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A THIRD ORDER RUNGE-KUTTA METHOD BASED ON A CONVEX COMBINATION OF LEHMER MEANS

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Abstract. In this paper we introduce a modification of the third order Runge-Kutta method based on a convex combination of Lehmer means. The error of this method is also presented. The stability of this method is similar to the stability of third order Runge-Kutta method based on arithmetic mean. We end the discussion with two numerical examples to justify the effectiveness of the method.

Keywords: convex combination; initial value problem; Lehmer means; Runge-Kutta method.

2010 AMS Subject Classification: 65L05, 65L06.

1. Introduction

We consider the first order initial value problem (*IVP*) in the form of

$$\left.\begin{aligned} Y'(t) &= f(t,Y(t)),\ \ t_0 \le t \le b, \\ Y(t_0) &= Y_0, \end{aligned}\right\} \tag{1}$$

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where $Y(t)$ is the exact solution, and $f(t,Y(t))$ is a continuous function in the domain D containing a point (t_0,Y_0). In the autonomous form, *IVP* (1) can be written as

$$\left.\begin{aligned} Y'(t) &= f(Y(t)),\ t_0 \le t \le b, \\ Y(t_0) &= Y_0. \end{aligned}\right\} \tag{2}$$

The numerical solution $y(t)$ of the problem (1) is found in the following set of discrete points:

$$t_0 < t_1 < t_2 < ... < t_N \le b.$$

The distance between these points denotes by h, so it can be written as

$$t_n = t_0 + nh,\ n = 0,1,...,N.$$

The notation for numerical solutions at the n-th point is denoted by

$$y(t_n) = y_h(t_n) = y_n,\ n = 0,1,...,N.$$

There are several numerical methods that can be used to solve the problem (1). One of which is the Runge-Kutta method. Evans [6] presents the third order Runge-Kutta method with the following formula:

$$y_{n+1} = y_n + \frac{h}{2}\left(\frac{k_1+k_2}{2} + \frac{k_2+k_3}{2}\right), \tag{3}$$

where

$$\begin{aligned} k_1 &= f(t_n,y_n), \\ k_2 &= f(t_n + \frac{2}{3}h, y_n + \frac{2}{3}hk_1), \\ k_3 &= f(t_n + \frac{2}{3}h, y_n - \frac{1}{3}hk_1 + hk_2), \end{aligned}$$

The equation (3) is also called the third order Runge-Kutta method based on arithmetic mean (RK3AM).

Arithmetic mean is one form of the Lehmer mean. In his research, Lehmer [7] explains that for $(v,w) > 0$ and $p \in \mathbb{R}$, the formula of Lehmer mean is given by

$$L_p(v,w) = \frac{v^p + w^p}{v^{p-1} + w^{p-1}}.$$

The Lehmer mean with $p = 1$ is the arithmetic mean, the Lehmer mean with $p = 0$ is the harmonic mean and the Lehmer mean with $p = 2$ is the contraharmonic mean.

Several modifications of the third order Runge-Kutta (RK3) method are the RK3 method based on the harmonic mean [12], the RK3 method based on geometry mean [6], the RK3 method based on the contraharmonic mean [1], and RK3 methods based on a linear combination of the arithmetic, harmonic, and geometric means [14].

In this article we present the RK3 method based on a convex combination of the Lehmer means with the value $p = 0$ and $p = 3$ (RK3L). In section 2, the derivation of RK3L is presented and local truncation error as describe in section 3. The stability analysis for the proposed method is described in section 4. We end the presentation by numerical comparisons using two problems.

For $(k_1, k_2) > 0$, the convex combination formulas of Lehmer means with $p = 0$ and $p = 3$ (CCL) is as follows

$$CCL(k_1,k_2) = (1-\alpha)\frac{2k_1k_2}{k_1+k_2} + \alpha\frac{k_1^3+k_2^3}{k_1^2+k_2^2}, \tag{4}$$

and CCL formula for $(k_2, k_3) > 0$ is given by

$$CCL(k_2,k_3) = (1-\alpha)\frac{2k_2k_3}{k_2+k_3} + \alpha\frac{k_2^3+k_3^3}{k_2^2+k_3^2}, \tag{5}$$

with $0 < \alpha < 1$.

2. Modified of Runge-Kutta Method

RK3L method is a modification of third order Runge-Kutta method which is obtained by replacing arithmetic mean in equation (3) with $CCL(k_1, k_2)$ and $CCL(k_2, k_3)$ in equations (4) and (5), respectively. So the formula is as follows:

$$y_{n+1} = y_n + \frac{h}{2}\left((1-\alpha)\left(\frac{2k_1k_2}{k_1+k_2} + \frac{2k_2k_3}{k_2+k_3}\right) + \alpha\left(\frac{k_1^3+k_2^3}{k_1^2+k_2^2} + \frac{k_2^3+k_3^3}{k_2^2+k_3^2}\right)\right), \tag{6}$$

with $0 < \alpha < 1$ and

$$\left.\begin{aligned} k_1 &= f(t_n, y_n), \\ k_2 &= f(t_n + a_1 h, y_n + a_1 h k_1), \\ k_3 &= f(t_n + (a_2+a_3)h, y_n + a_2 h k_1 + a_3 h k_2), \end{aligned}\right\} \tag{7}$$

with a_1, a_2, and a_3 are the parameters to be determined. To make presentation simple, we present the equation (7) in the autonomous form, which is as follows:

$$\left.\begin{aligned} k_1 &= f(y_n), \\ k_2 &= f(y_n + a_1hk_1), \\ k_3 &= f(y_n + a_2hk_1 + a_3hk_2). \end{aligned}\right\} \tag{8}$$

To obtain the value of parameters a_1, a_2, and a_3, firstly we expand $f(y)$ about $y = y_n$ using the Taylor series up to second order, so we have

$$f(y) = f(y_n) + (y - y_n)f'(y_n) + \frac{1}{2}(y - y_n)^2 f''(y_n) + O((y - y_n)^3). \tag{9}$$

Next the equation (9) is evaluated at $y = y_n + a_1hk_1$ for k_2 and $y = y_n + a_2hk_1 + a_3hk_2$ for k_3. Hence by writing $f(y_n) = f$, $f'(y_n) = f_y$, $f''(y_n) = f_{yy}$, we get

$$\left.\begin{aligned} k_1 &= f, \\ k_2 &= f + a_1ff_yh + \tfrac{1}{2}a_1^2f^2f_{yy}h^2 + O(h^3), \\ k_3 &= f + (a_3ff_y + a_2ff_y)h + \Big(\tfrac{1}{2}a_2^2f^2f_{yy} + a_1a_3ff_y^2 + a_2a_3f^2f_{yy} \\ &\quad + \tfrac{1}{2}a_3^2f^2f_{yy}\Big)h^2 + O(h^3). \end{aligned}\right\} \tag{10}$$

Furthermore by simplifying (6) we have

$$y_{n+1} = y_n + \frac{M}{N}, \tag{11}$$

where

$$\begin{aligned} M = h\Big(&\alpha k_1^4k_2^3 + \alpha k_1^4k_2^2k3 + \alpha k_1^4k_2k_3^2 + \alpha k_1^4k_3^3 - 2\alpha k_1^3k_2^3k_3 - \alpha k_1^3k_2^2k_3^2 - 2\alpha k_1^3k_2k_3^3 \\ &+\alpha k_1^3k_3^4 + \alpha k_1^2k_2^5 - \alpha k_1^2k_2^4k_3 - \alpha k_1^2k_2^2k_3^3 + \alpha k_1^2k_2k_3^4 - 2\alpha k_1k_2^5k_3 - \alpha k_1k_2^4k_3^2 \\ &-2\alpha k_1k_2^3k_3^3 + \alpha k_1k_2^2k_3^4 + 2\alpha k_2^7 + \alpha k_2^5k_3^2 + \alpha k_2^3k_3^4 + 2k_1^3k_2^4 + 4k_1^3k_2^3k_3 \\ &+2k_1^3k_2^2k_3^2 + 4k_1^3k_2k_3^3 + 2k_1^2k_2^4k_3 + 2k_1^2k_2^2k_3^3 + 2k_1k_2^6 + 4k_1k_2^5k_3 \\ &+2k_1k_2^4k_3^2 + 4k_1k_2^3k_3^3 + 2k_2^6k_3 + 2k_2^4k_3^3\Big), \end{aligned} \tag{12}$$

and

$$N = 2\,(k_1 + k_2)\left(k_1^2 + k_2^2\right)(k_2 + k_3)\left(k_2^2 + k_3^2\right). \tag{13}$$

On substituting the values of k_1, k_2, and k_3 in equations (10) into (12) and (13), we obtain respectively

$$\begin{aligned} M = {} & 32f^7h + (112a_1 + 56a_2 + 56a_3)f^7 f_y h^2 + ((56a_1^2 + 28a_2^2 + 56a_2a_3 \\ & + 28a_3^2)f^8 f_{yy} + (24\alpha a_1^2 - 24\alpha a_1a_2 - 24\alpha a_1a_3 + 12\alpha a_2^2 \\ & + 24\alpha a_2a_3 + 12\alpha a_3^2 + 176a_1^2 + 160a_1a_2 + 216a_1a_3 \\ & + 40a_2^2 + 80a_2a_3 + 40a_3^2)f^7 f_y^2)h^3 + O(h^4), \end{aligned} \tag{14}$$

and

$$\begin{aligned} N = {} & 32f^6 + (96a_1 + 48a_2 + 48a_3)f^6 f_y h + ((136a_1^2 + 104a_1a_2 + 152a_1a_3 + 32a_2^2 \\ & + 64a_2a_3 + 32a_3^2)f^6 f_y^2 + (48a_1^2 + 24a_2^2 + 48a_2a_3 + 24a_3^2)f^7 f_{yy})h^2 \\ & + ((112a_1^3 + 104a_1^2a_2 + 208a_1^2a_3 + 56a_1a_2^2 + 176a_1a_2a_3 + 120a_1a_3^2 \\ & + 8a_2^3 + 24a_2^2a_3 + 24a_2a_3^2 + 8a_3^3)f^6 f_y^3 + (136a_1^3 + 52a_1^2a_2 + 52a_1^2a_3 \\ & + 52a_1a_2^2 + 104a_1a_2a_3 + 52a_1a_3^2 + 32a_2^3 + 96a_2^2a_3 + 96a_2a_3^2 \\ & + 32a_3^3)f^7 f_y f_{yy})h^3 + O(h^4). \end{aligned} \tag{15}$$

Next expanding $y(t)$ at $t = t_n$ using the third order Taylor series and evaluated at $t = t_{n+1}$, the following equations is obtained

$$y_{n+1} = y_n + E, \tag{16}$$

where

$$E = fh + \frac{1}{2} f f_y h^2 + \frac{1}{6}\left(f f_y^2 + f^2 f_{yy}\right) h^3 + O(h^4), \tag{17}$$

and $h = t_{n+1} - t_n$. By matching the equation (11) and (16) we obtain

$$M = EN. \tag{18}$$

By substituting equation (15) and (17) into (18), we have

$$\begin{aligned} M = {} & 32f^7h + \Big(96a_1 + 48a_2 + 48a_3 + 16\Big)f^7 f_y h^2 + \Big(\Big(\frac{16}{3} + 152a_1a_3 + 32a_3^2 \\ & + 64a_2a_3 + 104a_1a_2 + 24a_3 + 32a_2^2 + 48a_1 + 24a_2 + 136a_1^2\Big)f^7 f_y^2 \\ & + \Big(24a_3^2 + \frac{16}{3} + 24a_2^2 + 48a_1^2 + 48a_2a_3\Big)f^8 f_{yy}\Big)h^3 + O(h^4). \end{aligned} \tag{19}$$

Then comparing the coefficients of h^j in equations (14) and (19) we obtain

$$\left.\begin{aligned} f^8 f_{yy} &: 4a_3^2+4a_2^2+8a_1^2+8a_2a_3=\tfrac{16}{3}, \\ f^7 f_y^2 &: -24\alpha a_1a_2+16a_2a_3+12\alpha a_2^2+12\alpha a_3^2+24\alpha a_2a_3 \\ &\quad +56a_1a_2+64a_1a_3-24\alpha a_3a_1-48a_1-24a_2 \\ &\quad -24a_3+8a_2^2+8a_3^2+24\alpha a_1^2+40a_1^2=\tfrac{16}{3}, \\ f^7 f_y &: 16a_1+8a_2+8a_3=16. \end{aligned}\right\} \tag{20}$$

Solving the equation (20) using Maple17 we have

$$\left.\begin{aligned} a_1 &= \tfrac{2}{3}, \\ a_2 &= \alpha-\tfrac{2}{3}, \\ a_3 &= -\alpha+\tfrac{4}{3}. \end{aligned}\right\} \tag{21}$$

Substituting a_1, a_2, and a_3 in (21) into (8), we obtain the formula of RK3L as follows:

$$y_{n+1}=y_n+\frac{h}{2}\left((1-\alpha)\left(\frac{2k_1k_2}{k_1+k_2}+\frac{2k_2k_3}{k_2+k_3}\right)+\alpha\left(\frac{k_1^3+k_2^3}{k_1^2+k_2^2}+\frac{k_2^3+k_3^3}{k_2^2+k_3^2}\right)\right), \tag{22}$$

with $0<\alpha<1$ and

$$\left.\begin{aligned} k_1 &= f(y_n), \\ k_2 &= f\left(y_n+\tfrac{2}{3}hk_1\right), \\ k_3 &= f\left(y_n+\left(\alpha-\tfrac{2}{3}\right)hk_1+\left(-\alpha+\tfrac{4}{3}\right)hk_2\right). \end{aligned}\right\} \tag{23}$$

Then by substituting the same a_1, a_2, and a_3 as found in equation (23) into (7), we get the formula RK3L method in a nonautonomous form as in equation (22) with the coefficients as follows:

$$k_1 = f(t_n, y_n),$$

$$k_2 = f\left(t_n+\frac{2}{3}h, y_n+\frac{2}{3}hk_1\right),$$

$$k_3 = f\left(t_n+\frac{2}{3}h, y_n+\left(\alpha-\frac{2}{3}\right)hk_1+\left(-\alpha+\frac{4}{3}\right)hk_2\right).$$

3. The Local Truncation Error of RK3L Method

We derive the local truncation error (LTE) of RK3L formulas in the autonomous form as in the equations (22) and (23). The LTE is obtained firstly by expanding $f(y)$ about $y=y_n$ using the third order Taylor series, it is given by

$$f(y)=f(y_n)+(y-y_n)f'(y_n)+\frac{1}{2}(y-y_n)^2f''(y_n)+\frac{1}{6}f'''(y_n)(y-yn)^3+O((y-yn)^4). \quad (24)$$

By evaluating the equation (24) at $y=y_n+\frac{2}{3}hk_1$ for k_2 and $y=y_n+\left(\alpha-\frac{2}{3}\right)hk_1+\left(-\alpha+\frac{4}{3}\right)hk_2$ for k_3 respectively, and by writing $f(y_n)=f$, $f'(y_n)=f_y$, $f''(y_n)=f_{yy}$, $f'''(y_n)=f_{yyy}$ we obtain

$$\left.\begin{aligned} k_1 &= f,\\ k_2 &= f+\tfrac{2}{3}ff_yh+\tfrac{2}{9}f^2f_{yy}h^2+\tfrac{4}{81}f^3f_{yyy}h^3+O(h^4),\\ k_3 &= f+\tfrac{2}{3}ff_yh+\left(-\tfrac{2}{3}\alpha ff_y^2+\tfrac{8}{9}ff_y^2+\tfrac{2}{9}f^2f_{yy}\right)h^2\\ &\quad+\left(\tfrac{4}{81}f^3f_{yyy}+\tfrac{8}{9}f^2f_yf_{yy}-\tfrac{2}{3}\alpha f^2f_yf_{yy}\right)h^3+O(h^4). \end{aligned}\right\} \quad (25)$$

Substituting the equation (25) into (22) we obtain

$$\begin{aligned} y_{n+1}=y_n+fh+\frac{1}{2}ff_yh^2+\frac{1}{6}\left(ff_y^2+f^2f_{yy}\right)h^3+\Bigg(\left(-\frac{473}{54}+\frac{17}{6}\alpha\right)ff_y^3\\ +\left(\alpha-\frac{131}{27}\right)f^2f_yf_{yy}-\frac{2}{9}f^3f_{yyy}\Bigg)h^4+O(h^5). \end{aligned} \quad (26)$$

Furthermore, expanding $y(t)$ about $t=t_n$ using the fourth order Taylor series and evaluating at $t=t_{n+1}$ we have

$$\begin{aligned} y_{n+1}=y_n+fh+\frac{1}{2}ff_yh^2+\frac{1}{6}\left(ff_y^2+f^2f_{yy}\right)h^3+\frac{1}{24}\big(f^3f_{yyy}\\ +4f^2f_yf_{yy}+ff_y^3\big)h^4+O(h^5), \end{aligned} \quad (27)$$

where $h=t_{n+1}-t_n$. Then subtracting the equations (26) from (27) we obtain the LTE of RK3L method as follows:

$$LTE=\left(\left(-\frac{1901}{216}+\frac{17}{6}\alpha\right)ff_y^3+\left(-\frac{271}{54}+\alpha\right)f^2f_yf_{yy}-\frac{19}{72}f^3f_{yyy}\right)h^4+O(h^5).$$

4. Stability of RK3L Method

The stability of the method is obtained by solving differential equation $Y'(t)=\lambda Y(t)$ with initial value $Y(0)=1$ as suggested by Dahlquist [5, p.374]. The first step to obtain the stability of RK3L method is by substituting $f(y_n) = \lambda y_n$ into the equation (23), then we get

$$\left.\begin{aligned} k_1 &= \lambda y_n, \\ k_2 &= \lambda y_n \left(1 + \tfrac{2}{3} h\lambda\right), \\ k_3 &= -\tfrac{1}{9}\lambda y_n \left(6\alpha h^2\lambda^2 - 8h^2\lambda^2 - 6h\lambda - 9\right). \end{aligned}\right\} \tag{28}$$

Substituting the equation (28) into (22), the following equation is obtained:

$$\frac{y_{n+1}}{y_n} = 1 + \frac{1}{54}\lambda^4 h^4 - \frac{1}{18}\alpha\lambda^4 h^4 + \frac{1}{6}\lambda^3 h^3 + \frac{1}{2}\lambda^2 h^2 + \lambda h. \tag{29}$$

Hence by writing $z = \lambda h$ and taking the right-hand side of the equation (29) up to z^3 we obtain the polynomial stability RK3L method as follows:

$$\frac{y_{n+1}}{y_n} = \frac{1}{6}z^3 + \frac{1}{2}z^2 + z + 1. \tag{30}$$

Polynomial stability of RK3L method (30) equals the stability of RK3AM method [6]. The stability area of the RK3L method is shown in Figure 1.

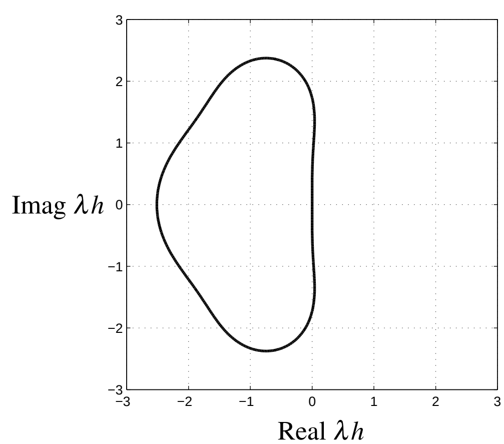

FIGURE 1. The stability area of RK3L method

Figure 1 shows that for the value of λ less than zero and real, the stability region of the RK3L method is obtained if $\lambda h > -2.5$. Therefore, in this case we must choose the step length $0 < h < \frac{-2.5}{\lambda}$.

5. Numerical Comparisons

To see the effectiveness of the method, RK3L method is used to solve the two following problems:

(i) **Problem 1**: $Y'(t) = (\cos(Y(t)))^2$, with $Y(0) = 0$, with the exact solution $Y(t) = \arctan(t)$ at $[0,1]$.

(ii) **Problem 2**: $Y'(t) = Y(t)^2 + (2tY(t)+2)\sin^3(2t)$, with $Y(1) = -1$, with the exact solution $Y(t) = -\frac{1}{t}$ at $[1,2]$.

Furthermore RK3L method is compared with RK3AM, RK3 based on harmonic mean (RK3HM), and RK3 based on geometry mean (RK3GM) methods for each problems. Computational results are presented in Table 1.

To find the best α for the problems, we vary $\alpha \in (0,1)$ and by looking into the error of RK3L method, we conclude that the best α is close to zero and $\alpha = 0.32$ for Problem 1 and Problem 2 respectively, as shown in Figure 2.

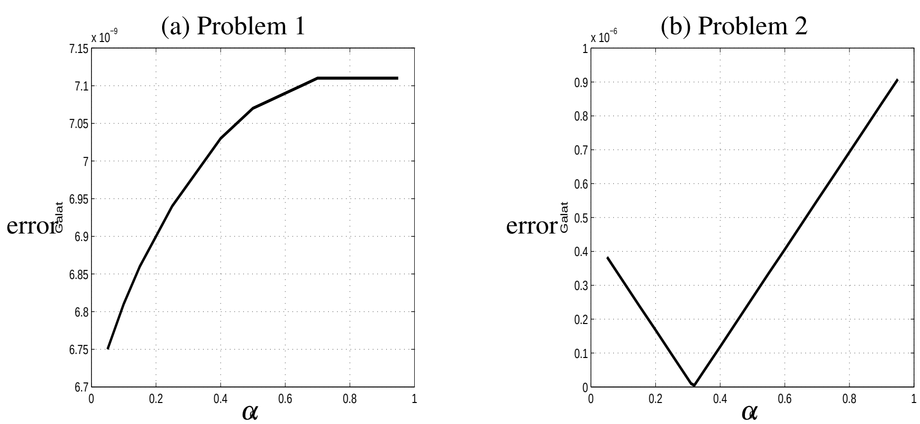

FIGURE 2. Error of RK3L method for $\alpha \in (0,1)$ for the Problem 1 and Problem 2

Table 1 shows that the error of RK3L method for solving Problem 1 and Problem 2, using $\alpha = 1/6$ and $\alpha = 0.32$ respectively. We can see that the error of RK3L method and is smaller than those of RK3AM and RK3GM methods for Problem 1. For Problem 2, the error of RK3L method is smaller than those of the other methods. Furthermore, the computational results of RK3L method also show that the smaller error is generated if the h is closer to zero. Hence RK3L method can be used as an alternative method of third order method.

TABLE 1. Errors of third order methods for Problem 1 and Problem 2

IVP	h	$\lvert Y(t_N) - y(t_N)\rvert$			
		RK3L	RK3AM	RK3HM	RK3GM
Problem 1	$h = \frac{1}{50}$	5.440026e−008	5.650845e−008	5.149991e−008	5.444816e−008
	$h = \frac{1}{100}$	6.873405e−009	7.003736e−009	6.694238e−009	6.876355e−009
	$h = \frac{1}{200}$	8.636510e−010	8.717527e−010	8.525188e−010	8.638341e−010
Problem 2	$h = \frac{1}{50}$	1.160229e−007	1.948537e−007	1.761876e−006	7.826091e−007
	$h = \frac{1}{100}$	4.558229e−009	2.374023e−008	4.549522e−007	2.155516e−007
	$h = \frac{1}{200}$	1.811856e−009	2.929544e−009	1.155525e−007	5.630814e−008

Conflict of Interests

The authors declare that there is no conflict of interests.

REFERENCES

[1] O. Y. Ababneh and R. Rozita, *New third order Runge Kutta based on contraharmonic mean for stiff problems*, Appl. Math. Sci. 3 (8) (2009), 365-376.

[2] K. E. Atkinson, *An Introduction to Numerical Analysis, Second Ed.*, John Wiley & Sons, New York, 1988.

[3] K. E. Atkinson, H. Weimin, and E. S. David, *Numerical Solution of Ordinary Differential Equation*, John Wiley & Sons, Canada, 2009.

[4] R. G. Bartle and D. R. Shebert, *Introduction to Real Analysis, Third Ed.*, John Wiley & Sons, Inc. New York, 2000.

[5] G. Dahlquist and A. Bjorck, *Numerical Method*, Prentice-Hall, New York, 1974.

[6] D. J. Evans, *New Runge-Kutta methods for initial value problems*, Appl. Math. Lett. 2 (1) (1989), 25–28.

[7] D. H. Lehmer, *On the compounding of certain means*, J. Math. Anal. Appl., 36 (1) (1971), 183-200.

[8] R. T. Rockafellar, *Convex Analysis*, Princeton University Press, Washington, 1970.

[9] K. A. Ross, *Elementary Analysis: The Theory of Calculus*, Springer, New York, 1980.

[10] L. P. Shampine, *Numerical of Ordinary Differential Equation*. Chapman and Hall, New York, 1994.

[11] L. P. Shampine and H.A. Watts, *Practical solution of ordinary differential equation by Runge-Kutta methods*, Report SAND76-0585, New Mexico: Sandia Laboratories, 1976.

[12] A. M. Wazwaz, *A modified third order Runge Kutta method*. Appl. Math. Lett. 3 (3) (1990), 123-125.

[13] A. M. Wazwaz, *A comparison of modified Runge Kutta formulas based on a variety of means*. Int. J. Comput. Math. 50 (1-2) (1994), 105-112.

[14] R. Yanti, M. Imran, and Syamsudhuha, *A third Runge Kutta method based on linear combination of arithmetic mean, harmonic mean and geometric mean*, Appl. Comput. Math. 3 (5) (2014), 231-234.