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GLOBAL EXISTENCE FOR SOME NEUTRAL FUNCTIONAL INTEGRODIFFERENTIAL EQUATIONS WITH FINITE DELAY

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Abstract. In this paper, we study a class of neutral partial functional integrodifferential equations with finite delay in Banach spaces. We are interested in the global existence, uniqueness and regularity of solutions with values in the subspace $D(A)$. The method used are based on Banach's fixed point theorem and on the technique of the graph norm. In our work the nonlinear term is treated as a perturbation of the linear equation. As an application, we consider a diffusive neutral partial functional integrodifferential equation.

Keywords: mild solutions; strict solutions; functional integrodifferential equations; neutral equations; semigroup of bounded linear operators; infinitesimal generator; finite delay; phase space.

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1. Introduction

Since Volterra's pionneering works on integrodifferential equations with delayed effets in population dynamics and materials with memory, the theory of delay differential equations progressed dramatically stimulated by the developpment of functional analysis and its numerous

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real world applications, wherever (in physics, chemistry, biology, medecine, economy etc, see e.g, [16]) the evolution of a process depends on its history in an essentiel way. In recent year, the theory of integrodifferential equations with delay has been studied deeply in the literature. For more details, we refer to [2], [3], [5], [6], [7], [8], [9], [10], [17] and the references therein.

In this paper, we are interested in the existence and regularity of solutions for the following neutral partial functional integrodifferential equation with finite delay

$$\begin{cases} \frac{d}{dt}g(t,u_t) & = Ag(t,u_t) + \int_0^t B[t-s, g(s,u_s)]ds + F(t,u_t) \quad \text{for} \quad t \geq 0 \\ u_0 & = \varphi \in \mathscr{C} = \mathscr{C}\left([-r,0];\mathrm{D}(A)\right), \end{cases} \tag{1.1}$$

where A is the infinitesimal generator of a strongly continuous semigroup of $(T(t))_{t\geq 0}$ on a Banach space E with domain $D(A)$, and B is, in general, a nonlinear operator from $\mathbb{R}^+ \times D(A)$ to E. The phase space $\mathscr{C}$ is the space of continuous functions from $[-r,0]$ into $D(A)$, where $D(A)$ is endowed with the graph norm, namely for $x \in D(A)$, $|x|_{D(A)} = |x|_E + |Ax|_E$. We know that $\left(D(A), |.|_{D(A)}\right)$ is a Banach space. Also g is a function defined from $\mathbb{R}^+ \times \mathscr{C}$ into $D(A)$ by

$$g(t,\varphi) = \varphi(0) - G(t,\varphi), \tag{1.2}$$

and $G,\ F$ are two continuous functions from $\mathbb{R}^+ \times \mathscr{C}$ into E.

For $u \in \mathscr{C}\left([-r,0];D(A)\right)$ and for every $t \geq 0$, the history function $u_t \in \mathscr{C}$ is defined by

$$u_t(\theta) = u(t+\theta) \quad \text{for} \quad \theta \in [-r,0].$$

In the case where $B = 0$ and A is the infinitesimal generator of a strongly continuous semigroup, the mild solution of $Eq.(1.1)$ is given by the following variation of constants formula

$$g(t,u_t) = T(t)g(0,\varphi) + \int_0^t T(t-s)F(s,u_s)ds \quad for \quad t \geq 0.$$

Our work is motivated by [23], where the author proved the existence of mild and strict solutions for a partial functional integrodifferential equation in the following from

$$\begin{cases} u'(t) & = Au(t) + \int_0^t g\left(t-s, u(s)\right)ds + f(t) \quad \text{for} \quad t \geq 0, \\ u_0 & = x \in E. \end{cases} \tag{1.3}$$

The goal of this work is to extend this problem to neutral type equations and to discuss the existence and regularity results of solutions for $Eq.(1.1)$ by using the semigroup theory. The result obtained is a generalization and a continuation of our first work, see reference [9] and [23].

The paper is organized as follows. In Section 2, we recall some preliminary results of $Eq.(1.3)$. In Section 3, we study the existence of mild solutions for the neutral system (1.1) using the theory of semigroup and Banach's fixed point theorem. Sufficient conditions for the existence of mild and strict solutions are also established. Finally, we present in Section 4 an example which illustrates our results.

2. Preliminary results

In this section, we recall some notions and results that we need in the following. Throughout the paper, E is a Banach space, $A : D(A) \subset E \to E$ is closed linear operator which generates a c_0-semigroup $(T(t))_{t\geq 0}$ on E. For more details, we refer to [20]. Recall that for such a semigroup, there exists $M > 0$ and $\omega \in \mathbb{R}$ such that

$$|T(t)| \leq Me^{\omega t}, \quad t \geq 0, \tag{2.1}$$

where $|T(t)|$ is the norm of the bounded linear operator $T(t)$.

We denote by Y the space $D(A)$ equipped with the graph norm defined by

$$|y|_{D(A)} = |y|_E + |Ay|_E . \tag{2.2}$$

It is well-known that $D(A)$ equipped with norm $|\cdot|_{D(A)}$ is a Banach space.

Definition 2.1. *A continuous function $u : [0,+\infty[\to E$ is said to be a strict solution of $Eq.(1.3)$ if*

(i) $u \in \mathscr{C}^1([0,+\infty[;E) \cap \mathscr{C}([0,+\infty[;Y)$;

(ii) *u satisfies $Eq.(1.3)$ for all $t \geq 0$.*

Remark 2.2. *From this definition, we deduce that $u(t) \in Y$ and the function $s \mapsto g(t-s,u(s))$ is integrable on $[0,t]$ for every $t \geq 0$.*

Proposition 2.3. [23] *If u is a strict solution of* $Eq.(1.3)$, *then u satisfies the integral equation*

$$u(t) = T(t)x + \int_0^t T(t-s)\int_0^s g(s-r,u(r))\,drds + \int_0^t T(t-s)f(s)ds. \tag{2.3}$$

Remark 2.4. *If u satisfies the equation* (2.3), *it is not necessarily a strict solution. That is why we give the next definition.*

Definition 2.5. *A continuous function* $u : [0,+\infty[\to E$ *is called a mild solution of* $Eq.(1.3)$ *if it* $Eq.(2.3)$.

3. Main results

In this section, we prove the global existence, uniqueness and regularity of the solution to $Eq.(1.1)$. Firstly, we show the existence and uniqueness of the mild solution. Secondly, we give sufficient conditions ensuring that the mild solution is a strict solution of the problem, in the sens of the following definition.

3.1. Global existence of mild solutions.

Definition 3.1. *We say that a continuous function* $u : [-r,+\infty[\to Y$ *is a strict solution of* $Eq.(1.1)$ *if the following conditions hold*

(i) $u \in \mathscr{C}^1([0,+\infty[,E)\cap\mathscr{C}([0,+\infty[,Y)$;

(ii) *u satisfies* $Eq.(1.1)$ *on* $[0,+\infty[$;

(iii) $u(\theta) = \varphi(\theta)$ *for* $-r \le \theta \le 0$.

Remark 3.2. *Form this definition, we deduce that* $u(t) \in Y$ *and the function* $s \mapsto g(t-s,u(s))$ *is integrable on* $[0,t]$ *for the every* $t \ge 0$.

From Proposition 2.3 we have the following.

Proposition 3.3. *If u is a strict solution of* $Eq.(1.1)$, *then u satisfies the integral equation*

$$\begin{aligned} u(t) &= T(t)g(0,\varphi) + G(t,u_t) + \int_0^t T(t-s)\int_0^s B(s-r,g(r,u_r))drds \\ &\quad + \int_0^t T(t-s)F(s,u_s)ds. \end{aligned} \tag{3.1}$$

Proof. *It is just a consequence of Proposition. 2.3 and of* $Eq.(1.2)$. ◊

Remark 3.4. *The converse in not true. In fact if* u *satisfies* $Eq.(3.1)$, u *it is necessary to be a strict solution. That is why we give the following definition.*

Definition 3.5. *We say that a continuous function* $u : [-r,+\infty[\to Y$ *is a mild solution of* $Eq.(1.1)$ *if it satisfies the equation* (3.1).

To establish the existence of the mild solution, we assume that the following conditions are satisfied.

$(\mathbf{H_1})$ $F,G : \mathbb{R}^+ \times \mathscr{C}([-r,0],Y) \to Y$ are continuous and lipschitzian with respect to the second argument, namely, then exist constants $L_F > 0$ and $L_G > 0$ such that

$$|F(t,\varphi) - F(t,\hat{\varphi})|_Y \le L_F\,|\varphi-\hat{\varphi}|_{\mathscr{C}([-r,0],Y)} \quad \text{for} \quad t \ge 0 \quad \text{and} \quad \varphi,\hat{\varphi} \in \mathscr{C}([-r,0],Y),$$

$$|G(t,\varphi) - G(t,\hat{\varphi})|_Y \le L_G\,|\varphi-\hat{\varphi}|_{\mathscr{C}([-r,0],Y)} \quad \text{for} \quad t \ge 0 \quad \text{and} \quad \varphi,\hat{\varphi} \in \mathscr{C}([-r,0],Y).$$

$(\mathbf{H_2})$ The derivative $\frac{\partial B}{\partial t}(t,u)$ exists and is continuous from $\mathbb{R}^+ \times Y$ into E, moreover there exist continuous and increasing functions $b : \mathbb{R}^+ \to \mathbb{R}^+$ and $c : \mathbb{R}^+ \to \mathbb{R}^+$ such that:

$$|B(t,u) - B(t,v)|_E \le b(t)\,|u-v|_Y$$

and

$$\left|\frac{\partial B}{\partial t}(t,u) - \frac{\partial B}{\partial t}(t,v)\right|_E \le c(t)\,|u-v|_Y$$

for all $t \in \mathbb{R}+$, and $u,v \in Y$.

Theorem 3.6. *Assume that* $(\mathbf{H_1})$ *and* $(\mathbf{H_2})$ *hold. If* $\varphi \in \mathscr{C}([-r,0];Y)$, *then there exist a unique continuous function* $u : [-r,+\infty[\to Y$ *which is a mild solution of* $Eq.(1.1)$

Proof. We define the set $[N_{t_1}(\varphi) := \{u \in \mathscr{C}([0,t_1];Y) : u(0) = \varphi(0)\}]$. Cleary $N_{t_1}(\varphi)$ is a closed subset of the space $\mathscr{C}([0,t_1];Y)$, where $\mathscr{C}([0,t_1];Y)$ is the space of continuous functions from $[0,t_1]$ to Y. Next, for each $u \in N_{t_1}(\varphi)$ we define

$$\tilde{u}(t) = \begin{cases} \varphi(t) & \text{for} \quad t \in [-r, 0], \\ u(t) & \text{for} \quad t \in [0, t_1]. \end{cases}$$

Consider the operator $P : N_{t_1}(\varphi) \to \mathscr{C}([-r, 0], E)$ defined by

$$(3.2)\quad \begin{aligned} (Pu)(t) &= G(t, \tilde{u}_t) + T(t)g(0, \varphi) + \int_0^t T(t-s) \int_0^s B(s-r, g(r, \tilde{u}_r))\, dr \\ &+ \int_0^t T(t-s)F(s, \tilde{u}_s) ds. \end{aligned}$$

The first step is to show that $P(N_{t_1}(\varphi)) \subset N_{t_1}(\varphi)$. From (3.2) we have

$$\begin{aligned} (APu)(t) &= AG(t, \tilde{u}_t) + AT(t)g(0, \varphi) + A \int_0^t T(t-s) \int_0^s B(s-r, g(r, \tilde{u}_r)) dr ds \\ &+ A \int_0^t T(t-s)F(s, \tilde{u}_s) ds \quad 0 \le t \le t_1. \end{aligned}$$

Since A is closed, then

$$\begin{aligned} (APu)(t) &= AG(t, \tilde{u}_t) + AT(t)g(0, \varphi) + A \int_0^t T(t-s) \int_0^s B(s-r, g(r, \tilde{u}_r)) dr ds \\ &+ \int_0^t T(t-s)AF(s, \tilde{u}_s) ds \quad 0 \le t \le t_1. \end{aligned}$$

For the next, we need the following result from $[18, p.486]$.

Lemma 3.7. *Let $k : [0, t_1] \to E$ be continuously differentiable, and q be defined by*

$$q(t) = \int_0^t T(t-s)k(s) ds, \quad \text{for} \quad t \in [0, t_1].$$

Then $q(t) \in Y$, for every $t \in [0, t_1]$, q is continuously differentialble, and

$$q(t) = q'(t) - k(t) = \int_0^t T(t-s)k'(s) ds + T(t)k(0) - k(t).$$

By virtue of the hypothesis we have on B, by Lemma 3.7, for $u \in Y$, we have

$$(3.3)\quad \begin{aligned} (APu)(t) &= AG(t, \tilde{u}_t) + AT(t)g(0, \varphi) + \int_0^t T(t-s)B(0, g(s, \tilde{u}_s))\, ds \\ &+ \int_0^t T(t-s) \int_0^s \frac{\partial B}{\partial s}(s-r, g(r, \tilde{u}_r))\, dr ds \\ &- \int_0^t B(t-s, g(s, \tilde{u}_s))\, ds + \int_0^t T(t-s)AF(s, \tilde{u}_s) ds. \end{aligned}$$

Using (2.2), thus, for $u \in N_{t_1}(\varphi)$, Pu and APu are both continuous from $[0,t_1]$ to E, and so P maps $N_{t_1}(\varphi)$ into itself.

Hence, in order to apply Banach's contraction principle, it remains to prove that P is a contraction on $N_{t_1}(\varphi)$ with respect to a suitable graph norm. To show this, consider two arbitrary functions $u,\ v \in N_{t_1}(\varphi)$ and any $t \in [0,t_1]$. Using (2.1), we have

$$\begin{aligned}
|(Pu)(t)-(Pv)(t)|_E &\leq |G(t,\tilde{u}_t)-G(t,\tilde{v}_t)|_E + \left|\int_0^t T(t-s)\int_0^s [B(s-r,g(r,\tilde{u}_r))-B(s-r,g(r,\tilde{v}_r)]\,drds\right|_E \\
&+\left|\int_0^t T(t-s)\,[F(s,\tilde{u}_s)-F(s,\tilde{v}_s)]\,ds\right|_E \\
&\leq |G(t,\tilde{u}_t)-G(t,\tilde{v}_t)|_E + M\int_0^t e^{w(t-s)}\int_0^s |B(s-r,g(r,\tilde{u}_r))-B(s-r,g(r,\tilde{v}_r))|_E\,drds \\
&+M\int_0^t e^{w(t-s)}\,|F(s,\tilde{u}_s)-F(s,\tilde{v}_s)|_E\,ds \\
&\leq |G(t,\tilde{u}_t)-G(t,\tilde{v}_t)|_Y + M\int_0^t e^{w(t-s)}\int_0^s |B(s-r,g(r,\tilde{u}_r))-B(s-r,g(r,\tilde{v}_r))|_Y\,drds \\
&+M\int_0^t e^{w(t-s)}\,|F(s,\tilde{u}_s)-F(s,\tilde{v}_s)|_Y\,ds
\end{aligned}$$

Without loss of generality, we assume that $w>0$. By $(\mathbf{H_1})$ and $(\mathbf{H_2})$, we obtain that

(3.4)
$$\begin{aligned}
|(Pu)(t)-(Pv)(t)|_E &\leq L_G\,|\tilde{u}_t-\tilde{v}_t|_{\mathscr{C}([-r,0],Y)} + Me^{wt_1}\int_0^t\int_0^s b(s-r)\,|\tilde{u}_r-\tilde{v}_r|_{\mathscr{C}([-r,0],Y)}\,drds \\
&+ML_F e^{wt_1}\int_0^t |\tilde{u}_s-\tilde{v}_s|_{\mathscr{C}([-r,0],Y)}\,ds,
\end{aligned}$$

(3.5)
$$\begin{aligned}
|(APu)(t)-(APv)(t)|_E &\leq |AG(t,\tilde{u}_t)-AG(t,\tilde{v}_t)| + M\int_0^t e^{w(t-s)}\,|B(0,g(s,\tilde{u}_s))-B(0,g\tilde{v}_s))|\,ds \\
&+M\int_0^t e^{w(t-s)}\int_0^s \left|\frac{\partial B}{\partial s}(s-r,g(r,\tilde{u}_r))-\frac{\partial B}{\partial s}(s-r,g(r,\tilde{v}_r))\right|drds \\
&+\int_0^t |B(t-s,g(s,\tilde{u}_s))-B(t-s,g(s,\tilde{v}_s))|\,ds \\
&+M\int_0^t e^{w(t-s)}\,|AF(s,\tilde{u}_s)-AF(s,\tilde{v}_s)|\,ds \\
&\leq L_G\,|\tilde{u}_t-\tilde{v}_t|_{\mathscr{C}([-r,0],Y)} + Mb(0)e^{wt_1}\int_0^t |\tilde{u}_s-\tilde{v}_s|_{\mathscr{C}([-r,0],Y)}\,ds \\
&+Me^{wt_1}\int_0^t\int_0^s c(s-r)\,|\tilde{u}_r-\tilde{v}_r|_{\mathscr{C}([-r,0],Y)}\,drds
\end{aligned}$$

$$+\int_0^t b(t-s)\,|\tilde{u}_s-\tilde{v}_s|_{\mathscr{C}([-r,0],Y)}\,ds+ML_Fe^{wt_1}\int_0^t|\tilde{u}_s-\tilde{v}_s|_{\mathscr{C}([-r,0],Y)}\,ds.$$

From (3.4) and (3.5), we have using (2.2)

$$\begin{aligned}|(Pu)(t)-(Pv)(t)|_Y\ &\leq L_G\,|\tilde{u}_t-\tilde{v}_t|_{\mathscr{C}([-r,0],Y)}+Mb(0)e^{wt_1}\int_0^t|\tilde{u}_s-\tilde{v}_s|_{\mathscr{C}([-r,0],Y)}\,ds\\ &+Me^{wt_1}\int_0^t\int_0^s[b(s-r)+c(s-r)]\,|\tilde{u}_r-\tilde{v}_r|_{\mathscr{C}([-r,0],Y)}\,drds\\ &+\int_0^t b(t-s)\,|\tilde{u}_s-\tilde{v}_s|_{\mathscr{C}([-r,0],Y)}+ML_Fe^{wt_1}\int_0^t|\tilde{u}_s-\tilde{v}_s|_{\mathscr{C}([-r,0],Y)}\,ds.\end{aligned}$$

Define $\alpha(t)=\displaystyle\int_0^t e^{-ws}(b(s)+c(s))ds$ and $\beta(t)=\max\limits_{0\leq s\leq t}e^{-ws}b(s)$ for $t\geq 0$.

Then

$$\begin{aligned}|(Pu)(t)-(Pv)(t)|_Y\ &\leq L_G\,|\tilde{u}_t-\tilde{v}_t|_{\mathscr{C}([-r,0],Y)}+Mb(0)e^{wt_1}\int_0^{t_1}|\tilde{u}_s-\tilde{v}_s|_{\mathscr{C}([-r,0],Y)}\,ds\\ &+Me^{wt_1}\alpha(t)\int_0^{t_1}|\tilde{u}_s-\tilde{v}_s|_{\mathscr{C}([-r,0],Y)}\,ds+Me^{wt_1}\beta(t)\int_0^{t_1}|\tilde{u}_s-\tilde{v}_s|_{\mathscr{C}([-r,0],Y)}\,ds\\ &+ML_Fe^{wt_1}\int_0^{t_1}|\tilde{u}_s-\tilde{v}_s|_{\mathscr{C}([-r,0],Y)}\,ds,\end{aligned}$$

and finally

$$|(Pu)(t)-(Pv)(t)|_Y\ \leq\left[L_G+Mt_1e^{wt_1}(b(0)+\alpha(t)+\beta(t)+L_F)\right]|\tilde{u}-\tilde{v}|_{\mathscr{C}([-r,0],Y)}.$$

If we choose t_1 small enough and $L_G<1$ such that $\left[L_G+Mt_1e^{wt_1}(b(0)+\alpha(t)+\beta(t)+L_F)\right]<1$, then P is a strict contraction in $N_{t_1}(\varphi)$ and by applying Banach's fixed point theorem, we deduce that there exists a unique fixed point $u=u(.,\varphi)$ for P in $N_{t_1}(\varphi)$, which implies that $Eq.(1.1)$ has a unique mild solution on $[-r,t_1]$. A similar argument can be used for $[t_1,2t_1]$,...,$[nt_1,(n+1)t_1]$, for all $n\geq 0$, which implies that the mild solution exists uniquely in $[-r,+\infty[$. This completes the proof. $\Diamond$

3.2. Existence of strict solutions. In this section we recall some fundamental results needed to establish our results. We consider the inhomogeneous initial value problem

$$\left\{\begin{array}{l}u'(t)=Au(t)+h(t)\quad\text{for}\quad t\geq 0,\\[2ex] u(0)=x\in E\end{array}\right.\tag{3.6}$$

where $h:[0,t_1]\longrightarrow E$, is continuous.

Definition 3.8. *A continuous function* $u : [0,+\infty[\to E$ *is said to be strict solution of* $Eq.(3.6)$ *if*

(i) $u \in \mathscr{C}^1([0,+\infty[;E) \cap \mathscr{C}([0,+\infty[;D(A))$

(ii) *u satisfies* $Eq.(3.6)$ *for all* $t \geq 0$.

Proposition 3.9. [21]. *If u is a strict solution of* $Eq.(3.6)$, *then u is given by*

$$u(t) = T(t)x + \int_0^{t_1} T(t-s)h(s)ds \quad \text{for} \quad t \in [0,t_1]. \tag{3.7}$$

The next Theorem provides sufficients conditions for the regularity of solution to $Eq.(3.6)$.

Theorem 3.10. [21]. *Let A be the infinitesimal generator of a* C_0*-semigroup* $(T(t))_{t\geq 0}$. *let* $h \in L^1(0,t_1;X)$ *be continuous on* $[0,t_1]$ *and let*

$$v(t) = \int_0^t T(t-s)h(s)ds \quad t \in [0,t_1].$$

The $Eq.(3.6)$ *has a strict solution u on* $[0,t_1]$ *for every* $x \in D(A)$ *if one of the following conditions is satisfied;*

(1) $v(t)$ *is continuously differentiable on* $[0,t_1]$.

(2) $v(t) \in D(A)$ *for* $0 < t < t_1$ *and* $Av(t)$ *is continuous on* $[0,t_1]$.

• *If* $Eq.(3.6)$ *has a strict solution u on* $[0,t_1]$ *for some* $x \in D(A)$ *then v satisfies both* (1) *and* (2).

From Theorem 3.11 we dram the following useful Lemma.

Lemma 3.11. [21]. *Let A be the infinitesimal generator of a* c_0*-semigroup* $(T(t))_{t\geq 0}$. *Let* $h \in L^1([0,t_1];D(A))$ *be continuous on* $[0,t_1]$. *If* $h(s) \in D(A)$ *for* $0 < s < t_1$ *and* $Ah \in L^1([0,t_1];D(A))$ *then for every* $x \in D(A)$ *the* $Eq.(3.6)$ *has a strict solution u on* $[0,t_1]$.

Now we give some sufficient conditions for the existence of a strict solution. To do this, we suppose the following condition on G.

$(\mathbf{H_3})$ $G : \mathbb{R}^+ \times \mathscr{C}([-r,0],Y) \to Y$ is continuously differentiable with respect to the first variable on $\mathbb{R}^+$.

we posit $h(t) = \displaystyle\int_0^t B(t-s, g(s,u_s))ds + F(t,u_t) \quad \text{for} \quad t \geq 0.$

Theorem 3.12. *Let $u \in \mathscr{C}([0,t_1],Y)$ the mild solution of $Eq.(1.1)$. If $g(.,\varphi) \in Y$ and $h \in L^1([0,t_1];D(A))$ is continuous from $[0,t_1]$ to $D(A)$, then u is a strict solution of $Eq.(1.1)$.*

Proof. It is just a consequence of Theorem 3.10.

Here

$$h(t) = \int_0^t B(t-s, g(s,u_s))ds + F(t,u_t)ds \quad \text{for} \quad t \geq 0$$

and

$$v(t) = \int_0^t T(t-s)\int_0^s B(s-r, g(s,u_r))drds + \int_0^t T(t-s)F(s,u_s)ds \quad \text{for} \quad t \geq 0.$$

We show that v satisfies the following two conditions

(i) $v(t)$ is continuously differentiable on $[0,t_1]$;

(ii) $v(t) \in Y$ on $[0,t_1]$ and $Av(t) \in L^1([0,t_1],X)$.

Based on the formula (3.1) we have: $v(t) = u(t) - T(t)g(0,\varphi) - G(t,u_t)$ is differentiable for $t > 0$ as sum of three differentiable functions and $\frac{d}{dt}v(t) = \frac{d}{dt}u(t) - T(t)Ag(0,\varphi) - \frac{d}{dt}G(t,u_t)$ is obviously continuous on $]0,t_1[$. Therefore (i) is satisfied. Also if $g(0,\varphi) \in Y$ then $T(t)g(0,\varphi) \in Y$ for $t \geq 0$ and therefore $v(t) = u(t) - T(t)g(0,\varphi) - G(t,u_t) \in Y$ for $t > 0$ and $Av(t) = Au(t) - AT(t)g(0,\varphi) - AG(t,u_t) = \frac{d}{dt}u(t) - \int_0^t B(t-s, g(s,u_s))ds - F(t,u_t) - T(t)Ag(0,\varphi) - AG(t,u_t)$ is continuous on $]0,t_1[$. Thus also (ii) is satisfaied.

On the other hand, it is easy to verify for $h > 0$ the identify

$$\left(\frac{T(h)-I}{h}\right)v(t) = \frac{v}{(}t+h) - v(t)h - \frac{1}{h}\int_t^{t+h} T(t+h-s)\left[k(s) + F(s,u_s\right]ds. \tag{3.8}$$

From the continuity of k and F it is clear that the second therm on the right-hand side of (3.8) has the limit $k(t) + F(t,u_t)$ as $h \to 0$. If $v(t)$ is continuously differentiable on $]0,t_1[$ then it follows from (3.8) that $v(t) \in Y$ for $0 < t < t_1$ and $Av(t) = \dfrac{d}{dt}v(t) - [k(t) + F(t,u_t)]$. Since $v(0) = 0$ it follows that $u(t) = T(t)g(0,\varphi) + v(t) + G(t,u_t)$ is the solution of $Eq.(1.1)$ for $g(0,\varphi) \in Y$. If $v(t) \in Y$ it follows from (3.8) that $v(t)$ is differentiable from the right at t and the right derivative $D^+v(t)$ of v satisfies $D^+v(t) = Av(t) + k(t) + F(t,u_t)$. Since $D^+v(t)$ is continuous, $v(t)$ is continuously differentiable and $\frac{d}{dt}v(t) = Av(t) + k(t) + F(t,u_t)$. Since $v(0) = 0$, $u(t) = T(t)g(0,\varphi) + G(t,u_t) + v(t)$ is the solution of $Eq.(1.1)$ for $g(0,\varphi) \in Y$ and the proof is complete.

4. Application

To illustrate the previous results, we consider the following reaction diffusion model of neutral type

$$
(4.1) \quad \begin{cases}
\frac{\partial}{\partial t}\left[w(t,\xi) - \int_{-r}^{0}\delta(t,w(t+\theta,\xi))d\theta\right] = \frac{\partial^2}{\partial \xi^2}\left[w(t,\xi) - \int_{-r}^{0}\delta(t,w(t+\theta,\xi))d\theta\right] \\
+\int_0^t \beta\left(t-s, \frac{\partial^2}{\partial \xi^2}\left(w(t,\xi) - \int_{-r}^{0}\delta(t,w(t+\theta,\xi))d\theta\right)\right)ds \\
+\int_{-r}^{0} f(t,w(t+\theta,\xi))d\theta \quad \text{for} \quad t \geq 0 \quad \text{and} \quad \xi \in [0,\pi] \\
w(t,0) - \int_{-r}^{0}\delta(t,w(t+\theta,0))d\theta = 0 \quad \text{for} \quad t \geq 0 \\
w(t,\pi) - \int_{-r}^{0}\delta(t,w(t+\theta,\pi))d\theta = 0 \quad \text{for} \quad t \geq 0 \\
w(\theta,\xi) = w_0(\theta,\xi) \quad \text{for} \quad \theta \in [-r,0] \quad \text{and} \quad \xi \in [0,\pi],
\end{cases}
$$

where $\delta, f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ are continuous functions, $w_0 : [-r,0] \times [0,\pi] \to \mathbb{R}$ is a given continuous function and $\beta : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ is continuous. We take space $\mathrm{E} = \mathscr{C}_0([0,\pi];\mathbb{R})$, the space of continuous functions from $[0,\pi]$ to $\mathbb{R}$ vanishing at 0 and π, endowed with the uniform norm topology.

We suppose that

(i) For $t \geq 0$, $\delta(t,0) = f(t,0) = 0$.

(ii) There exists a constant L_δ and L_f such that

$$|\delta(t,u) - \delta(t,v)|_Y \leq L_\delta\, |u-v|_{\mathscr{C}([-r,0];Y)} \quad \text{for} \quad t \geq 0 \quad \text{and} \quad u,v \in \mathscr{C}([-r,0];Y),$$

$$|f(t,u) - f(t,v)|_Y \leq L_f\, |u-v|_{\mathscr{C}([-r,0];Y)} \quad \text{for} \quad t \geq 0 \quad \text{and} \quad u,v \in \mathscr{C}([-r,0];Y).$$

(iii) $\beta : \mathbb{R}^+ \times Y \to E$ β is continuous, $\beta(t,u)$ is continuously differentiable in its first variable, $\beta(t,u)$ and the derivative $\frac{\partial \beta(t,u)}{\partial t}$ are Lipschitzian, moreover there exist continuous and increasing functions $b:\mathbb{R}^+\to\mathbb{R}^+$ and $c:\mathbb{R}^+\to\mathbb{R}^+$ such that:

$$|\beta(t,u)-\beta(t,v)|_E \le b(t)\,|u-v|_Y$$

and

$$\left|\frac{\partial\beta}{\partial s}(t,u)-\frac{\partial\beta}{\partial t}(t,v)\right|_E \le c(t)\,|u-v|_Y$$

for all $t\in\mathbb{R}+$, and $u,v\in Y$.

We defined the operators $\mathrm{G},\mathrm{F}:\mathbb{R}^+\times\mathscr{C}([-r,0];Y)\to \mathrm{Y}$ by

$$\mathrm{G}(t,\varphi)(\xi)=\int_{-r}^{0}\delta(t,\varphi(\theta)(\xi))d\theta \quad \text{for} \quad \xi\in[0,\pi] \quad \text{and} \quad \varphi\in\mathscr{C}([-r,0];Y),$$

$$\mathrm{F}(t,\varphi)(\xi)=\int_{-r}^{0}f(t,\varphi(\theta)(\xi))d\theta \quad \text{for} \quad \xi\in[0,\pi] \quad \text{and} \quad \varphi\in\mathscr{C}([-r,0];Y).$$

Consider the linear operator $A:\mathrm{D}(A)\subset \mathrm{E}\to\mathrm{E}$ defined by

$$\begin{cases} \mathscr{D}(A)=\left\{z\in \mathrm{E}: z''\in\mathrm{E}, z(0)=z(1)=0\right\}, \\ \\ Az=z''. \end{cases}$$

It is well known in [21] that A is the infinitesimal generator of strongly continuous semigroup on E.

Let $\mathrm{B}:\mathbb{R}^+\times\mathrm{D}(A)\to\mathrm{E}$ by $\mathrm{B}(t,z)=\beta(t,Az)$ for $t\ge 0$.

If we put

$$\begin{cases} u(t)=w(t,\xi) \quad \text{for} \quad t\ge 0 \quad \text{and} \quad \xi\in[0,\pi] \\ \varphi(\theta)(\xi)=w_0(\theta,\xi) \quad \text{for} \quad \theta\in[-r,0] \quad \text{and} \quad \xi\in[0,\pi]. \end{cases}$$

Then $Eq.(4.1)$ takes the following abstract from

$$\text{(4.2)} \qquad \begin{cases} \dfrac{d}{dt}[u(t)-G(t,u_t)]=A[u(t)-G(t,u_t)]+\displaystyle\int_0^t B[t-s,u(t)-G(t,u_s)]ds+F(t,u_t) \\ u_0=\varphi. \end{cases}$$

Consequently, the conditions of Theorem 3.6 are fulfilled. Then, we obtain the following result.

Proposition 4.1. *Under the assumptions (i), (ii) and (iii), problem (4.1) has a unique solution $u\in C([0,\infty),Y)$ and the function v defined by $w(t,\xi)=u(t)(\xi)$ for $t\ge 0$ and $\xi\in[0,\pi]$ is a solution of $Eq.(4.1)$.*

Conflict of Interests

The authors declare that there is no conflict of interests.

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