Available online at http://scik.org
J. Math. Comput. Sci. 8 (2018), No. 6, 689-704
https://doi.org/10.28919/jmcs/3815
ISSN: 1927-5307

EXISTENCE OF MILD SOLUTION FOR NONLOCAL IMPULSIVE FRACTIONAL SEMILINEAR DIFFERENTIAL INCLUSION IN BANACH SPACE

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Abstract. In this paper, we deal with the existence of PC-mild solutions of nonlocal impulsive differential inclusions in Banach space when the values of the orient field is convex (P). By using methods and results of semilinear differential inclusions, and techniques of fixed point theorems, we establish sufficient conditions that guarantee the existence of PC-mild solutions of (P). Our results develop and extend various results proved recently.

Keywords: impulsive differential inclusions; nonlocal conditions; fixed point theorems; mild solutions.

2010 AMS Subject Classification: 34A60, 34B37, 34G10.

1. Introduction

Fractional differential equations and fractional differential inclusions have been an object of interest since two decades due to their wide applications in various fields, such as physics, biology, mechanics and engineering, medical field, industry and technology. Also fractional differential equations are used as an excellent tool for the description of hereditary properties of

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Received July 26, 2018

various materials and processes. For instance, we refer to [18], [9], [11], [15] and the references therein.

In particular, impulsive differential equations and impulsive differential inclusions have gained much more attention because they serve as an appropriate model to describe processes which can not be described by classical differential equations, such as processes which at certain moments change their state rapidly. For some of these applications, one can see, [5], [1] and the references therein. During the last ten years, impulsive differential inclusions with different conditions have been intensely investigated by numerous researchers, we refer readers to [4], [13], [20], [17], [3], [8], [19], [10], [12], [22], [7], [24]. Moreover, a strong motivation for studying the nonlocal Cauchy problems comes from physical problems. For instance, using nonlocal Cauchy problems in determining unknown parameters in some inverse heat condition problems, see [4], [13].

In this study, we are concerned with the existence of mild solution for the following impulsive nonlocal Cauchy problem of fractional order $\alpha \in (0,1)$ driven by a semilinear differential inclusion in a real separable Banach space E of the form

$$\begin{cases} {}^cD^{\alpha}x(t) \in Ax(t) + F(t,x(t)),\ t \in J = [0,b],\ t \neq t_i, i = 1,...,m, \\ x(t_i^+) = x(t_i) + I_i(x(t_i)), i = 1,...,m, \\ x(0) = g(x), \end{cases} \tag{P}$$

where ${}^cD^{\alpha}$ is the Caputo derivative of order α, $A : D(A) \subseteq E \to E$ is the infinitesimal generator of a C_0-semigroup $\{T(t), t \geq 0\}$ on a Banach space E, $F : J \times E \to 2^E$ be a multifunction, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = b$, for every $i = 1,2,...,m$ $I_i : E \to E$ impulsive functions which characterize the jump of the solutions at impulse points, $g : PC(J,E) \to E$, is a nonlinear function related to the nonlocal condition at the origin and $x(t_i^+), x(t_i^-)$ are the right and left limits of x at the point t_i respectively and $PC(J,E)$ will be defined later.

The concept of mild solution was firstly introduced by Mophou [19], inspired by Jaradat et.al. [17]. Since then, the main goal for many mathematicians has been to establish sufficient conditions regarding the existence of mild solution for differential equations or inclusions problems. In [3], R. Al-Omair and A. Ibrahim studied (P) without impulsive, when $\alpha = 1$. Also Fan

[12] considered a nonlocal Cauchy problem in the presence of impulses, and controlled by autonomous semilinear differential equation. While Wang et.al. [23] introduced a new concept of PC-mild solutions for (P) and obtained existence and uniqueness results concerning the PC-mild solutions for (P) when F is a Lipschitz single-valued function or continuous and maps bounded sets into bounded sets and $\{T(t)\}_{t>0}$ is compact. Cardinali et.al. [7] proved the existence of mild solutions to the problem (P) when $\alpha = 1$ and the multivalued function F satisfies the lower Scorza-Dragoni property and $\{A(t)\}_{t\geq 0}$ is a family of linear operator, generating a strongly continuous evolution operators. Recently, A. G. Ibrahim and N. Almoulhim [16] discussed (p) without impulsive.

Motivated by the above works, by using properties of multifunctions, some methods and results semilinear differential inclusions, and fixed point theorems, we develop the results shown in [16] as well as we extend the results in [23] to the case when (P) is taken with impulsive and nonlocal conditions.

This paper is organized as: Section 2 recalls some basic foundations related to multifunctions and fractional calculus to be used later. In section 3, the existence of mild solution for (P) is proved. We adopt the definition of mild solution introduced by Wang et.al. [23]. We used the properties of multifunctions, methods and results regarding semilinear differential inclusions, and fixed point techniques to obtain the results.

2. Preliminaries

During this section, we state some previous known results so that we can use them later throw out this paper. Let $C(J,E)$ be the Banach space of all $E-$valued continuous functions from J into E with the uniform norm $\|x\| = sup\{\|x(t)\|, t\in J\}$, $L^1(J,E)$ the space of $E-$valued Bochner integrable functions on J with the norm $\|f\|_{L^1(J,E)} = \int_0^b \|f(t)\|dt$,

$P_b(E) = \{B \subseteq E : B$ is nonempty and bounded $\}$,

$P_{cl}(E) = \{B \subseteq E : B$is nonempty and closed $\}$,

$P_k(E) = \{B \subseteq E : B$ is nonempty and compact $\}$,

$P_{cl,cv}(E) = \{B \subseteq E : B$ is nonempty, closed and convex $\}$,

$P_{ck}(E) = \{B \subseteq E : B$ is nonempty, convex and compact $\}$,

$conv(B)$, $\overline{conv}(B)$ be the convex hull and convex closed hull in E of subset B.

Considering a division of $[0,b]$, i.e a finite set $\{t_0,\cdots,t_m,t_{m+1}\} \subset [0,b]$ such that $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = b$, we put $J_0 = [0,t_1]$, $J_i =]t_i,t_{i+1}]$ and $x(t_i^+) = \lim_{s\to t_i^+} x(s)$, $1 \le i \le m$.

We also introduce the set of functions

$$PC(J,E) = \{x : J \to E : x_{|J_i} \in C(J_i,E) \text{ and } x(t_i^+) \text{ and } x(t_i^-) \text{ exist}\}.$$

It is easy to check that $PC(J,E)$ is a Banach space endowed with the norm

$$\|x\|_{PC(J,E)} = max\{\|x(t)\| : t \in J\}$$

For any subset $B \subseteq PC(J,E)$ and for any $i = 0,1,...,m$, let

$$B_{|\overline{J_i}} = \{x^* : \overline{J_i} \longrightarrow E : x^*(t) = x(t), t \in J_i \text{ and } x^*(t_i) = x(t_i^+), x \in B\}.$$

Of course $B_{|\overline{J_0}} = \{x_{|\overline{J_0}} : x \in B\}$.

Definition 2.1. [21] A semigroup $T(t), 0 \le t < \infty$, of bounded linear operators on a Banach space X is said to be

(1) uniformly continuous if $\lim_{t\to 0^+} \|T(t) - I\| = 0$, where I is the identity operator;
(2) strongly continuous if $\lim_{t\to 0^+} T(t)x = x$, for every $x \in X$.

A strongly continuous semigroup of bounded linear operators on X will be called a semigroup of class C_0 or simply a C_0-semigroup. It is known that if $T(t), 0 \le t < \infty$ is a C_0-semigroup, then there exist constant $\omega \ge 0$ and $M \ge 1$ such that $\|T(t)\| \le Me^{\omega t}$, for $0 \le t < \infty$. A C_0-semigroup $T(t), 0 \le t < \infty$ is called compact if for every $t > 0, T(t)$ is compact. It is known that ([21], Theorem 3.2) every compact C_0-semigroup is uniformly continuous.

Definition 2.2. [21] Let $T(t), 0 \le t < \infty$, be a semigroup of bounded linear operators an a Banach space X. The linear operator A defined by

$$D(A) = \{x \in X : \lim_{t\to 0^+} \frac{T(t) - x}{t} \text{ exists}\}$$

is called the infinitesimal generator of the semigroup $T(t), D(A)$ is the domain of A.

Definition 2.3. ([2], [14]) Let X and Y be two topological spaces. A multifunction $F : X \to P(Y)$ is said to be

(i) upper semicontinuous $(u.s.c)$ if $F^{-1}(V) = \{x \in X : F(x) \subseteq V\}$ is an open subset of X for every open $V \subseteq Y$;

(ii) closed if its graph $\Gamma_F = \{(x,y) \in X \times Y : y \in F(x)\}$ is closed subset of the topological space $X \times Y$;

(iii) completely continuous if $F(B)$ is relatively compact for every bounded subset B of X.

Note that if the multifunction F is completely continuous with non empty compact values, then F is $u.s.c.$ if and only if F is closed.

Lemma 2.1. [14] Let X,Y be (not necessarily separable) Banach spaces, and let $F : J \times X \to P_k(Y)$ be such that

(i) for every $x \in X$ the multifunction $F(\cdot,x)$ has a strongly measurable selection;

(ii) for $a.e.\ t \in J$ the multifunction $F(t,\cdot)$ is upper semicontinuous.

Then for every strongly measurable function $z : J \to X$ there exists a strongly measurable function $f : J \to Y$ such that $f(t) \in F(t,z(t))\ a.e.$

Remark 2.1. [14] For single-valued or compact valued multifunction acting on a separable Banach space the notions measurability and strongly measurable coincide. So, if X,Y be separable Banach spaces, we can replace strongly measurable with measurable in the previous lemma.

Definition 2.4. A sequence $\{f_n : n \in \mathbb{N}\} \subset L^1(J,E)$ is said to be semi-compact if:

(i) It is integrably bounded i.e. there is $q \in L^1(J,\mathbb{R}^+)$ such that

$$\|f_n(t)\| \leq q(t)\ a.e.\ t \in J.$$

(ii) The set $\{f_n : n \in \mathbb{N}\}$ is relatively compact in $E\ \ a.e.\ t \in J$.

Lemma 2.2. [14] Every semi-compact sequence in $L^1(J,E)$ is weakly compact in $L^1(J,E)$.

Lemma 2.3. [23] For $\tau \in (0,1]$ and $0 < e \leq c$, we have $|e^\tau - c^\tau| \leq (c-e)^\tau$.

Definition 2.5. According to the Riemann-Liouville approach, the fractional integral of order $\alpha \in (0,1)$ of a function $f \in L^1(J,E)$ is defined by

$$I^\alpha f(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds,\ t > 0,$$

provided the right side is defined on J, where Γ is the Euler gamma function defined by $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}dt$.

Definition 2.6. The Caputo derivative of order $\alpha \in (0,1)$ of continuously differentiable function $f : J \to E$ is defined by

$$^{c}D^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f^{(1)}(s)ds = I^{(1-\alpha)} f^{(1)}.$$

Note that the integral appear in the two previous definitions are taken in Bochner'sense and $^{c}D^{\alpha} I^{\alpha} f(t) = f(t)$ for all $t \in J$. For more informations about the fractional calculus we refer to ([11], [18]).

Definition 2.7. A function $x \in PC(J,E)$ is an impulsive mild solution for (p) if

$$x(t) = \begin{cases} K_1(t)g(x) + \int_0^t (t-s)^{\alpha-1} K_2(t-s) f(s)ds, t \in J_0, \\ K_1(t)g(x) + \sum_{i=1}^{i=m} K_1(t-t_i)y_i + \int_0^t (t-s)^{\alpha-1} K_2(t-s) f(s)ds, t \in J_i, \end{cases}$$

where $y_i = I_i(x(t_i^-)), i = 1,2,...,m$, f is an integrable selection for $F(\cdot, x(\cdot))$,

$$K_1(t) = \int_0^{\infty} \xi_\alpha(\theta) T(t^\alpha \theta) d\theta, K_2(t) = \alpha \int_0^{\infty} \theta \xi_\alpha(\theta) T(t^\alpha \theta) d\theta,$$

$$\xi_\alpha(\theta) = \frac{1}{\alpha} \theta^{-1-\frac{1}{\alpha}} \varpi_\alpha(\theta^{\frac{-1}{\alpha}}) \geq 0, \varpi_\alpha(\theta) = \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^{n-1} \theta^{-\alpha n-1} \frac{\Gamma(n\alpha+1)}{n!} \sin(n\pi\alpha),$$

where $\theta \in (0,\infty)$ and ξ is a probability density function defined on $(0,\infty)$, that is $\int_0^\infty \xi_\alpha(\theta) d\theta = 1$.

Remark 2.2. Since $K_1(\cdot), K_2(\cdot)$ are associated with the number α, there are no analogue of the semigroup property, i.e. $K_1(t+s) \neq K_1(t)K_1(s)$, $K_2(t+s) \neq K_2(t)K_2(s)$.

In the following we recall the properties of $K_1(\cdot), K_2(\cdot)$.

Lemma 2.4. [25]

(i) For any fixed $t \geq 0, K_1(t), K_2(t)$ are linear bounded operators.
(ii) For $\gamma \in [0,1], \int_0^\infty \theta^\gamma \xi_\alpha(\theta) d\theta = \dfrac{\Gamma(1+\gamma)}{\Gamma(1+\alpha\gamma)}$.
(iii) If $\|T(t)\| \leq M, t \geq 0$, then for any $x \in E, \|K_1(t)x\| \leq M\|x\|$ and $\|K_2(t)x\| \leq \dfrac{M}{\Gamma(\alpha)} \|x\|$.
(iv) For any fixed $t \geq 0, K_1(t), K_2(t)$ are strongly continuous.
(v) If $T(t), t > 0$ is compact, then $K_1(t), K_2(t)$ are compact.

Theorem 2.1. [6] Let B be a nonempty subset of a Banach space E, which is bounded, closed and convex. Suppose $G : B \to 2^E$ is u.s.c. with closed and convex value such that $G(B) \subseteq B$ and $G(B)$ is compact. Then G has a fixed point.

3. Main results

Now, after the preliminaries are laid, we encounter our main problem (P). The following theorem proves the existence of mild solution for (P).

Theorem 3.1. Let $F : J \times E \to P_{ck}(E)$ be a multifunction. Assume the following conditions:

(C$_1$) A is the infinitesimal generator of a C_0-semigroup $T(t) : t \geq 0$ and $T(t), t > 0$ is compact.

(C$_2$) For every $x \in E, t \to F(t,x)$ is measurable, for almost $t \in J, x \to F(t,x)$ is upper semicontinuous.

(C$_3$) There exist a function $\varphi \in L^{\frac{1}{q}}(J, \mathbb{R}^+), 0 < q < \alpha$ such that for any $x \in E$

$$\|F(t,x)\| \leq \varphi(t)\|x\|, \ a.e. t \in J.$$

(C$_4$) $g : PC(J,E) \to E$ is continuous, compact and there exist two positive numbers a, d such that

$$\|g(x)\| \leq a\|x\| + d, \forall x \in PC(J,E).$$

(C$_5$) For every $i = 1,2,...,m$, I_i is continuous, compact and there exists a positive constant h_i such that

$$\|I_i(x)\| \leq h_i\|x\|, \ x \in E.$$

Then the problem (P) has a mild solution provided that there is $r > 0$ such that

$$M(d + r(a + h + \gamma)) \leq r \tag{3.1}$$

where $M > 0$ such that $\sup_{t\in J} \|T(t)\| \leq M$, and

$$h = \sum_{i=1}^{i=m} h_i, \gamma = \frac{b^{\alpha-q}}{\Gamma(\alpha)(\varpi+1)^{1-q}} \|\varphi\|_{L^{\frac{1}{q}}(J,\mathbb{R}^+)},$$

where $\varpi = \dfrac{\alpha-1}{1-q}$.

Proof. From C_2, Lemma 2.1 and Remark 2.1 the set

$$S^1_{F(\cdot,x(\cdot))} = \{f \in L^1(J,E) : f(t) \in F(t,x(t)), \ a.e.\}$$

is nonempty, for any $x \in PC(J,E)$. Thus, we can consider the operator $R : PC(J,E) \to 2^{PC(J,E)}$, which defined by $y \in R(x)$ if and only if

$$y(t) = \begin{cases} K_1(t)g(x) + \int_0^t (t-s)^{\alpha-1} K_2(t-s) f(s) ds, t \in J_0, \\ K_1(t)g(x) + \sum_{k=1}^{k=i} K_1(t-t_k) I_k(x(t_k^-)) \\ + \int_0^t (t-s)^{\alpha-1} K_2(t-s) f(s) ds, t \in J_i, i = 1,...,m. \end{cases} \tag{3.2}$$

Where $f \in S^1_{F(\cdot,x(\cdot))}$. It can be easily to see that any fixed point for R is a mild solution for the problem (P). So, our goal is to prove that R satisfies the conditions of Theorem 2.1 in the preliminaries. The proof will be given in six steps.

Step 1. We show that the values of R are convex subset in $PC(J,E)$.

Let $x \in PC(J,E), y_1, y_2 \in R(x), \lambda \in (0,1)$ and let $t \in J_0$. From the definition of R we have

$$\lambda y_1(t) + (1-\lambda) y_2(t) = K_1(t)g(x) + \int_0^t (t-s)^{\alpha-1} K_2(t-s)[f_1(s) + (1-\lambda) f_2(s)] ds,$$

where $f_1, f_2 \in S^1_{F(t,x(t))}$. It is easy to see that $S^1_{F(t,x(t))}$ is convex since F has convex values. So $\lambda f_1 + (1-\lambda) f_2 \in S^1_{F(t,x(t))}$. Therefore, $\lambda y_1(t) + (1-\lambda) y_2(t) \in R(x)$. By the same way we can show that $\lambda y_1(t) + (1-\lambda) y_2(t) \in R(x)$ for $t \in J_i, i = 1,2,...,m$. Which means that $R(x)$ is convex for each $x \in PC(J,E)$.

Step 2. We prove that $R(x)$ is closed for every $x \in PC(J,E)$. To prove that the values of R are closed, let $x \in PC(J,E)$ and $(z_n)_{n\geq 1}$ be a sequence in $R(x)$ such that $z_n \to z$ in $PC(J,E)$. Then, we need to prove that $z \in R(x)$. According to the definition of R there is a sequence $(f_n)_{n\geq 1}$, in $S^1_{F(t,x(t))}$ such that for any $t \in J_i$, $i = 0,1,..,m$.

$$z_n(t) = \begin{cases} K_1(t)g(x) + \int_0^t (t-s)^{\alpha-1} K_2(t-s) f_n(s) ds, \ t \in J_0, \\ K_1(t)g(x) + \sum_{k=1}^{k=i} K_1(t-t_k) I_k(x(t_k^-)) \\ + \int_0^t (t-s)^{\alpha-1} K_2(t-s) f_n(s) ds, t \in J_i, \ i = 1,...,m. \end{cases} \tag{3.3}$$

From C_3, for any $n \geq 1$,

$$\|f_n(t)\| \leq \varphi(t)\|x\| \ a.e. \ t \in J.$$

This show that the set $\{f_n : n \geq 1\}$ is integrably bounded. Also, because $\{f_n(t) : n \geq 1\} \subset F(t,x(t))$, for $a.e.t \in J$, the set $\{f_n(t) : n \geq 1\}$ is relatively compact in E for $a.e.t \in J$. Therefore, the set $\{f_n : n \geq 1\}$ is semicompact and then, by Lemma 2.2 it is weakly compact in $L^1(J,E)$. We can assume that f_n converges weakly to a function $f \in L^1(J,E)$. From Mazur's lemma, there is a sequence $(g_n), n \geq 1$ such that $\{g_n(t) : n \geq 1\} \subseteq \overline{Conv}\{f_n(t) : n \geq 1\}; t \in J$ and g_n converges strongly to f. Since, the values of F are convex, $g_n \in S^1_{F(t,x(t))}$ and hence, by the compactness of $F(t,x(t)), f \in S^1_{F(t,x(t))}$. Moreover, for every $t,s \in J$ and for every $n \geq 1$,

$$\|(t-s)^{\alpha-1}K_2(t-s)f_n(s)\| \leq | t-s |^{\alpha-1} \frac{M}{\Gamma(\alpha)}\varphi(s)\|x\| \in L^1(J,\mathbb{R}^+).$$

Therefore, based on the Lebesgue dominated convergence theorem, taking $n \to \infty$ on both sides of (3.3) we get for every $i = 0,1,..,m$,

$$z(t) = \begin{cases} K_1(t)g(x) + \int_0^t (t-s)^{\alpha-1}K_2(t-s)f(s)ds, t \in J_0, \\ K_1(t)g(x) + \sum_{k=1}^{k=i} K_1(t-t_k)I_k(x(t_k^-)) \\ + \int_0^t (t-s)^{\alpha-1}K_2(t-s)f(s)ds, t \in J_i,\ i = 1,...,m. \end{cases}$$

Which means that $z \in R(x)$.

Step 3. For each positive number $r > 0$, let $B_r = \{x \in PC(J,E) : \|x\| \leq r\}$.

We claim that $R(B_r) \subseteq B_r$. To prove that, let $x \in B_r, y \in R(x)$ and let $t \in J_0$ we have

$$\begin{aligned} \|y(t)\| &\leq \|K_1(t)g(x) + \int_0^t (t-s)^{\alpha-1}K_2(t-s)f(s)ds\| \\ &\leq \|K_1(t)g(x)\| + \|\int_0^t (t-s)^{\alpha-1}K_2(t-s)f(s)ds\|, \end{aligned}$$

by Lemma 2.4, C_3, C_4 and (3.1) we get

$$\begin{aligned} \|y(t)\| &\leq M(ar+d) + \frac{M}{\Gamma(\alpha)}\|x\| \int_0^t (t-s)^{\alpha-1}\varphi(s)ds \\ &\leq M(ar+d) + \frac{M}{\Gamma(\alpha)} r\|\varphi\|_{L^{\frac{1}{q}}(J,\mathbb{R}^+)} [\int_0^t (t-s)^{\frac{\alpha-1}{1-q}}ds]^{1-q} \\ &\leq M(ar+d) + Mr\frac{b^{\alpha-q}}{\Gamma(\alpha)(\varpi+1)^{1-q}}\|\varphi\|_{L^{\frac{1}{q}}(J,\mathbb{R}^+)} \\ &\leq M(ar+d+r\gamma) \leq r. \end{aligned}$$

Similarly, by using Lemma 2.4, C_3, C_4, C_5 and (3.1) we have for every $t \in J_i, i = 1,2,...,m$,

$$\|y(t)\| \leq M(d + r(a+h+\gamma)) \leq r.$$

Therefore, we conclude that $R(B_r) \subseteq B_r$.

step 4. R sends bounded sets into equicontinuous sets of $PC(E,J)$. Put $B = R(B_r)$. We claim that B is equicontinuous, let $x \in B_r$ and $y \in R(B_r)$. According to (3.2) we have

$$y(t) = \begin{cases} K_1(t)g(x) + \int_0^t (t-s)^{\alpha-1} K_2(t-s) f(s) ds, t \in J_0, \\ K_1(t)g(x) + \sum_{k=1}^{k=i} K_1(t-t_k) I_k(x(t_k^-)) \\ + \int_0^t (t-s)^{\alpha-1} K_2(t-s) f(s) ds, t \in J_i, i = 1, ..., m, \end{cases}$$

where $f \in S^1_{F(\cdot,x(\cdot))}$. To show that $B_{|J}$ is equicontinuous it suffices to verify that $B_{|\overline{J_i}}$ is equicontinuous for every $i = 0, 1, ..., m$, where

$$B_{|\overline{J_i}} = \{x^* \in C(\overline{J_i}, E) : x^*(t) = x(t), t \in J_i =]t_i, t_{i+1}], x^*(t_i) = x(t_i^+), x \in B\}.$$

We study the following cases:

case 1. If $i = 0$. Let $t, t+\lambda$ be two points in $\overline{J_0} = J_0$, then

$$\|y^*(t+\lambda) - y^*(t)\| = \|y(t+\lambda) - y(t)\| \leq \|K_1(t+\lambda)g(x) - K_1(t)g(x)\|$$
$$+ \| \int_0^{t+\lambda} (t+\lambda-s)^{\alpha-1} K_2(t+\lambda-s) f(s) ds - \int_0^t (t-s)^{\alpha-1} K_2(t-s) f(s) ds\|$$
$$\leq R_1 + R_2 + R_3 + R_4,$$

where

$$R_1 = \|K_1(t+\lambda)g(x) - K_1(t)g(x)\|,$$
$$R_2 = \| \int_t^{t+\lambda} (t+\lambda-s)^{\alpha-1} K_2(t+\lambda-s) f(s) ds\|,$$
$$R_3 = \| \int_0^t [(t+\lambda-s)^{\alpha-1} - (t-s)^{\alpha-1}] K_2(t+\lambda-s) f(s) ds\|,$$
$$R_4 = \|(t-s)^{\alpha-1} [K_2(t+\lambda-s) - K_2(t-s)] f(s) ds\|.$$

We only need to check $R_i \to 0$ as $\lambda \to 0$ for every $i = 1, 2, 3, 4$. Since $K_1(t), t > 0$, is uniformly continuous on J. So, $R_1 \to 0$ as $\lambda \to 0$ independently of $x \in B_r$. For R_2, by the Holder inequality we have

$$\lim_{\lambda\to 0} R_2 = \lim_{\lambda\to 0} \| \int_t^{t+\lambda} (t+\lambda-s)^{\alpha-1} K_2(t+\lambda-s) f(s)ds\|$$
$$\leq \frac{M}{\Gamma(\alpha)} \lim_{\lambda\to 0} \int_t^{t+\lambda} (t+\lambda-s)^{\alpha-1} \|f(s)\|ds$$
$$\leq \frac{Mr}{\Gamma(\alpha)} \lim_{\lambda\to 0} [\frac{\lambda^{(\varpi+1)}}{\varpi+1}]^{1-q} \|\varphi\|_{L^{\frac{1}{q}}(J,\mathbb{R}^+)} = 0,$$

independently of $x\in B_r$.

For R_3 we have $\varpi = \dfrac{\alpha-1}{1-q} \in (-1,0)$, then for $s < t \leq t+\lambda$, we have $(t-s)^{\varpi} \geq (t+\lambda-s)^{\varpi}$. By applying Lemma 2.3 and taking into account $1-q\in(0,1)$ we get

$$\lim_{\lambda\to 0} R_3 \leq \frac{Mr}{\Gamma(\alpha)} \lim_{\lambda\to 0} [\int_0^t ((t-s)^{\varpi} - (t+\lambda-s)^{\varpi})ds]^{1-q} \|\varphi\|_{L^{\frac{1}{q}}(J,\mathbb{R}^+)}$$
$$\leq \frac{Mr}{\Gamma(\alpha)} \lim_{\lambda\to 0} [\frac{1}{\varpi+1}(t^{\varpi+1} + \lambda^{\varpi+1} - (t+\lambda)^{\varpi+1})]^{1-q} \|\varphi\|_{L^{\frac{1}{q}}(J,\mathbb{R}^+)} = 0,$$

independently of $x\in B_r$.

For R_4, by using the Lebesgue dominated convergence theorem and the condition C_1, we obtain

$$\lim_{\lambda\to 0} R_4 \leq \lim_{\lambda\to 0} \| \int_0^t (t-s)^{\alpha-1} [K_2(t+\lambda-s) - K_2(t-s)] f(s)ds\|$$
$$\leq \lim_{\lambda\to 0} \sup_{s\in[0,t]} \|K_2(t+\lambda-s) - K_2(t-s)\| \int_0^t (t-s)^{\alpha-1} \|f(s)\|ds.$$

Since $K_2(t), t>0$, is uniformly continuous, so, we conclude that $R_4 \to 0$ as $\lambda \to 0$, independently of $x\in B_r$.

Case 2. If $i\in\{1,2,...,m\}$, let $t, t+\lambda$ be two points in J_i. Recalling (3.2) we have

$$\|y^*(t+\lambda) - y^*(t)\| = \|y(t+\lambda) - y(t)\|$$
$$\leq \|K_1(t+\lambda)g(x) - K_1(t)g(x)\| + \sum_{k=1}^{k=i} \|K_1(t+\lambda-t_k)I_k(x(t_k^-)) - K_1(t-t_k)I_k(x(t_k^-))\|$$
$$+ \| \int_0^{t+\lambda} (t+\lambda-s)^{\alpha-1} K_2(t+\lambda-s) f(s)ds - \int_0^t (t-s)^{\alpha-1} K_2(t-s) f(s)ds\|.$$

Arguing as in the first case we get

$$\lim_{\lambda\to 0} \|y(t+\lambda) - y(t)\| = 0.$$

Case 3. If $t=t_i, i=1,2,..,m$ let $\lambda>0$ be such that $t_i+\lambda\in J_i$ and $\delta>0$ such that $t_i<t_i+\lambda\leq t_{i+1,}$ then we have

$$\|y^*(t_i+\lambda)-y^*(t_i)\|=\lim_{\delta\to t_i^+}\|y(t_i+\lambda)-y(\delta)\|.$$

Based on (3.2) we get

$$\|y(t_i+\lambda)-y(\delta)\|\leq\|K_1(t+\lambda)g(x)-K_1(\delta)g(x)\|$$
$$+\sum_{k=1}^{k=i}\|K_1(t_i+\lambda-t_k)I_k(x(t_k^-))-K_1(\delta-t_k)I_k(x(t_k^-))\|$$
$$+\|\int_0^{t_i+\lambda}(t_i+\lambda-s)^{\alpha-1}K_2(t+\lambda-s)f(s)ds-\int_0^{\delta}(\delta-s)^{\alpha-1}K_2(\delta-s)f(s)ds\|.$$

Arguing as in the first case we can see that

$$\lim_{\substack{\lambda\to 0\\ \delta\to t_i^+}}\|y(t_i+\lambda)-y(\delta)\|=0.$$

Now we can say that, $B_{|\overline{J_i}}$ is equicontinuous for every $i=0,1,2,...,m$. Thus, B is equicontinuous on J.

Step 5. We prove that for any $t\in[0,b]$ the set $\Delta(t)=\{y(t):y\in R(x), x\in B_r\}$ is relatively compact in E. Let us defined

$$R_1:PC(J,E)\to PC(J,E), R_2:PC(J,E)\to 2^{PC(J,E)}$$

such that

$$R_1(x)(t)=\begin{cases}0, t\in[0,t_1]\\ \sum_{k=1}^{k=i}K_1(t-t_k)I_k(x(t_k^-)), t\in J_i, i=1,2,...,m,\end{cases}$$

and $y\in R_2(x)$ iff

$$y(t)=\{K_1(t)g(x)+\int_0^t(t-s)^{\alpha-1}K_2(t-s)f_y(s)ds, t\in J-\{t_1,t_2,...,t_m\},$$

where $f_y\in S^1_{F(\cdot,x(\cdot))}$. clearly, $R=R_1+R_2$. The set $\Delta_1(t)=\{R_1(x)(t):x\in B_r\}$ is relatively compact in E because the functions $I_k, k=1,2,..,m$ are compact. Now, we have to prove that the set $\Delta_2(t)=\{y(t):y\in R_2(x), x\in B_r\}$ is relatively compact in E. Let $0<\varepsilon<1$ and $l\in(0,t)$ for each $t\in J-\{t_1,t_2,...,t_m\}$, we define

$$y_{l,\varepsilon}(t)=\int_\varepsilon^\infty\zeta_\alpha(\theta)T(t^\alpha\theta)g(x)d\theta+\alpha\int_0^{t-l}(t-s)^{\alpha-1}\int_\varepsilon^\infty\theta\zeta_\alpha(\theta)T((t-s)^\alpha\theta)f_y(s)d\theta ds.$$

We can write $y_{l,\varepsilon}(t)$ in the form

$$y_{l,\varepsilon}(t) = T(l^\alpha \varepsilon)\int_\varepsilon^\infty \zeta_\alpha(\theta)T(t^\alpha\theta - l^\alpha\varepsilon)g(x)d\theta$$
$$+T(l^\alpha\varepsilon)\int_0^{t-l}(t-s)^{\alpha-1}(\alpha\int_\varepsilon^\infty \theta\zeta_\alpha(\theta)T((t-s)^\alpha\theta - l^\alpha\varepsilon)d\theta)f_y(s)ds.$$

The set $\{y_{l,\varepsilon}(t) : y \in R_2(x), x \in B_r\}$ is relatively compact in E because $T(t), t > 0$ and g are compact. Also, by C_3, C_4 and Holder's inequality we get

$$\|y(t) - y_{l,\varepsilon}(t)\| \leq \|\int_0^\varepsilon \zeta_\alpha(\theta)T(t^\alpha\theta)g(x)d\theta\|$$
$$+\alpha\|\int_0^t\int_0^\infty \theta(t-s)^{\alpha-1}\zeta_\alpha(\theta)T((t-s)^\alpha\theta)f_y(s)d\theta ds$$
$$-\int_0^{t-l}\int_\varepsilon^\infty \theta(t-s)^{\alpha-1}\zeta_\alpha(\theta)T((t-s)^\alpha\theta)f_y(s)d\theta ds\|$$
$$\leq M(ar+d)\int_0^\varepsilon \zeta_\alpha(\theta)d\theta + \alpha Mr\|\varphi\|_{L^{\frac{1}{q}}(J,\mathbb{R}^+)}\frac{l^{\alpha-q}}{(\varpi+1)^{1-q}}\int_\varepsilon^\infty \theta\zeta_\alpha(\theta)d\theta$$
$$+\alpha Mr\|\varphi\|_{L^{\frac{1}{q}}(J,\mathbb{R}^+)}\frac{b^{\alpha-q}}{(\varpi+1)^{1-q}}\int_0^\varepsilon \theta\zeta_\alpha(\theta)d\theta.$$

Clearly, by Lemma 2.4, the right hand side of the previous inequality tend to zero as $\varepsilon, l \to 0$. Therefore, we can say that there exists a relatively compact set which can be arbitrary close to the set $\Delta_2(t), t \in (0,b]$. Hence, this set is relatively compact in E. So, $\Delta(t), t \in J$ is relatively compact.

From step 4 and step 5 we conclude that B is relatively compact.

Step 6. R is closed i.e. its graph is closed. Let $x_n \in B_r, x_n \to x$ in $PC(J,E)$ and $y_n \in R(x_n) \forall n \geq 1$ with $y_n \to y$ in B. We will prove that $y \in R(x)$. By the definition of R, for any $n \geq 1$ there exists $f_n \in S^1_{F(\cdot,x_n(\cdot))}$ such that

$$y_n(t) = \begin{cases} K_1(t)g(x_n) + \int_0^t (t-s)^{\alpha-1}K_2(t-s)f_n(s)ds,\ t \in J_0, \\ K_1(t)g(x_n) + \sum_{k=1}^{k=i} K_1(t-t_k)I_k(x_n(t_k^-)) \\ +\int_0^t (t-s)^{\alpha-1}K_2(t-s)f_n(s)ds, t \in J_i, i = 1,...,m. \end{cases} \tag{3.4}$$

We will prove that the sequence $(f_n), n \geq 1$ is semicompact. The assumption C_3 implies

$$\|f_n(t)\| \leq \varphi(t)\|x_n\| \leq r\varphi(t),\ a.e.\ t \in J.$$

This shows that the family $\{f_n : n \geq 1\}$ is integrably bonded. Moreover $F(t,\cdot)$ is u.s.c with compact values, then for every ε, we can find a natural number $k(\varepsilon)$ such that for every $n \geq k$

$$f_n(t) \in F(t,x_n(t)) \subseteq F(t,x(t)) + \varepsilon B(0,1),\ a.et \in J.$$

Where $B(0,1) = \{x \in E : \|x\| \leq 1\}$. Then, the set $\{f_n : n \geq 1\}$ is relatively compact in E $a.e.t \in J$. Then, by Definition 2.4 $\{f_n : n \geq 1\}$ is semicompact, hence, by Lemma 2.2 it is weakly compact in $L^1(J,E)$. So, by Mazur's theorem we can say there is a sequence $(\alpha_n)_{n\geq 1}$ such that $\{\alpha_n(t) : n \geq 1\} \subset \overline{conv}\{f_n(t) : n \geq 1\}$ such that α_n converges strongly to f. Since the values of F are convex and compact, the set $S_{F(\cdot,x_n(\cdot))}$ is convex and compact. Therefore, $f \in S_{F(\cdot,x(\cdot))}$. By taking the limit in (3.4) with taking into account that g and I_i for every $i = 1,2,\cdots,m$ are continuous, we obtain.

$$y(t) = \lim_{n\to\infty} y_n(t) = \begin{cases} K_1(t)g(x) + \int_0^t (t-s)^{\alpha-1} K_2(t-s) f(s)ds, t \in J_0, \\ K_1(t)g(x) + \sum_{k=1}^{k=i} K_1(t-t_k) I_k(x(t_k^-)) \\ + \int_0^t (t-s)^{\alpha-1} K_2(t-s) f(s)ds, t \in J_i, i = 1,...,m. \end{cases}$$

Which means that $y \in R(x)$. This proves that R has closed graph on B. Therefore, R verifies the hypotheses of Theorem 2.1, so R has a fixed point x which is a mild solution of problem (P). This completes the proof.

CONCLUSION

This research paper dealt with the existence of PC-mild solutions of nonlocal impulsive differential inclusions in Banach space when the values of the orient field is convex (P). We used methods and results of semilinear differential inclusions, and techniques of fixed point theorems in order to establish sufficient conditions that guarantee the existence of PC-mild solutions of (P). The result in this paper developed and extended some previous results.

Conflict of Interests

The authors declare that there is no conflict of interests.

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