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ESTIMATION OF EIGEN FUNCTIONS TO THE NEW TYPE OF SPECTRAL PROBLEM

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Abstract: In this paper, we study some properties of eigenvalues and the corresponding eigen functions of new type of spectral problem (1)-(4).

Keywords: Spectral problem, eigenvalues, eigen functions.

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1. Introduction

In this paper, we study the new type of spectral problem T_o which is defined by:

$$-y''(x) + y'(x) = \lambda^2 \rho(x) y(x), x \in [0, a], \tag{1}$$

$$y(0) = y'(0) = y(a) + y'(a) = 0, \tag{2}$$

$$\int_0^a y'(x)\bar{y}(x)dx = \tau^2 \ (\tau \ is \ \ constant), \tag{3}$$

$$\left(\int_0^a \rho(x)|y(x)|^2 dx \right)^{\frac{1}{2}} = 1, \tag{4}$$

where λ is a spectral parameter, and $\lambda = \delta + i\sigma$, where $\delta, \sigma \in R$, and $i = \sqrt{-1}$. Let $a > 0$, we assume that $\rho(x) = \rho$ is a constant and let m and M be fixed such that $0 < m \leq M$. Let

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$V^+[0,a]$ denotes the family of allpositive integrable functions $\rho(x)$on the closed interval $[0,a]$ that satisfy the condition$0 < m \leq \rho(x) \leq M$, equipped with usual L_1 metric. In what follows we refer to these functions as weight functions. Here we attempt to specify the properties of eigenvalues of the spectral problem T_o and estimating the eigen functions corresponding to the eigenvalues.For the first time an Italian physicist T.Regge[7] has studied thedifferential equation $-y'' + q(x)y = \lambda^2\rho(x)y(x), x \in (0,a)$with the boundary condition $y(0) = 0, y'(a) - i\lambda$y(a)=0, andwas considered by who showed that the system of eigen functions of this problem are completed and studied asymptoticbehavior of eigenvalues of this problem $\rho(x) = 1$. Kravitsky [6] specified a class of functions that allowed expansion in uniformly convergent series in eigen functions and associated functions in the Regge problem when$\rho(x) \equiv 1$. The present time they are many Arthurs studied the estimation of eigen functions to the equation$-y'' + q(x)y = \lambda^2\rho(x)y(x)$ but with different boundary conditions for more known about their works see [1-5].

2. Features of Eigenvalues of the problem T_o

Here we determine the properties of the eigenvalues of our problem T_o with the given boundary conditions.

Theorem 1: *Let y(x) be an eigen function corresponding to the eigenvalue λ of the problem* T_o*, and*$\rho(x) = \rho$ *is a constant, then: (i) If* $\delta \neq 0$*, then λ is real.*

(ii) If $\sigma \neq 0$*, then* λ *is complex.*

Proof: Multiplying equation (1) by $\bar{y}(x)$ and integrating the obtained equation

from0 to a, yields:

$$-\int_0^a y''(x)\bar{y}(x)dx + \int_0^a y'(x)\bar{y}(x)dx = \lambda^2 \int_0^a \rho(x)y(x)\,\bar{y}(x)dx$$

$$-\bar{y}(x)y'(x) \quad \Big]_0^a + \int_0^a y'(x)\bar{y}'(x)dx + \int_0^a y'(x)\bar{y}(x)dx = \lambda^2 \int_0^a \rho(x)|y(x)|^2 dx$$

$$-\bar{y}(a)y'(a) + \ \bar{y}(0)y'(0) + \int_0^a y'(x)\bar{y}'(x)dx + \int_0^a y'(x)\bar{y}(x)dx = \lambda^2 \int_0^a \rho(x)|y(x)|^2 dx$$

By using boundary conditions (2), we get:

$$\bar{y}(a)y(a) + \int_0^a |y'(x)|^2 dx \ + \int_0^a y'(x)\bar{y}(x)dx = \lambda^2 \int_0^a \rho(x)|y(x)|^2 dx$$

In view of condition (3) and normalized condition (4), we have:

$$|y(a)|^2 + \int_0^a |y'(x)|^2 dx \ + \tau^2 = \lambda^2 \qquad (5)$$

From equation (1)and the conditions (2)-(4) replace$y(x)$ by$\bar{y}(x)$, we get:

$$-\bar{y}''(x) + \ \bar{y}'(x) = \overline{\lambda^2}\rho(x)\bar{y}(x)$$

$$\bar{y}(0) = \bar{y}'(0) = \bar{y}(a) + \ \bar{y}'(a) = 0, \int_0^a \bar{y}'(x)y(x)dx = \tau^2.$$

Multiplying theabove differential equation by $y(x)$ and integrate from 0 up to a, we obtain:

$$|y(a)|^2 + \int_0^a |y'(x)|^2 dx \ + \tau^2 = \overline{\lambda^2} (6)$$

Subtracting equation (6) from equation (5) yields:

$\lambda^2 - \overline{\lambda^2} = 0 \ \to (\lambda - \bar{\lambda})(\lambda + \bar{\lambda}) = 0, (\lambda - \bar{\lambda}) = 0 \ \ or (\lambda + \bar{\lambda}) = 0$, then:

(i) If $\delta \neq 0$, $\therefore (\lambda + \bar{\lambda}) \neq 0, thus (\lambda - \bar{\lambda}) = 0 \ \to \lambda = \bar{\lambda}$, then λ is real.

(ii) If $\sigma \neq 0, so (\lambda - \bar{\lambda}) \neq 0, hence (\lambda + \bar{\lambda}) = 0 \ \to \lambda = -\bar{\lambda}$,then λ is complex.

3. Estimation of Eigen functions of problem T_o

In this section, we estimate the eigen function $y(x)$ corresponding to eigenvalue λ of problem T_o.

Theorem 2: *Let λ be an eigenvalue corresponding to the eigen function $y(x)$ of problem T_o, and $\rho(x) \in L^+[0,a]$, and $\delta \neq 0$, then*

$$\lim_{n\to\infty} \frac{\max_{x\in[0,a]} |y(x)|}{|\lambda|^{\frac{1}{2}}} = A, \text{where} A = \frac{\sqrt{2}}{\sqrt[4]{m}}.$$

Proof:

Let us consider the identity:

$$|y(x)|^2 = y(x)\bar{y}(x) = \int_0^x [\bar{y}(t)y'(t) + y(t)\bar{y}'(t)]dt + |y(0)|^2$$

$$= \int_0^x \frac{\sqrt{\rho(t)}[\bar{y}(t)y'(t) + y(t)\bar{y}'(t)]}{\sqrt{\rho(t)}} dt + |y(0)|^2$$

From inequality $(t) \geq m$, we get:

$$|y(x)|^2 \leq \int_0^x \frac{\sqrt{\rho(t)}|\bar{y}(t)y'(t) + y(t)\bar{y}'(t)|}{\sqrt{m}} dt + |y(0)|^2$$

$$\leq \frac{1}{\sqrt{m}}\left[\int_0^x \sqrt{\rho(t)}|\bar{y}(t)y'(t)|dt + \int_0^x \sqrt{\rho(t)}|y(t)\bar{y}'(t)|dt\right] + |y(0)|^2$$

$$\leq \frac{1}{\sqrt{m}}\left[\int_0^x \sqrt{\rho(t)}|\bar{y}(t)||y'(t)|dt + \int_0^x \sqrt{\rho(t)}|y(t)||\bar{y}'(t)|dt\right] + |y(0)|^2$$

$$= \frac{2}{\sqrt{m}}\int_0^x \sqrt{\rho(t)}|y(t)||y'(t)| + |y(0)|^2$$

And from boundary condition (2), $y(0) = 0$, therefore

$$|y(x)|^2 \leq \frac{2}{\sqrt{m}} \int_0^x \sqrt{\rho(t)}|y(t)||y'(t)|$$

$$\leq \frac{2}{\sqrt{m}} \int_0^a \sqrt{\rho(t)}|y(t)||y'(t)|.$$

Using Bunyakovsky's inequality on the last inequality, we shall obtain:

$$|y(x)|^2 \leq \frac{2}{\sqrt{m}} \left[\int_0^a \rho(t)\,|y(t)|^2 dt\right]^{\frac{1}{2}} \left[\int_0^a \quad |y'(t)|^2 dt\right]^{\frac{1}{2}}$$

From normality condition (4) we have: $\left[\int_0^a \rho(t)\,|y(t)|^2 dt\right]^{\frac{1}{2}} = 1,$ hence

$$|y(x)|^2 \leq \frac{2}{\sqrt{m}} \left[\int_0^a \quad |y'(t)|^2 dt\right]^{\frac{1}{2}} \tag{7}$$

From equation (5), we have:

$\int_0^a |y'(x)|^2 dx = \lambda^2 - |y(a)|^2 - \tau^2$,therefore equation (7) becomes:

$$|y(x)|^2 \leq \frac{2}{\sqrt{m}} [\lambda^2 - |y(a)|^2 - \tau^2]^{\frac{1}{2}} = \frac{2}{\sqrt{m}} [\lambda^2 - (\,|y(a)|^2 + \tau^2\,)\,]^{\frac{1}{2}}$$

And since $\delta \neq 0$, so by theorem (1) λ is real, hence $\lambda^2 = |\lambda\ \ |^2$, thus the last inequality becomes:

$$|y(x)|^2 \leq \frac{2}{\sqrt{m}} \left[|\lambda\ \ |^2 - (\,|y(a)|^2 + \tau^2\,)\,\right]^{\frac{1}{2}} = \frac{2|\lambda\ \ |}{\sqrt{m}} \left[1 - \frac{(\,|y(a)|^2 + \tau^2\,)}{|\lambda\ \ |^2}\right]^{\frac{1}{2}}$$

Or

$$|y(x)|^2 \leq \frac{2}{\sqrt{m}} |\lambda\ \ | \rightarrow |y(x)| \leq |\lambda\ \ |^{\frac{1}{2}} \sqrt{\frac{2}{\sqrt{m}}}$$

And since x is any value in the interval $[0, a]$, thus

$$\max_{x\in[0,a]}|y(x)| \leq |\lambda \quad |^{\frac{1}{2}}\sqrt{\frac{2}{\sqrt{m}}} \rightarrow \frac{\max_{x\in[0,a]}|y(x)|}{|\lambda \quad |^{\frac{1}{2}}} \leq \frac{\sqrt{2}}{\sqrt[4]{m}}$$

Hence

$$\lim_{n\to\infty}\frac{\max_{x\in[0,a]}|y(x)|}{|\lambda|^{\frac{1}{2}}} \quad = A, \text{where} A = \frac{\sqrt{2}}{\sqrt[4]{m}}.$$

Theorem 2.3.2: *Let ρ be a constant in the problem T_o and if $y(x)$ is an eigen function of the problemT_o, then $y(x)$satisfy the inequality*

$$\frac{1}{\sqrt{|\lambda|}}K_1 \leq \max_{x\in[0,a]}|y(x)| \leq \frac{1}{\sqrt{|\lambda|}}K_2,$$

Where K_1 and K_2 are constants.

Proof:

From equation (1), we have $y''(x) - y'(x) + \lambda^2\rho y(x) = 0$this is second order linear differential equation with constant coefficients, and then general solution is:

$$y(x) = e^{\frac{1}{2}x}\left[c_1 e^{i\sqrt{\lambda^2\rho-\frac{1}{4}}x} + c_2 e^{-i\sqrt{\lambda^2\rho-\frac{1}{4}}x}\right]$$

Applying the condition $y(0) = 0$, yields $c_2 = -c_1$, then we have

$$y(x) = c_1\left[e^{\left(\frac{1}{2}+i\sqrt{\lambda^2\rho-\frac{1}{4}}\right)x} - e^{\left(\frac{1}{2}-i\sqrt{\lambda^2\rho-\frac{1}{4}}\right)x}\right]$$

Then

$$y'(x) = c_1\left[\left(\frac{1}{2} + i\sqrt{\lambda^2\rho - \frac{1}{4}}\right)e^{\left(\frac{1}{2}+i\sqrt{\lambda^2\rho-\frac{1}{4}}\right)x} - \left(\frac{1}{2} - i\sqrt{\lambda^2\rho - \frac{1}{4}}\right)e^{\left(\frac{1}{2}-i\sqrt{\lambda^2\rho-\frac{1}{4}}\right)x}\right]$$

From the boundary condition$y(a) + y'(a) = 0$, we obtain:

$$c_1\left[e^{\left(\frac{1}{2}+i\sqrt{\lambda^2\rho-\frac{1}{4}}\right)a} - e^{\left(\frac{1}{2}-i\sqrt{\lambda^2\rho-\frac{1}{4}}\right)a}\right] +$$

$$c_1\left[\left(\frac{1}{2}+i\sqrt{\lambda^2\rho-\frac{1}{4}}\right)e^{\left(\frac{1}{2}+i\sqrt{\lambda^2\rho-\frac{1}{4}}\right)a} - \left(\frac{1}{2}-i\sqrt{\lambda^2\rho-\frac{1}{4}}\right)e^{\left(\frac{1}{2}-i\sqrt{\lambda^2\rho-\frac{1}{4}}\right)a}\right] = 0$$

Dividing both sides of the above equation by c_1,we get:

$$\frac{\frac{3}{2}-i\sqrt{\lambda^2\rho-\frac{1}{4}}}{\frac{3}{2}+i\sqrt{\lambda^2\rho-\frac{1}{4}}} = e^{2i\sqrt{\lambda^2\rho-\frac{1}{4}}\,a} \qquad (8)$$

The resulting equation (8) is used for specifying the eigenvalues of our problem.

To find the coefficient c_1, we use the normalization condition (4)

$$\int_0^a \rho\,|c_1|^2\left|e^{\frac{1}{2}x}\left[e^{i\sqrt{\lambda^2\rho-\frac{1}{4}}\,x} - e^{-i\sqrt{\lambda^2\rho-\frac{1}{4}}\,x}\right]\right|^2 dx = 1,$$

Or

$$\rho\,|c_1|^2\int_0^a \left|e^{\frac{1}{2}x}\left[e^{i\sqrt{\lambda^2\rho-\frac{1}{4}}\,x} - e^{-i\sqrt{\lambda^2\rho-\frac{1}{4}}\,x}\right]\right|^2 dx = 1.$$

We introduce the notation $\alpha + i\beta = i\sqrt{\lambda^2\rho-\frac{1}{4}}$ (where α and β $are\ real\ numbers$),

then

$$\rho\,|c_1|^2\int_0^a \left|e^{\frac{1}{2}x}\left[e^{(\alpha+i\beta)\,x} - e^{-(\alpha+i\beta)\,x}\right]\right|^2 dx = 1,$$

Or

$$\rho\,|c_1|^2\int_0^a \left|e^{\left(\frac{1}{2}+\alpha\right)x+i\beta x} - e^{\left(\frac{1}{2}-\alpha\right)x-i\beta x}\right|^2 dx = 1 \qquad (9)$$

Since

$$\left|e^{\left(\frac{1}{2}+\alpha\right)x+i\beta x}-e^{\left(\frac{1}{2}-\alpha\right)x-i\beta x}\right|^2=2\,e^{x}(cosh2\alpha x-cos2\beta x)$$

Thus equation (9) becomes:

$$\rho\,|c_1|^2\int_0^a 2\,e^{x}(cosh2\alpha x-cos2\beta x)\,dx=1.$$

By integrating the last equation by parts, we obtain

$$2\rho\,|c_1|^2\,[\,\frac{1}{2(1+2\alpha)}\left(e^{(1+2\alpha)a}-1\right)+\frac{1}{2(1-2\alpha)}\left(e^{(1-2\alpha)a}-1\right)$$

$$-\frac{(2\beta sin2\beta a+cos2\beta a)}{(4\beta^2+1)}e^{a}+\frac{2\beta}{(4\beta^2+1)}\,]=1$$

After some algebraic operations, we get

$$|c_1|^2=(1-4\alpha^2)(4\beta^2+1)\,/\,2\rho[\,e^{a}(cosh2\alpha a-2\alpha sinh2\alpha a)(4\beta^2+1)$$

$$-(4\beta^2+1)+2\beta(1-4\alpha^2)-e^{a}(2\beta sin2\beta a+cos2\beta a)(1-4\alpha^2)\,]$$

Or

$$|c_1|=\frac{1}{\sqrt{2\rho}}\frac{1}{\sqrt{\frac{1}{(1-4\alpha^2)}[e^{a}(cosh2\alpha a-2\alpha sinh2\alpha a)-1]+\frac{1}{(4\beta^2+1)}[2\beta-e^{a}(2\beta sin2\beta a+cos2\beta a)\,]}}$$

By substituting $|c_1|$in equation $y(x)$,we conclude that:

$$y(x)=c_o\frac{1}{\sqrt{2\rho}}\frac{1}{\sqrt{\frac{1}{(1-4\alpha^2)}[e^{a}(cosh2\alpha a-2\alpha sinh2\alpha a)-1]+\frac{1}{(4\beta^2+1)}[2\beta-e^{a}(2\beta sin2\beta a+cos2\beta a)\,]}}$$

$$\left[e^{\left(\frac{1}{2}+i\sqrt{\lambda^2\rho-\frac{1}{4}}\right)x} - e^{\left(\frac{1}{2}-i\sqrt{\lambda^2\rho-\frac{1}{4}}\right)x}\right]. \qquad (10)$$

Where $c_o$arbitrary complex number with module is one (i.e.$|c_o| = 1$).

If λ satisfies equation (8) (i.e. λ eigenvalue), then equation (10) gives eigen functions for our problem T_o (corresponding to the eigenvalueλ).

Now we determine $\max_{x\in[0,a]}|y(x)|$ and its behaviour depends on, $\alpha$and β.

From

$$\left|e^{\left(\frac{1}{2}+\alpha\right)x+i\beta x} - e^{\left(\frac{1}{2}-\alpha\right)x-i\beta x}\right|^2 = 2\,e^x(cosh2\alpha x - cos2\beta x),$$

We conclude that

$$\left|e^{\left(\frac{1}{2}+\alpha\right)x+i\beta x} - e^{\left(\frac{1}{2}-\alpha\right)x-i\beta x}\right| = \sqrt{2\,e^x(cosh2\alpha x - cos2\beta x)}$$

Therefore,

$$|y(x)| = \left|\frac{c_o}{\sqrt{2\rho}}\frac{e^{\left(\frac{1}{2}+i\sqrt{\lambda^2\rho-\frac{1}{4}}\right)x} - e^{\left(\frac{1}{2}-i\sqrt{\lambda^2\rho-\frac{1}{4}}\right)x}}{\sqrt{\frac{1}{(1-4\alpha^2)}[e^a(cosh2\alpha a - 2\alpha sinh2\alpha a) - 1] + \frac{1}{(4\beta^2+1)}[2\beta - e^a(2\beta sin2\beta a + cos2\beta a)]}}\right|$$

Or

$$|y(x)| = \frac{1}{\sqrt{\rho}}\sqrt{\frac{e^x(cosh2\alpha x - cos2\beta x)}{\frac{1}{(1-4\alpha^2)}[e^a(cosh2\alpha a - 2\alpha sinh2\alpha a) - 1] + \frac{1}{(4\beta^2+1)}[2\beta - e^a(2\beta sin2\beta a + cos2\beta a)]}}$$

Then

$$\frac{1}{\sqrt{\rho}}\sqrt{\frac{e^x(cosh2\alpha x-1)}{\frac{1}{(1-4\alpha^2)}[e^a(cosh2\alpha a-2\alpha sinh2\alpha a)-1]+\frac{1}{(4\beta^2+1)}[2\beta-e^a(2\beta sin2\beta a+cos2\beta a)\,]}}\ \le |y(x)|$$

$$\le \frac{1}{\sqrt{\rho}}\sqrt{\frac{e^x(cosh2\alpha x+1)}{\frac{1}{(1-4\alpha^2)}[e^a(cosh2\alpha a-2\alpha sinh2\alpha a)-1]+\frac{1}{(4\beta^2+1)}[2\beta-e^a(2\beta sin2\beta a+cos2\beta a)\,]}}$$

Or

$$\sqrt{\frac{e^x(cosh2\alpha x-1)}{\rho\left[\left(1-e^a(cosh2\alpha a-2\alpha sinh2\alpha a)\right)+\left(2\beta+e^a(2\beta+1)\right)\right]}}\ \le |y(x)| \le$$

$$\sqrt{\frac{e^x(cosh2\alpha x+1)}{\rho\left[\left(e^a(cosh2\alpha a-2\alpha sinh2\alpha a)-1\right)\ +\frac{1}{(4\beta^2+1)}\left(2\beta-e^a(2\beta+1)\right)\right]}}\ .$$

Let $\max_{x\in[0,a]}|y(x)|$ be achieved at the point of x_o, then

$$\max_{x\in[0,a]}|y(x)| = |y(x_o)|$$

$$\le \sqrt{\frac{e^{x_o}(cosh2\alpha x_o+1)}{\rho\left[\left(e^a(cosh2\alpha a-2\alpha sinh2\alpha a)-1\right)\ +\frac{1}{(4\beta^2+1)}\left(2\beta-e^a(2\beta+1)\right)\right]}}$$

$$\le \sqrt{\frac{e^{a}(cosh2\alpha a+1)}{\rho\left[\left(e^a(cosh2\alpha a-2\alpha sinh2\alpha a)-1\right)\ +\frac{1}{(4\beta^2+1)}\left(2\beta-e^a(2\beta+1)\right)\right]}}$$

(Since e^x and cosh$2\alpha x$ are monotonic increasing on$[0, a]$), on the other hand

$|y(x_o)| = \max_{x\in[0,a]} |y(x)| \geq |y(a)| \geq$

$$\sqrt{\frac{e^a(cosh2\alpha a - 1)}{\rho\left[\left(1 - e^a(cosh2\alpha a - 2\alpha sinh2\alpha a)\right) + \left(2\beta + e^a(2\beta + 1)\right)\right]}}$$

Therefore

$$\sqrt{\frac{e^a(cosh2\alpha a - 1)}{\rho\left[\left(1 - e^a(cosh2\alpha a - 2\alpha sinh2\alpha a)\right) + \left(2\beta + e^a(2\beta + 1)\right)\right]}} \leq \max_{x\in[0,a]} |y(x)|$$

$$\leq \sqrt{\frac{e^a(cosh2\alpha a + 1)}{\rho\left[(e^a(cosh2\alpha a - 2\alpha sinh2\alpha a) - 1) + \frac{1}{(4\beta^2+1)}\left(2\beta - e^a(2\beta + 1)\right)\right]}}$$

Or

$$\sqrt{\frac{(cosh2\alpha a - 1)}{\rho\left[\left(\frac{1}{e^a} - (cosh2\alpha a - 2\alpha sinh2\alpha a)\right) + \left(\frac{2\beta}{e^a} + (2\beta + 1)\right)\right]}} \leq \max_{x\in[0,a]} |y(x)|$$

$$\leq \sqrt{\frac{(cosh2\alpha a+1)}{\rho\left[\left((cosh2\alpha a-2\alpha sinh2\alpha a)-\frac{1}{e^a}\right)+\frac{1}{(4\beta^2+1)}\left(\frac{2\beta}{e^a}-(2\beta+1)\right)\right]}} \qquad (11)$$

Now, in the obtained equation (11) used parameters α and β clearly are not parts of the equation (1) and the boundary and normalized conditions (2)-(4). Therefore we express α and β through ρ.

Suppose arg $\lambda = \theta$, then $\lambda^2 = |\lambda|^2(cos2\theta + isin2\theta)$.

$$\lambda^2\rho - \frac{1}{4} = \rho|\lambda|^2 cos2\theta - \frac{1}{4} + i\rho|\lambda|^2 sin2\theta$$

On the other hand $-\left(\lambda^2\rho - \frac{1}{4}\right) = (\alpha + i\beta)^2 = \alpha^2 - \beta^2 + i2\alpha\beta$, Hence

$$\alpha^2-\beta^2=-\rho|\lambda\ \ |^2 cos2\theta+\frac{1}{4}$$

$$2\alpha\beta=-\rho|\lambda\ \ |^2 sin2\theta$$

Or

$$\alpha^2-\beta^2=-\rho|\lambda\ \ |^2 cos2\theta+\frac{1}{4}$$

$$4\alpha^2\beta^2=\rho^2|\lambda\ \ |^4\ \ \sin^2 2\theta$$

Solving these two last systems of equations, we get

$$\alpha^2=\frac{-\rho|\lambda\ \ |^2 cos2\theta+\frac{1}{4}+\sqrt{\left(\rho|\lambda\ \ |^2 cos2\theta-\frac{1}{4}\right)^2+\rho^2|\lambda\ \ |^4\ \ \sin^2 2\theta}}{2}$$

and

$$\beta^2=\frac{\rho^2|\lambda\ \ |^4\ \ \sin^2 2\theta}{2[-\rho|\lambda\ \ |^2 cos2\theta+\frac{1}{4}+\sqrt{\left(\rho|\lambda\ \ |^2 cos2\theta-\frac{1}{4}\right)^2+\rho^2|\lambda\ \ |^4\ \ \sin^2 2\theta}\ ]}$$

(Since$\alpha^2\geq 0$, then chose non negative root). Separating out the factor $\rho|\lambda\ \ |^2$ from the last relations, we deduce

$$\alpha^2=\rho|\lambda\ \ |^2\left(\frac{-cos2\theta+\frac{1}{4\rho|\lambda\ \ |^2}+\sqrt{1-\frac{1}{2\rho|\lambda\ \ |^2}cos2\theta+\left(\frac{1}{4\rho|\lambda\ \ |^2}\right)^2}}{2}\right)$$

and

$$\beta^2 = \frac{\rho \quad |\lambda \quad|^2 \quad \sin^2 2\theta}{2\left[-cos2\theta + \frac{1}{4\rho|\lambda \quad|^2} + \sqrt{1 - \frac{1}{2\rho|\lambda \quad|^2} cos2\theta + \left(\frac{1}{4\rho|\lambda \quad|^2}\right)^2}\right]}$$

Or

$$\alpha = |\lambda \quad| \sqrt{\frac{-\rho cos2\theta + \frac{1}{4|\lambda \quad|^2} + \rho\sqrt{1 - \frac{1}{2\rho|\lambda \quad|^2} cos2\theta + \left(\frac{1}{4\rho|\lambda \quad|^2}\right)^2}}{2}}$$

and

$$\beta = \frac{\sqrt{\rho} \quad |\lambda \quad| \quad sin\, 2\theta}{\sqrt{-2cos2\theta + \frac{1}{2\rho|\lambda \quad|^2} + \ 2\sqrt{1 - \frac{1}{2\rho|\lambda \quad|^2} cos2\theta + \left(\frac{1}{4\rho|\lambda \quad|^2}\right)^2}}}$$

(We take the positive root and for negative root we proceed by similar way).

By substituting α and βin equation (11) and making some algebraic operations we get:

$$\frac{1}{\sqrt{|\lambda\ |}}\sqrt{\frac{cosh2a\left(|\lambda\ |\sqrt{\frac{-\rho cos2\theta+\frac{1}{4|\lambda\ |^2}+\rho\sqrt{1-\frac{1}{2\rho|\lambda\ |^2}cos2\theta+\left(\frac{1}{4\rho|\lambda\ |^2}\right)^2}}{2}}\right)-1}{\begin{array}{l}\frac{\rho e^{-a}}{|\lambda\ |}-\frac{\rho}{|\lambda\ |}cosh2a\left(|\lambda\ |\sqrt{\frac{-\rho cos2\theta+\frac{1}{4|\lambda\ |^2}+\rho\sqrt{1-\frac{1}{2\rho|\lambda\ |^2}cos2\theta+\left(\frac{1}{4\rho|\lambda\ |^2}\right)^2}}{2}}\right)\\ +\rho\sqrt{2}\sqrt{-\rho cos2\theta+\frac{1}{4|\lambda\ |^2}+\rho\sqrt{1-\frac{1}{2\rho|\lambda\ |^2}cos2\theta+\left(\frac{1}{4\rho|\lambda\ |^2}\right)^2}}\ *\\ sinh2a\left(|\lambda\ |\sqrt{\frac{-\rho cos2\theta+\frac{1}{4|\lambda\ |^2}+\rho\sqrt{1-\frac{1}{2\rho|\lambda\ |^2}cos2\theta+\left(\frac{1}{4\rho|\lambda\ |^2}\right)^2}}{2}}\right)\\ \frac{2e^{-a}\sqrt{\rho^3}\,sin2\theta}{\sqrt{-2cos2\theta+\frac{1}{2\rho|\lambda\ |^2}+2\sqrt{1-\frac{1}{2\rho|\lambda\ |^2}cos2\theta+\left(\frac{1}{4\rho|\lambda\ |^2}\right)^2}}}+\\ \frac{2\sqrt{\rho^3}\,sin2\theta}{\sqrt{-2cos2\theta+\frac{1}{2\rho|\lambda\ |^2}+2\sqrt{1-\frac{1}{2\rho|\lambda\ |^2}cos2\theta+\left(\frac{1}{4\rho|\lambda\ |^2}\right)^2}}}+\frac{\rho}{|\lambda\ |}\end{array}}}$$

$$\leq \max_{x\in[0,a]}|y(x)| \leq$$

$$\frac{1}{\sqrt{|\lambda|}}\sqrt{\frac{\cosh2a\left(|\lambda|\sqrt{\frac{-\rho\cos2\theta+\frac{1}{4|\lambda|^2}+\rho\sqrt{1-\frac{1}{2\rho|\lambda|^2}\cos2\theta+\left(\frac{1}{4\rho|\lambda|^2}\right)^2}}{2}}\right)+1}{\begin{array}{l}\frac{\rho}{|\lambda|}\cosh2a\left(|\lambda|\sqrt{\frac{-\rho\cos2\theta+\frac{1}{4|\lambda|^2}+\rho\sqrt{1-\frac{1}{2\rho|\lambda|^2}\cos2\theta+\left(\frac{1}{4\rho|\lambda|^2}\right)^2}}{2}}\right)\\ -\rho\sqrt{2}\sqrt{-\rho\cos2\theta+\frac{1}{4|\lambda|^2}+\rho\sqrt{1-\frac{1}{2\rho|\lambda|^2}\cos2\theta+\left(\frac{1}{4\rho|\lambda|^2}\right)^2}}\;*\\ \sinh2a\left(|\lambda|\sqrt{\frac{-\rho\cos2\theta+\frac{1}{4|\lambda|^2}+\rho\sqrt{1-\frac{1}{2\rho|\lambda|^2}\cos2\theta+\left(\frac{1}{4\rho|\lambda|^2}\right)^2}}{2}}\right)-\frac{\rho e^{-a}}{|\lambda|}\\ +\frac{\rho}{|\lambda|}\left(\frac{-\rho\cos2\theta+\frac{1}{4|\lambda|^2}+\rho\sqrt{1-\frac{1}{2\rho|\lambda|^2}\cos2\theta+\left(\frac{1}{4\rho|\lambda|^2}\right)^2}}{2\rho|\lambda|^2\quad\sin^2 2\theta-\cos2\theta+\frac{1}{4\rho|\lambda|^2}+\sqrt{1-\frac{1}{2\rho|\lambda|^2}\cos2\theta+\left(\frac{1}{4\rho|\lambda|^2}\right)^2}}\right)\\ *\left(\frac{2e^{-a}\sqrt{\rho}\,\sin2\theta}{\sqrt{-2\cos2\theta+\frac{1}{2\rho|\lambda|^2}+2\sqrt{1-\frac{1}{2\rho|\lambda|^2}\cos2\theta+\left(\frac{1}{4\rho|\lambda|^2}\right)^2}}}-\frac{2|\lambda|^{\sqrt{\rho}\,\sin2\theta}}{\sqrt{-2\cos2\theta+\frac{1}{2\rho|\lambda|^2}+2\sqrt{1-\frac{1}{2\rho|\lambda|^2}\cos2\theta+\left(\frac{1}{4\rho|\lambda|^2}\right)^2}}}-1\right)\end{array}}}$$

Let$K_1 =$

$$\sqrt{\frac{\cosh 2a\left(|\lambda\ |\sqrt{\frac{-\rho\cos 2\theta+\frac{1}{4|\lambda\ |^2}+\rho\sqrt{1-\frac{1}{2\rho|\lambda\ |^2}\cos 2\theta+\left(\frac{1}{4\rho|\lambda\ |^2}\right)^2}}{2}}\right)-1}{\frac{\rho e^{-a}}{|\lambda\ |}-\frac{\rho}{|\lambda\ |}\cosh 2a\left(|\lambda\ |\sqrt{\frac{-\rho\cos 2\theta+\frac{1}{4|\lambda\ |^2}+\rho\sqrt{1-\frac{1}{2\rho|\lambda\ |^2}\cos 2\theta+\left(\frac{1}{4\rho|\lambda\ |^2}\right)^2}}{2}}\right)+\rho\sqrt{2}\sqrt{-\rho\cos 2\theta+\frac{1}{4|\lambda\ |^2}+\rho\sqrt{1-\frac{1}{2\rho|\lambda\ |^2}\cos 2\theta+\left(\frac{1}{4\rho|\lambda\ |^2}\right)^2}}*\sinh 2a\left(|\lambda\ |\sqrt{\frac{-\rho\cos 2\theta+\frac{1}{4|\lambda\ |^2}+\rho\sqrt{1-\frac{1}{2\rho|\lambda\ |^2}\cos 2\theta+\left(\frac{1}{4\rho|\lambda\ |^2}\right)^2}}{2}}\right)\frac{2e^{-a}\sqrt{\rho^3}\sin 2\theta}{\sqrt{-2\cos 2\theta+\frac{1}{2\rho|\lambda\ |^2}+2\sqrt{1-\frac{1}{2\rho|\lambda\ |^2}\cos 2\theta+\left(\frac{1}{4\rho|\lambda\ |^2}\right)^2}}}+\frac{2\sqrt{\rho^3}\sin 2\theta}{\sqrt{-2\cos 2\theta+\frac{1}{2\rho|\lambda\ |^2}+2\sqrt{1-\frac{1}{2\rho|\lambda\ |^2}\cos 2\theta+\left(\frac{1}{4\rho|\lambda\ |^2}\right)^2}}}+\frac{\rho}{|\lambda\ |}}}$$

And

$K_2 =$

$$\sqrt{\frac{\cosh 2a\left(|\lambda|\sqrt{\frac{-\rho\cos2\theta+\frac{1}{4|\lambda|^2}+\rho\sqrt{1-\frac{1}{2\rho|\lambda|^2}\cos2\theta+\left(\frac{1}{4\rho|\lambda|^2}\right)^2}}{2}}\right)+1}{\frac{\rho}{|\lambda|}\cosh 2a\left(|\lambda|\sqrt{\frac{-\rho\cos2\theta+\frac{1}{4|\lambda|^2}+\rho\sqrt{1-\frac{1}{2\rho|\lambda|^2}\cos2\theta+\left(\frac{1}{4\rho|\lambda|^2}\right)^2}}{2}}\right)-\rho\sqrt{2}\sqrt{-\rho\cos2\theta+\frac{1}{4|\lambda|^2}+\rho\sqrt{1-\frac{1}{2\rho|\lambda|^2}\cos2\theta+\left(\frac{1}{4\rho|\lambda|^2}\right)^2}}\ast\sinh 2a\left(|\lambda|\sqrt{\frac{-\rho\cos2\theta+\frac{1}{4|\lambda|^2}+\rho\sqrt{1-\frac{1}{2\rho|\lambda|^2}\cos2\theta+\left(\frac{1}{4\rho|\lambda|^2}\right)^2}}{2}}\right)-\frac{\rho e^{-a}}{|\lambda|}+\frac{\rho}{|\lambda|}\left(\frac{-\rho\cos2\theta+\frac{1}{4|\lambda|^2}+\rho\sqrt{1-\frac{1}{2\rho|\lambda|^2}\cos2\theta+\left(\frac{1}{4\rho|\lambda|^2}\right)^2}}{2\rho|\lambda|^2\quad \sin^2 2\theta-\cos2\theta+\frac{1}{4\rho|\lambda|^2}+\sqrt{1-\frac{1}{2\rho|\lambda|^2}\cos2\theta+\left(\frac{1}{4\rho|\lambda|^2}\right)^2}}\right)\ast\left(\frac{2e^{-a}\sqrt{\rho}\,\sin2\theta}{\sqrt{-2\cos2\theta+\frac{1}{2\rho|\lambda|^2}+2\sqrt{1-\frac{1}{2\rho|\lambda|^2}\cos2\theta+\left(\frac{1}{4\rho|\lambda|^2}\right)^2}}}-\frac{2|\lambda|^{\sqrt{\rho}}\sin2\theta}{\sqrt{-2\cos2\theta+\frac{1}{2\rho|\lambda|^2}+2\sqrt{1-\frac{1}{2\rho|\lambda|^2}\cos2\theta+\left(\frac{1}{4\rho|\lambda|^2}\right)^2}}}-1\right)}}$$

Then

$$\frac{1}{\sqrt{|\lambda|}}K_1 \leq \max_{x\in[0,a]}|y(x)| \leq \frac{1}{\sqrt{|\lambda|}}K_2.$$

Thus the proof of theorem is completed.

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