ for some } z_3 \in S_{i_3}\\
&\vdots\\
x &= x_mt_{m+1} \text{ for some } z_{m+1} \in S_{i_{m+1}}
\end{aligned} \tag{2}
$$

Observe that equation (1) can be written as:

$$1 = y(w_1w_2 \dots w_{n+1}) \text{ for some } w_k \in S_{i_k}, k = 1, 2, \dots, n+1. \tag{3}$$

This implies that

$$y \in [S]_\ell \tag{4}$$

Similarly, equation (2) can be written as:

$$x = y(v_1v_2 \dots v_{m+1}) \text{ for some } v_k \in S_{i_k}, k = 1, 2, \dots, m+1. \tag{5}$$

From equations (4) and (5), we have

$$x \in [S]_\ell[S]. \tag{6}$$

Since x is arbitrary, $G = [S]_\ell[S]$.

Conversely, assume that $G = [S]_\ell[S]$. Let x and y be two arbitrary vertices in G. Let $y = xz$. Then $z \in G$. This implies that $z \in [S]_\ell[S]$. Then there exits $z_1 \in [S]_\ell$ and $z_2 \in [S]$ such that $z = z_1z_2$. $z_1 \in [S]_\ell$ implies that there exits $t_k \in S_{i_k}$ such that $1 = z_1(t_1t_2 \dots t_m)$.

That is,$1 = (((z_1r_1)r_2)\dots r_m)$ for some $r_m \in S_{i_k}, k = 1, 2, \dots, m$. This implies that $(z_1, z_1r_1, z_1r_1r_2, \dots, 1)$ is a path from z_1 to 1. That is,
$(yz_1, yz_1r_1, yz_1r_1r_2, \dots, y)$ is a path from yz_1 to y. Similarly, $z_2 \in [S]$ implies that there exits $a_k \in S_{i_k}$ such that $z_2 = a_1a_2\dots a_m$. Observe that $(z_2, a_1a_2, a_1a_2a_3, \dots, 1)$ is a path from z_2 to 1. That is, $(z_1z_2, z_1a_1a_2, a_1a_2a_3, \dots, z_1)$ is a path from z to yz_1. That is, $(yz, yz_1a_1a_2, ya_1a_2a_3, \dots, yz_1)$ is a path from x to yz_1. This implies that the digraph structure Cay$(G; S_1, S_2, \dots, S_n)$ is $E_1E_2\cdots E_n$- quasi connected.

Proposition 2.13 *Cay$(G; S_1, S_2, \dots, S_n)$ is E_i quasi connected if and only if $G =< S_i >_\ell< S_i >$.*

Proposition 2.14 *Cay$(G; S_1, S_2, \dots, S_n)$ is $E_1E_2\cdots E_n$- locally connected if and only if $[S] = [S]_\ell$.*

Proof.

Assume that Cay$(G; S_1, S_2, \dots, S_n)$ is $E_1E_2\cdots E_n$- locally connected. Let $x \in [S]$. Then $x \in A_m$ for some m. Then $x = s_is_j\dots s_m$. Let $x_0 = 1, x_1 = s_i, x_2 = s_is_j, \dots, x_m = s_is_j\dots s_m$. Then

$$(x_0, x_1, x_2, \dots, x_m)$$

is a path leading from 1 to x. Since Cay$(G; S_1, S_2, \dots, S_m)$- is locally connected, there exits a path from x to 1, say:

$$(x, y_1, y_2, \dots, y_m, 1)$$

This implies that

$$y_1 = xt_1 \text{ for some } t_1 \in S_{i_1}$$

$$y_2 = y_1t_2 \text{ for some } t_2 \in S_{i_2}$$

$$\vdots$$

$$1 = y_mt_{m+1} \text{ for some } t_{m+1} \in S_{i_n}$$

This implies that $1 = x(z_1z_2\cdots z_m)$ for some $z_k \in S_{i_k}, k = 1, 2, 3, \dots (m+1)$. That is $x \in [S]_\ell$. Hence $[S] \subseteq [S]_\ell$. Similarly, one can prove that $[S]_\ell \subseteq [S]$. Hence $[S] = [S]_\ell$.

Conversily, if $[S]=[S]_\ell$, one can easily verify that $\mathrm{Cay}(G;S_1,S_2,\ldots,S_n)$ is $E_1E_2\cdots E_n$- locally connected

Proposition 2.15 *Cay*$(G;S_1,S_2,\ldots,S_n)$ *is* E_i*- locally connected if and only if* $<S_i>=<S_i>_\ell$.

Proposition 2.16 *Cay*$(G;S_1,S_2,\ldots,S_n)$ *is* $E_1E_2\cdots E_n$*- semi connected if and only if* $G=[S]\cup[S]_\ell$.

Proof. Assume that $\mathrm{Cay}(G;S_1,S_2,\ldots,S_n)$ is $E_1E_2\cdots E_n$- semi connected and let $x\in G$. Then there is a path from 1 to x, say: $(1,x_1,x_2,\cdots,x_k,x)$ or a path from x to 1, say: $(x,y_1,y_2,\cdots,y_m,1)$. This implies that $x\in[S]$ or $x\in[S]_\ell$. This implies that $G=[S]\cup[S]_\ell$. Similarly, if $G=[S]\cup[S]_\ell$, then one can prove that $\mathrm{Cay}(G;S_1,S_2,\ldots,S_n)$ is $E_1E_2\cdots E_n$- semi connected.

Proposition 2.17 *Cay*$(G;S_1,S_2,\ldots,S_n)$ *is* E_i*- semi connected if and only if* $G=<S_i>\cup<S_i>_\ell$.

Proposition 2.18 *Cay*$(G;S_1,S_2,\ldots,S_n)$ *is an* $E_1E_2\cdots E_n$*- quasi ordered set if and only if*

$$(i)\quad 1\in S_1\cup S_2\cdots\cup S_n,$$
$$(ii)\quad \textit{for every}(i,j), S_iS_j\subseteq S_k \textit{ for some } k.$$

Proposition 2.19 *Cay*$(G;S_1,S_2,\ldots,S_n)$ *is an* E_i*- quasi ordered set if and only if* $1\in S_i$*, and* $S_i^2\subseteq S_i$.

Proposition 2.20 *Cay*$(G;S_1,S_2,\ldots,S_n)$ *if an* $E_1E_2\cdots E_n$*- partially ordered set if and only if*

$$(i)1\in S_1\cup S_2\cdots\cup S_n,$$
$$(ii)\textit{for every}(i,j), S_iS_j\subseteq S_k \textit{ for some } k,$$
$$(iii)\cup(S_i\cap S_{i_\ell})=\{1\}.$$

Proof. Observe that

$$\begin{aligned} x \in \cup(S_i \cap S_{i_\ell}) &\Leftrightarrow x \in (S_i \cap S_{i_\ell}) \text{ for some } i \\ &\Leftrightarrow x \in S_i \text{ and } x \in S_{i_\ell} \\ &\Leftrightarrow (1, x) \in E_i \text{ and } (x, 1) \in E_i \\ &\Leftrightarrow x = 1 \end{aligned}$$

From these equivalences, the result follows.

Proposition 2.21 *Cay*$(G; S_1, S_2, \ldots, S_n)$ *if an* E_i*- partially ordered set if and only if*

$$(i) 1 \in S_i,$$
$$(ii) S_i^2 \subseteq S_i$$
$$(iii) S_i \cap S_{i_\ell} = \{1\}$$

Proposition 2.22 *Let* A_m $(m \geq 2)$ *be the set of all* m *products of the form* $S_{i_1} S_{i_2} \cdots S_{i_m}$*. Then Cay*$(G; S_1, S_2, \ldots, S_n)$ *is an* $E_1 E_2 \cdots E_n$*- hasse diagram if and only if* $C \cap S_i = \emptyset$ *for all* i *and for all* $C \in A_m$*.*

Proof. Suppose the condition holds. Let $x_0, x_1, \ldots, x_m$ be $(m+1)$ elements in G such that $(x_i, x_{i+1}) \in \cup E_i$ for $i = 0, 1, \ldots, m-1$. This implies that

$$\begin{aligned} x_1 &= x_0 t_1 \text{ for some } t_1 \in S_{i_1} \\ x_2 &= x_1 t_2 \text{ for some } t_2 \in S_{i_2} \\ x_3 &= x_2 t_3 \text{ for some } t_3 \in S_{i_3} \\ &\vdots \\ x_m &= x_{m-1} t_n \text{ for some } t_m \in S_{i_m} \end{aligned}$$

The last equation can be written as:

$$\begin{aligned} x_n &= ((x_{n-2}t_{m-1}))t_m \\ &= ((x_0t_1)t_2)\cdots t_n \\ &= x_0(z_1z_2\dots z_m) \text{ for some } z_k \in S_{i_k}, k=1,2,\dots,m \\ &= x_0t, \text{ where } t = z_1z_2\dots z_m \in A_m \end{aligned}$$

Since $C \cap S_i = \emptyset$ for all i and for all $C \in A_m$, $(x_0, x_m) \notin \cup E_i$.

Conversely, assume that Cay$(G; S_1, S_2, \dots, S_n)$ is an $E_1E_2\cdots E_n$ -hasse diagram. We will show that $C \cap S_i = \emptyset$ for all i and for all $C \in A_m$. Let $S_{i_1}S_{i_2}S_{i_3}\cdots S_{i_n}$ be any element in A_m. Let $x \in S_{i_1}S_{i_2}S_{i_3}\cdots S_{i_m}$. Then $x = s_{i_1}s_{i_2}s_{i_3}\dots s_{i_m}$ for some $s_{i_k} \in S_{i_k}$. This implies that $(1, s_{i_1}, s_{i_2}s_{i_3}, \dots, x)$ is a path from 1 to x. Since Cay$(G; S_1, S_2, \dots, S_n)$ is an $E_1E_2\cdots E_n$ - hasse diagram, $x \notin S_i$ for any i. That is, $A_m \cap S_i = \emptyset$ for all i.

Proposition 2.23 *Let A_m $(m \geq 2)$ be the set of all m products of the form $S_{i_1}S_{i_2}\cdots S_{i_m}$. Then Cay$(G; S_1, S_2, \dots, S_n)$ is an E_i- hasse diagram if and only if $S_i^m \cap S = \emptyset$, for all $m \geq 2$.*

Proposition 2.24 *The $E_1E_2\cdots E_n$ out-degree of Cay$(G; S_1, S_2, \dots, S_n)$ is the cardinal number $|S_1 \cup S_2 \cup \cdots \cup S_n|$.*

Proof. Since by Theorem 2.2, Cay$(G; S_1, S_2, \dots, S_n)$ is vertex- transitive it suffices to consider the out degree of the vertex $1 \in G$. Observe that

$$\begin{aligned} \rho(1) &= \{u : (1,u) \in \cup E_i\} \\ &= \{u : u \in S_i \text{ for some } i\} \\ &= S_1 \cup S_2 \cup \cdots \cup S_n \end{aligned}$$

Hence $|\rho(1)| = |S_1 \cup S_2 \cup \cdots \cup S_n|$.

Proposition 2.25 *The E_i out-degree of Cay$(G; S_1, S_2, \dots, S_n)$ is the cardinal number $|S_i|$.*

Proposition 2.26 *The* $E_1E_2\cdots E_n$ *in-degree of Cay*$(G; S_1, S_2, \ldots, S_n)$ *is the cardinal number* $|S_{1\ell} \cup S_{2\ell} \cup \cdots \cup S_{n\ell}|$.

Proof. Since $\text{Cay}(G; S_1, S_2, \ldots, S_n)$ is vertex- transitive it suffices to consider the in degree of the vertex $1 \in G$. Observe that

$$\begin{aligned}
\sigma(1) &= \{u : (u, 1) \in \cup E_i\} \\
&= \{u : (u, 1) \in E_i\} \\
&= \{u : 1 = uz \text{ for some } z \in S_i\} \\
&= \{z_\ell : z_\ell \in S_{i_\ell} \text{ for some } i\} \\
&= S_{1\ell} \cup S_{2\ell} \cup \cdots \cup S_{n\ell}.
\end{aligned}$$

Hence $|\sigma(1)| = |S_{1\ell} \cup S_{2\ell} \cup \cdots \cup S_{n\ell}|$.

Proposition 2.27 *The* E_i *in-degree of Cay*$(G; S_1, S_2, \ldots, S_n)$ *is the cardinal number* $|S_{i_\ell}|$.

Proposition 2.28 *For* $k \geq 1$, *let* A_k *be the set of all* k *products of the form* $S_{i_1}S_{i_2}S_{i_3}\cdots S_{i_k}$. *If Cay*$(G; S_1, S_2, \ldots, S_n)$ *has finite diameter, then the* $E_1E_2\cdots E_n$ *diameter of the Cayley digraph structure Cay*$(G; S_1, S_2, \ldots, S_n)$ *is the least positive integer* m *such that* $G = A_m$.

Proof. Let m be the smallest positive integer such that $G = A_m$. We will show that the diameter of $\text{Cay}(G; S_1, S_2, \ldots, S_n)$ is m. Let x and y be any two arbitrary elements in G such that $y = xz$. Then $z \in G$. This implies that $x \in A_m$. But then z has a representation of the form $z = s_{i_1}s_{i_2}\cdots s_{i_m}$. This implies that$(1, s_{i_1}, s_{i_1}s_{i_2}, \ldots, z)$ is path of m edges from 1 to z. That is,$(x, xs_{i_1}, xs_{i_1}s_{i_2}, \ldots, y)$ is a path of length m from x to y. This shows that $d_{1,2,\ldots,n}(x, y) \leq m$. Since x and y are arbitrary, $\max_{x,y\in G}\{d_{1,2,\cdots,n}(x, y)\} \leq m$. Therefore the diameter of $\text{Cay}(G; S_1, S_2, \ldots, S_n)$ is less than or equal to m. On the other hand let the diameter of $\text{Cay}(G; S_1, S_2, \ldots, S_n)$ be k. Let $x \in G$ and $d_{1,2,\cdots,n}(1, x) = k$. Then we have $x \in B$ for some $B \in A_k$. That is,$G = A_k$. Now by the minimality of k, we have $m \leq k$. Hence $k = m$.

Proposition 2.29 *If Cay*$(G; S_1, S_2, \ldots, S_n)$ *has finite diameter, then the* E_i *diameter of the Cayley digraph structure Cay*$(G; S_1, S_2, \ldots, S_n)$ *is the least positive integer* m *such that* $G = S_i^m$.

Proposition 2.30 *The vertex* 1 *is an* $E_1E_2\cdots E_n$*-source of Cay*$(G; S_1, S_2, \ldots, S_n)$ *if and only if* $G = [S]$.

Proof. First, assume that 1 is an $E_1E_2\cdots E_n$ -source of Cay$(G; S_1, S_2, \ldots, S_n)$. Then for any vertex $x \in G$, there is an $E_1E_2\cdots E_n$- path from 1 to x. This implies that $G = [S]$. Conversely, if $G = [S]$, one can prove that 1 is an $E_1E_2\cdots E_n$- source.

Proposition 2.31 *The vertex* 1 *is an* E_i *source of Cay*$(G; S_1, S_2, \ldots, S_n)$ *if and only if* $G =< S_i >$.

Proposition 2.32 *The vertex* 1 *is an* $E_1E_2\cdots E_n$*- sink of Cay*$(G; S_1, S_2, \ldots, S_n)$ *if and only if* $G = [S]_\ell$.

Proof. First, assume that 1 is an $E_1E_2\cdots E_n$ -sink of Cay$(G; S_1, S_2, \ldots, S_n)$. Then for each $x \in G$, there is an $E_1E_2\cdots E_n$- path from x to 1. This implies that $x \in [S]_\ell$. Hence $G = [S]_\ell$.
Conversely, if $G = [S]_\ell$, one can easily prove that 1 is an $E_1E_2\cdots E_n$- sink of the Cayley digraph structure Cay$(G; S_1, S_2, \ldots, S_n)$.

Proposition 2.33 *The vertex* 1 *is an* E_i *sink of Cay*$(G; S_1, S_2, \ldots, S_n)$ *if and only if* $G =< S_i >_\ell$.

Proposition 2.34 *The* $E_1E_2\cdots E_n$ *reachable set* $R_{1,2,\ldots,n}(1)$ *of the vertex* 1 *is the set* $[S]$.

Proof. By definition, $R(1) = \{x : \text{there exits an } E_1E_2\cdots E_n \text{ - path from 1 to } x\}$.
Observe that

$$\begin{aligned} x \in R_{1,2,\ldots,n}(1) &\Leftrightarrow \text{there exits an } E_1E_2\cdots E_n \text{ - path from 1 to } x, \text{ say } (1, x_1, x_2, \ldots, x_n, x) \\ &\Leftrightarrow x \in [S]. \end{aligned}$$

Therefore, $R_{1,2,3,\cdots,n}(1) = [S]$.

Proposition 2.35 *The* E_i *reachable set* $R_i(1)$ *of the vertex* 1 *is the set* $< S_i >$.

Proposition 2.36 *The* $E_1E_2\cdots E_n$ *antecedent set* $Q_{1,2,\ldots,n}(1)$ *of the vertex* 1 *is the set* $[S]_\ell$.

Proof. Observe that

$$x \in Q_{1,2,\ldots,n}(1) \Leftrightarrow \text{there exits an } E_1E_2\cdots E_n \text{ -path from } x \text{ to } 1, \text{ say } (x, x_1, x_2, \ldots, x_n, 1)$$
$$\Leftrightarrow x \in [S]_\ell$$

Therefore, $Q_{1,2,\ldots,n}(1) = [S]_\ell$.

Proposition 2.37 *The* E_i *antecedent set* $Q_i(1)$ *of the vertex* 1 *is the set* $< S_i >_\ell$.

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