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THE SPIN NUMBER OF HOMOTOPY $E(2n)$ SURFACES UNDER THE ACTION OF $\mathbf{Z}_p$

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Abstract. In this paper, we study the spin number of homotopy $E(2n)$ surfaces under the action of $\mathbf{Z}_p$. By discussing the spin number as positive or negative number, we obtain restrictions on the coefficients of the $G-$index operator $\mathrm{Ind}_G D$.

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1. Introduction

Let $X = E(2n)(n \geq 1)$ be the relatively minimal elliptic surface with rational base. An elliptic surface $E(2n)$ is defined as the $2n$-fold fiber sum of copies of $E(1)$, where $E(1)$ denotes $CP^2\sharp 9\overline{CP^2}$ being equipped with an elliptic fibration. Let b_i be the i-th Betti number of X, and b_2^+ (resp. b_2^-) be the rank of the maximal positive (resp. negative) definite subspace $H^+(X;\mathbf{R})$ (resp. $H^-(X;\mathbf{R})$) of $H^2(X;\mathbf{R})$. $b_2^+(X/\tau)$ represents the rank of $H^2(X/\tau;\mathbf{R})$. The signature of X is denoted by $\sigma(X)$.

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It is well known that the spin number is defined as $\mathrm{Spin}(\hat{\tau}, X) = \mathrm{Ind}_{\hat{\tau}} D = \mathrm{tr}(\hat{\tau}|_{kerD}) - \mathrm{tr}(\hat{\tau}|_{cokerD})$. In this paper, we study the spin number of homotopy $E(2n)$ surfaces under the action of $\mathbf{Z}_p$.

At first, for the periodic diffeomorphism τ of odd prime order p on the homotopy elliptic surface $E(2n)$, we prove the following result.

Theorem 1.1. *Let X be a homotopy $E(2n)$ surface, and let X admit a periodic diffeomorphism τ of odd prime order p satisfying $b_2^+(X/\tau) = b_2^+(X)$. Then we have $k_0 \leq b_2^+(X) - 1$, where k_0 is one of the coefficients of the G-index operator $\mathrm{Ind}_G D$.*

Then by discussing the spin number as positive or negative number, we obtain the following theorems.

Theorem 1.2. *Let X be a homotopy $E(2n)$ surface and let X admit a periodic diffeomorphism τ of odd prime order p satisfying $b_2^+(X/\tau) = b_2^+(X)$. Assume that the spin number Spin$(\hat{\tau}, X)$ is both rational and non-negative. Then we have*

$$k_1 = k_2 = \cdots = k_{p-1}, \ \frac{2s}{p} \leq k_0 \leq l - 1.$$

Theorem 1.3. *Let X be a homotopy $E(2n)$ surface and let X admit a periodic diffeomorphism τ of odd prime order p satisfying $b_2^+(X/\tau) = b_2^+(X)$. Assume that the spin number Spin$(\hat{\tau}, X)$ is both rational and negative. Then we have*

$$k_1 = k_2 = \ldots = k_{p-1} = \frac{2s - k_0}{p - 1} \geq 1, \ k_0 < \frac{2s}{p}.$$

We organize this paper as follows. In section 2, we give some preliminaries about the group action of $\mathbf{Z}_p$ and the character formula for the K-theory. In section 3, we study the spin number and prove the main results.

2. Preliminaries

In this section, we review some basic knowledge about group representation theory and the T. tom Dieck's character formula.

Since any spin actions of odd prime order are of even type, we only consider spin actions of even type. For details, we refer to [3], [7].

Let $R(\mathbf{Z}_p)$ be the representation ring of $\mathbf{Z}_p$. Then $R(\mathbf{Z}_p)$ is generated by $1, \xi, \ldots, \xi^{p-1}$, where ξ is the one-dimensional representation such that $\xi^p = 1$. The group Pin(2) has a non-trivial 1-dimensional representation $\tilde{1}$ and a countable series of 2-dimensional irreducible representations $h_1, h_2, \cdots$. The representation $h_1 = h$ is the restriction of the standard representation of $SU(2)$ to $\mathrm{Pin}(2) \subset SU(2)$. Since the representation ring $R(G_{ev}) = R(\mathrm{Pin}(2)) \otimes R(\mathbf{Z}_p)$ then any general element $\alpha \in R(G_{ev})$ must have the form

$$\alpha = \alpha_0(\xi)1 + \widetilde{\alpha}_0(\xi)\widetilde{1} + \sum_{i=1}^{\infty} \alpha_i(\xi)h_i.$$

Denote $V_{\lambda,\mathbf{C}} = V_\lambda \otimes \mathbf{C}$, $W_{\lambda,\mathbf{C}} = W_\lambda \otimes \mathbf{C}$. Then the virtual representation $[V_{\lambda,\mathbf{C}}] - [W_{\lambda,\mathbf{C}}] \in R(\tilde{G})$ is the same as $\mathrm{Ind}D = [kerD] - [CokerD]$. And Furuta shows

$$r(\mathrm{Ind}D) = k(\xi)h - t(\xi)\tilde{1},$$

where $r : R(\tilde{G}) \to R(\mathrm{Pin}(2))$, $k(\xi)$ and $t(\xi)$ are polynomials of ξ such that $k(1) = -\sigma(X)/8$ and $t(1) = b_2^+(X)$. Thus $\mathrm{Ind}D = k(\xi)h - t(\xi)\tilde{1}$.

Let V and W be complex G representations for some compact Lie group G. BV and BW denote balls in V and W. $f : BV \to BW$ is a G-map preserving the boundaries SV and SW. Applying the K-theory functor to f, we get a map

$$f^* : K_G(W) \to K_G(V),$$

where $K_G(V)$ denotes $K_G(BV, SV)$. This map defines an unique element $\alpha_f \in R(G)$ by the equation $f^*(\lambda(W)) = \alpha_f \cdot \lambda(V)$, where $\lambda(V)$ is the Bott class which is the generator of $K_G(V)$. The element α_f is called the K-theory degree of f.

Let V_g and W_g denote the subspaces of V and W fixed by an element $g \in G$ and let $V_g^\perp$ and $W_g^\perp$ be the orthogonal complements. Let $f^g : V_g \to W_g$ be the restriction of f and let $d(f^g)$ denote the ordinary topological degree of f^g. For any $\beta \in R(G)$, let $\lambda_{-1}\beta$ denote the alternating sum $\Sigma(-1)^i\lambda^i\beta$ of exterior powers. The T. tom Dieck's character formula is that:

$$\mathrm{tr}_g(\alpha_f) = d(f^g)\mathrm{tr}_g(\lambda_{-1}(W_g^\perp - V_g^\perp)), \tag{1}$$

where tr_g is the trace of the action g on α_f. Note that when $\dim V_g \neq \dim W_g$, $d(f^g) = 0$.

Recall that $\lambda_{-1}(\Sigma_i a_i r_i) = \prod_i (\lambda_{-1} r_i)^{a_i}$ and that for a one dimensional representation r, we have $\lambda_{-1} r = (1 - r)$.

3. Main results

Let X be a homotopy $E(2n)$ surface and τ be a periodic diffeomorphism of odd prime order p. From the Atiyah-Singer G-spin theorem ([4], [2], Theorem 8.35 in [1], Theorem 3.1 in [5] and Theorem 14.11 in [6]), we can obtain the following lemma about the spin numbers in the case of $E(2n)$.

lemma 3.1. *Let X be the homotopy $E(2n)$ surface and let $g : X \to X$ is a cyclic group action on X with odd prime order p. Let $\hat{g}$ be the lift of g which preserves trivial Spin^c structure. Assume the fixed point set X^g is composed of isolated points P_j and connected 2-manifolds F_k. Then we have the following formula for Spin number*

$$\begin{aligned} \mathrm{Spin}(\hat{g}, X) = &-\frac{1}{4}\sum_{P_j} \epsilon(P_j, \hat{g}) csc(\alpha_j/2) csc(\beta_j/2) \\ &+ \frac{1}{4}\sum_{F_k} \epsilon(F_k, \hat{g}) cos(\theta_k/2) csc^2(\theta_k/2) \langle [F_k] \cdot [F_k] \rangle \end{aligned} \tag{2}$$

where α_j(resp β_j) denotes $2\pi l_{\alpha_j}/p$ (resp $2\pi l_{\beta_j}/p$) ($0 < \alpha_j,\ \beta_j < \pi$), $\theta_k = 2\pi l_{\theta_j}/p$ ($0 < \theta_k < \pi$), $\epsilon(P_j, \hat{g})$ and $\epsilon(F_k, \hat{g})$ are ± 1. And the signal depends on the action of g on the Spin bundle.

From the above theorem, we can easily show the following results.

Lemma 3.2. *Let X be a homotopy $E(2n)$ surface, and let τ be a periodic diffeomorphism of odd prime order p on X. Let $\hat{\tau}$ be a lifting of τ and preserve the trivial spin^c structure. Then*

(1). The spin number is always real.

(2). When $p = 3$, the spin number $\mathrm{Spin}(\hat{\tau}, X)$ is always rational.

Proof. (1). Since every part in formula (2) is real, we have $\mathrm{Spin}(\hat{\tau}, X)$ is always real.

(2). Since $p = 3$, α_j, β_j and θ_k are all equal to $2\pi/3$. Thus $csc(\alpha_j/2)csc(\beta_j/2) = 4/3$ and $cos(\theta_k/2)csc^2(\theta_k/2) = 2/3$. From formula (2) we get

$$\mathrm{Spin}(\hat{\tau}, X) = \frac{1}{6}\sum_{F_k} \pm\langle [F_k]\cdot[F_k]\rangle + \frac{1}{3}\sum_{P_j} \pm 1,$$

which is always rational. This completes the proof.

For $i = 0, 1, 2, \ldots, p-1$, let m_i (resp. n_i) denote the dimensions of the ν^i-eigenspaces of a generator of the $\mathbf{Z}_p$-action on V_Λ (resp. W_Λ). Then $\dim_{\mathbf{C}} V_\Lambda = m_0 + m_1 + \cdots + m_{p-1}$ and $\dim_{\mathbf{C}} W_\Lambda = n_0 + n_1 + \cdots + n_{p-1}$. Let $k_i = m_i - n_i$, for $i = 0, 1, 2, \ldots, p-1$. Then $\mathrm{Ind}_G D = k_0 + k_1\xi + \cdots + k_{p-1}\xi^{p-1}$. Note that the intersection form of $E(2n)$ is isomorphic to $2sE8 \oplus lH$, where $2s = -\frac{\sigma(X)}{8}$, $l = b_2^+(X)$ and $s, l > 0$.

Next, we prove the main theorems.

Theorem 3.3. *Let X be a homotopy $E(2n)$ surface, and let X admit a periodic diffeomorphism τ of odd prime order p satisfying $b_2^+(X/\tau) = b_2^+(X)$. Then we have $k_0 \leq b_2^+(X) - 1$, where k_0 is one of the coefficients of the G-index operator $\mathrm{Ind}_G D$.*

Proof. Since τ is of odd prime order, the spin action generated by τ is of even type. Recall that $[V] - [W] = k(\xi)h - t(\xi)\tilde{1}$, where

$$k(\xi) = k_0 + k_1\xi + \cdots + k_{p-1}\xi^{p-1}$$

and

$$t(\xi) = t_0 + t_1\xi + \cdots + t_{p-1}\xi^{p-1}$$

satisfying

$$t(1) = t_0 + t_1 + \cdots + t_{p-1} = b_2^+(X) = l$$

$$k(1) = k_0 + k_1 + \cdots + k_{p-1} = -\frac{\sigma(X)}{8} = 2s.$$

Note that τ is of odd prime order and $b_2^+(X) = b_2^+(X/\tau)$. Then $t_0 = b_2^+(X/\tau) = l$ and $t_1 + \cdots + t_{p-1} = 0$.

Let $\alpha = \alpha_0(\xi) + \tilde{\alpha}_0(\xi)\tilde{1} + \sum_{i=1}^{\infty}\alpha_i(\xi)h_i$ be the K-theoretic degree of f. Next, we compute α under the action of ϕ, $\phi\nu$ and $J\nu^j$ for $j = 1, 2, \ldots, p-1$.

Recall that $\phi \in S^1$ is an element generating a dense subgroup of S^1, ν is a generator of $\mathbf{Z}_p$ and $J \in \mathrm{Pin}(2)$ comes from the quaternion. Since ϕ and $\phi\nu$ act non-trivially on h and trivially on $\widetilde{1}$,

$$\mathrm{dim}V_{\phi} - \mathrm{dim}W_{\phi} = -(t_0 + t_1 + \ldots + t_{p-1}) = -l < 0,$$

$$\mathrm{dim}V_{\phi\nu} - \mathrm{dim}W_{\phi\nu} = -t_0 = -l < 0.$$

From the K-theoretic degree formula, we get $d(f^{\phi}) = d(f^{\phi\nu}) = 0$. Thus $\mathrm{tr}_{\phi}(\alpha) = \mathrm{tr}_{\phi\nu}(\alpha) = 0$. Furthermore

$$\alpha_0(\nu) + \widetilde{\alpha}_0(\nu) = 0, \alpha_i(\nu) = 0, \ i \geq 1.$$

Hence $\widetilde{\alpha}_0 = -\alpha_0$ and $\alpha_i = 0$ for all $i \geq 1$. moreover $\alpha = \alpha_0(\xi)(1 - \tilde{1})$.

Let $\alpha_0(\xi) = a_0 + a_1\xi + \ldots + a_{p-1}\xi^{p-1}$. Since the action of J on $\widetilde{1}$ is -1, we have

$$tr_{J\nu^j}(\alpha) = tr_{J\nu^j}(\alpha_0(\xi)(1 - \tilde{1})) = 2(a_0 + a_1\nu^j + \ldots + a_{p-1}\nu^{j(p-1)}). \tag{3}$$

On the other hand, since the action of J on h is $\begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}$, $J\nu$ acts non-trivially on $\xi\tilde{1}$, ξh and h. Thus $\mathrm{dim}V_{J\nu} - \mathrm{dim}W_{J\nu} = 0$, and $d(f^{J\nu}) = 1$. Then from the K-theoretic degree formula, we have

$$\begin{aligned} tr_{J\nu^j}(\alpha) &= tr_{J\nu^j}(\lambda_{-1}(W_g^{\perp} - V_g^{\perp})) \\ &= tr_{J\nu^j}(\lambda_{-1}((t_0 + t_1\xi + \cdots + t_{p-1}\xi^{p-1})\tilde{1} \\ &- (k_0 + k_1\xi + \cdots + k_{p-1}\xi^{p-1})h)) \\ &= 2^{t_0}(1+\nu^j)^{t_1}(1+\nu^{2j})^{t_2}\cdots(1+\nu^{(p-1)j})^{t_{p-1}} \\ & 2^{-k_0}(1+\nu^{2j})^{-k_1}(1+\nu^{4j})^{-k_2}\cdots(1+\nu^{2(p-1)j})^{-k_{p-1}} \end{aligned} \tag{4}$$

for $j = 1, 2, \ldots, p-1$.

We multiply $p-1$ equations from (3) and (4). Noticing that $t_0 = l$, we obtain

$$2^{(l-k_0)(p-1)}(\prod_{j=1}^{p-1}(1+\nu^j))^{t_1}\cdots(\prod_{j=1}^{p-1}(1+\nu^{2(p-1)j}))^{-k_{p-1}} = 2^{p-1}\prod_{j=1}^{p-1}(\sum_{i=0}^{p-1}a_i\nu^{ji}). \tag{5}$$

Since $\prod_{j=1}^{p-1}(1+\nu^j)=1$, (5) becomes

$$2^{(l-1-k_0)(p-1)}=\prod_{j=1}^{p-1}(\sum_{i=0}^{p-1}a_i\nu^{ji})=c_0+c_1\nu+\ldots+c_{p-1}\nu^{p-1},$$

where $c_i\in\mathbf{Z},(i=0,1,\ldots,p-1)$. If $k_0\geq l$ then the above equation means

$$1=2^{(k_0-l+1)(p-1)}(c_0+c_1\nu+\ldots+c_{p-1}\nu^{p-1})$$

which is a contradiction for $\nu^p=1$ and $c_i\in\mathbf{Z},(i=0,1,\ldots,p-1)$. Thus we have $k_0\leq l-1$. This completes the proof.

Next, we discuss k_i $(i=1,2,\cdots,p-1)$ when the spin number is non-negative, and we get the following theorem.

Theorem 3.4. *Let X be a homotopy $E(2n)$ surface and let X admit a periodic diffeomorphism τ of odd prime order p satisfying $b_2^+(X/\tau)=b_2^+(X)$. Assume that the spin number Spin$(\hat{\tau},X)$ is both rational and non-negative. Then we have*

$$k_1=k_2=\cdots=k_{p-1},\ \frac{2s}{p}\leq k_0\leq l-1.$$

proof. Note that

$$\text{Spin}(\hat{\tau},X)=k_0+k_1\nu+\ldots+k_{p-1}\nu^{p-1}.$$

Then by $1+\nu+\ldots+\nu^{p-1}=0$, we obtain

$$\begin{aligned}\text{Spin}(\hat{\tau},X)&=k_0+k_1\nu+\ldots+k_{p-1}\nu^{p-1}\\&=(k_0-k_{p-1})+(k_1-k_{p-1})\nu+\cdots+(k_{p-2}-k_{p-1})\nu^{p-2}.\end{aligned}$$

Since the spin number is rational, $k_1-k_{p-1}=\ldots=k_{p-2}-k_{p-1}=0$. Thus we have $k_1=k_2=\ldots=k_{p-1}$. Then

$$\text{Spin}(\hat{\tau},X)=k_0-k_1. \tag{6}$$

Note that $\text{Spin}(1,X)=k_0+(p-1)k_1=2s$. By (6), we have

$$0\leq\text{Spin}(\hat{\tau},X)=\frac{pk_0}{p-1}-\frac{2s}{p-1}. \tag{7}$$

Then we have $\frac{2s}{p}\leq k_0\leq l-1$ by theorem 1.1. This completes the proof.

Remark 3.5. When $p = 3$ and $X = E(2n) = E(4)$, we have the following five cases.

Case 1. $k_0 = 2$, $k_1 = k_2 = 1$ and $\mathrm{Spin}(\hat{\tau}, X) = 1$.

Case 2. $k_0 = 4$, $k_1 = k_2 = 0$ and $\mathrm{Spin}(\hat{\tau}, X) = 4$.

Case 3. $k_0 = 6$, $k_1 = k_2 = -1$ and $\mathrm{Spin}(\hat{\tau}, X) = 7$.

When the spin number is rational and negative, we get the following theorem.

Theorem 3.6. *Let X be a homotopy $E(2n)$ surface and let X admit a periodic diffeomorphism τ of odd prime order p satisfying $b_2^+(X/\tau) = b_2^+(X)$. Assume that the spin number $Spin(\hat{\tau}, X)$ is both rational and negative. Then we have*

$$k_1 = k_2 = \ldots = k_{p-1} = \frac{2s - k_0}{p-1} \geq 1,\ \ k_0 < \frac{2s}{p}.$$

proof. On the one hand, we can obtain $k_1 = k_2 = \ldots = k_{p-1}$ by the same methods as Theorem 1.2. On the other hand, from (7) and the assumptions of this lemma, we have

$$0 > \mathrm{Spin}(\hat{\tau}, X) = \frac{pk_0}{p-1} - \frac{2s}{p-1}.$$

Thus $k_0 < \frac{2s}{p}$. Besides, since $k_0 + k_1 + \ldots + k_{p-1} = -\frac{\sigma(X)}{8} = 2s$, we have

$$k_1 = k_2 = \ldots = k_{p-1} = \frac{2s - k_0}{p-1} \geq 1.$$

This completes the proof.

Remark 3.7. In particular, if $p = 3$ and $X = E(2n) = E(4)$ then $k_0 \leq 1$ and $k_1 = k_2 = \frac{4-k_0}{2} \geq 2$.

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