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AN UPPER BOUND ON THE NUMBER OF EDGES OF GRAPHS CONTAINING NO r VERTEX-DISJOINT ODD CYCLES

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Abstract. In [5], we found an upper bound on the number of edges, $\mathscr{E}(G)$, of a graph G containing no r vertex-disjoint cycles of length 3. In this paper we generalize this result to graphs containing no r vertex-disjoint cycles of length $2k+1$. We showed that $\mathscr{E}(G) \leqslant \lfloor \frac{(n-r+1)^2}{4} \rfloor + (r-1)(n-r+1)$ for every $G \in \mathscr{G}(n, V_{r,2k+1})$, the class of all graphs on n vertices containing no r vertex-disjoint cycles of length $2k+1$. Determination of the maximum number of edges in a given graph that contains no specific subgraphs is one of the important problems in graph theory. Solving such problems has attracted the attention of many researchers in graph theory.

Keywords: upper bound; number of edges; graph theory.

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1. Introduction

In this paper, we only consider simple graphs. That is, graphs that has no loops or multiple edges. Let $V(G)$ denote the set of vertices of a graph G and $E(G)$ be the set of edges of G. If an edge $e \in E(G)$ is incident with the two vertices u and v in $V(G)$, we write $e = uv = vu$. For

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a vertex $u \in V(G)$ we denote the *neighborhood* of u by $N_G(u)$, which is the set of all vertices $v \in V(G)$ such that $uv \in E(G)$. For a vertex $u \in V(G)$, we define the *degree* $d_G(u)$ to be the number of edges incident with u.

For vertex-disjoint subgraphs H_1 and H_2 of G, we let $E(H_1,H_2)$ to be the set of all edges that are incident to a vertex in H_1 and a vertex in H_2. That is $E(H_1,H_2) = \{uv \in E(G) \mid u \in V(H_1), v \in H_2\}$. We also define $\mathscr{E}(G)$ to be the number of edges of G. That is, $\mathscr{E}(G)$ equals the $|E(G)|$ and $\mathscr{E}(H_1,H_2) = |E(H_1,H_2)|$. The cycle on n vertices is denoted by C_n and the complete tripartite graph with partitioning sets of order m, n and k is denoted by $K_{m,n,k}$. For given graphs G_1 and G_2 we denote the union of G_1 and G_2 by $G_1 + G_2$ such that $V(G_1+G_2) = V(G_1) \cup V(G_2)$ and $E(G_1+G_2) = E(G_1) \cup E(G_2)$. We also denote the joint of G_1 and G_2 by $G_1 \vee G_2$ such that $V(G_1 \vee G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \vee G_2) = E(G_1) \cup E(G_2) \cup E(G_1,G_2)$.

The determination of maximum number of edges in a given graph that has no specific subgraphs has attracted the attention of many graph theorists. For example, Höggkvist et al in [6] proved that $\mathscr{E}(G) \leqslant \lfloor \frac{(n-1)^2}{4} \rfloor + 1$ for a non bipartite graph G with n vertices that contains no odd cycle C_{2k+1} for all positive integers k, Jia in [7] proved that $\mathscr{E}(G) \leqslant \lfloor \frac{(n-1)^2}{4} \rfloor + 3$ for a nonpartite graph G with n vertices such that contains no odd cycle for $n \geqslant 10$, and Hailat in [5] proved that $\mathscr{E}(G) \leqslant \lfloor \frac{(n-r+1)^2}{4} \rfloor + (r-1)(n-r+1)$ for every $G \in \mathscr{G}(n, V_r, 3)$.

In [2], M. Bataineh and M. Jaradat proved that $\mathscr{E}(G) \leqslant \lfloor \frac{n^2}{4} \rfloor + r - 1$ for any graph $G \in \mathscr{S}(n;r,2k+1)$ for large n and $r \geqslant 2$, $k \geqslant 1$, where $\mathscr{S}(n;r,2k+1)$ is the set of all graphs on n vertices containing no r edge-disjoint cycles of length $2k+1$.

In this paper, we generalize the result of [5] to the case where G is a graph that contains no r vertex-disjoint cycle of length $2k+1$. This result is parallel to the result of [1] in which the author considered the case of vertex-disjoint cycles instead of edge-disjoint cycles that was addressed in [2].

2. **Important Lemmas and Theorems**

In this section, we introduce the following results that will be used to prove the main theorem of this paper.

2.1. **Theorem** (Jia [7]). *Let $G \in \mathscr{G}(n,5)$, $n \geqslant 10$. Then $\mathscr{E}(G) \leqslant \lfloor \frac{(n-2)^2}{4} \rfloor + 3$.*

2.2. **Theorem** (Batineh [1]). *Let $k \geqslant 3$ be a positive integer and $G \in \mathscr{S}(n;2k+1)$. Then for large n, $\mathscr{E}(G) \leqslant \lfloor \frac{(n-2)^2}{4} \rfloor + 3$.*

Let $\mathscr{G}(n,r,2k+1)$ denote the class of graphs on n vertices containing no r edge-disjoint cycles of length $2k+1$, and $\mathscr{G}(n,V_r,2k+1)$ denote the class of graphs on n vertices containing no r vertex-disjoint cycles of length $2k+1$. Note that $\mathscr{G}(n,V_r,2k+1) \subseteq \mathscr{G}(n,r,2k+1)$.

2.3. **Theorem** (Batineh and Jaradat [2]). *Let $G \in \mathscr{G}(n,2,3)$. Then for large n, $\mathscr{E}(G) \leqslant \lfloor \frac{n^2}{4} \rfloor + 1$. Furthermore. equality holds if and only if $G \in \Omega(n,2) = K_{1,\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$.*

2.4. **Lemma** (Bondy and Murty [3]). *Let G be a graph on n vertices. If $\mathscr{E}(G) > \frac{n^2}{4}$, then G contains a cycle of length $2k+1$ for each $1 \leqslant k \leqslant \lfloor \frac{n+3}{4} \rfloor - \frac{1}{2}$.*

2.5. **Theorem** (Batineh and Jaradat [2]). *Let $k \geqslant 1$, $r \geqslant 2$ be two integers and $g \in \mathscr{G}(n;r,2k+1)$. For large n, $\mathscr{E}(G) \leqslant \lfloor \frac{n^2}{4} \rfloor + r - 1$. Furthermore. equality holds if and only if $G \in \Omega(n,r) = K_{r-1,\lfloor \frac{n-r+1}{2} \rfloor, \lceil \frac{n-r+1}{2} \rceil}$.*

Let $\mathscr{S}(n,V_{2k+1})$ denote the class of graphs on n vertices containing no vertex disjoint cycles of length $2k+1$.

2.6. **Theorem** (Batineh [1]). *Let $k \geqslant 1$ be an integer and $G \in \mathscr{S}(n,V_{2k+1})$. Then for $n > \max\{\frac{4k^3+15k^2+11k-5}{2}, 4(4k^2+8k-3)+1\}$, $\mathscr{E}(G) \leqslant \lfloor \frac{(n-1)^2}{4} \rfloor + n - 1$. Furthermore, equality holds if and only if $G = \Omega(n,2)$.*

2.7. **Theorem** (Hailat [5]). *Let $G \in \mathscr{S}(n,V_r,3)$. Then for large n, $\mathscr{E}(G) \leqslant \lfloor \frac{(n-r+1)^2}{4} \rfloor + (r-1)(n-r+1)$. Furthermore, equality holds if and only if $G = \Omega(n,r)$.*

3. Main Result

In this section, we generalize the result of Theorem 2.7 to the case where $G \in \mathscr{S}(n,V_r,2k+1)$. That is to the case where G is a graph on n vertices containing no r vertex-disjoint cycles of length $2k+1$. We prove our main result using induction on r and we start with $r = 2$.

3.1. **Theorem.** *Let k be a positive integer and $G \in \mathscr{S}(n,2,2k+1)$. Then for large n, $\mathscr{E}(G) \leqslant \lfloor \frac{(n-1)^2}{4} \rfloor + n - 1$. Furthermore, equality holds if and only if $G = \Omega(n,2)$.*

Proof. Since $G \in \mathscr{S}(n,2,2k+1)$, then G has no two vertex-disjoint cycles of length $2k+1$. Suppose first that G has no cycle of length $2k+1$. The for $n \geqslant 4k-1$, we have $3 \leqslant 2k+1 \leqslant \frac{1}{2}(4k+2) \leqslant \lfloor \frac{n+3}{3} \rfloor$, so that, using Lemma 2.4 (Bondy and Murty [3])

$$\begin{aligned}
\mathscr{E}(G) &\leqslant \left\lfloor \frac{n^2}{4} \right\rfloor \\
&= \left\lfloor \frac{((n-1)+1)^2}{4} \right\rfloor \\
&\leqslant \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + \frac{2(n-1)}{4} + \frac{1}{4} + 1 \\
&\leqslant \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + (n-1) \qquad \text{for } n \geqslant 4k-1
\end{aligned}$$

Suppose second that G has a cycle of length $2k+1$. Then for large n, $\mathscr{E}(G) \leqslant \lfloor \frac{(n-1)^2}{4} \rfloor + n - 1$ by Theorem 2.6. Note that if $G = \Omega(n,2) = K_{1,\lfloor \frac{n-1}{2} \rfloor, \lceil \frac{n-1}{2} \rceil}$ then

$$\mathscr{E}(G) = \left\lceil \frac{n-1}{2} \right\rceil + \left\lfloor \frac{n-1}{2} \right\rfloor + \left\lceil \frac{n-1}{2} \right\rceil \left\lfloor \frac{n-1}{2} \right\rfloor = \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + (n-1)$$

Therefore equality holds if and only if $G = \Omega(n,2)$. □

To prove the main result we need to introduce Turán graphs, since these graphs play a major role in the proof.

3.2. **Definition.** The complete s-partite graph on n vertices with part sizes being $\lceil \frac{n}{s} \rceil$ or $\lfloor \frac{n}{s} \rfloor$ is called *Turán graph*. We denote this graph by $T_{n,s}$.

Note that Turán graph is K_{s+1} free, where K_{s+1} is the complete graph on $(s+1)$-vertices. In [4], David Conlon introduced the following statement of Turán's theorem.

3.3. **Theorem.** *(Turán) If G is an n-vertex K_{s+1}-free graph, then it contains at most $\mathscr{E}(T_{n,s})$ edges.*

In addition, Conlon introduced three different proofs of Turáns Theorem. In this paper we use the result of 2 (Zykovs Symmetrization). In this proof it was concluded that the set of vertices of a K_{s+1}-free graph G on n vertices with maximum number of edges can be partitioned into s equivalence classes. In these classes, vertices in the same class are non-adjacent and vertices in different classes are adjacent. Since the graph G is K_{s+1}-free, it must be a complete s-partite graph. Note that $T_{n,s}$ is the unique graph that maximizes the number of edges among such graphs.

3.4. **Theorem.** *Let G be a graph that has $(r-1)$ vertex-disjoint cycles C_1, C_2,...,C_{r-1}, but has no r vertex disjoint cycles of length $2k+1$ and let $H = G - \bigcup_{i=1}^{r-1} G(C_i)$. Then $\mathscr{E}(\bigcup_{i=1}^{r-1} G(C_i), H) \leqslant 2(r-1)(n-r+1) - 4k(r-1)^2$ and $\mathscr{E}(\bigcup_{i=1}^{r-1} G(C_i)) \leqslant (2k+1)(k+1)(r-1)^2$.*

Proof. Note that H is K_{2k+1} free graph since, otherwise, G would have r vertex-disjoint cycles of length $2k+1$, a contradiction to the assumption. Let H' be a graph on the vertices of H with a maximum number of edges. Note that $|V(H)| = |V(H')| = n-(2k+1)(r-1) = (n-r+1) + 2k(r-1)$, $\mathscr{E}(H) \leqslant \mathscr{E}(H')$, and $\mathscr{E}(\bigcup_{i=1}^{r-1} G(C_i), H) = \mathscr{E}(\bigcup_{i=1}^{r-1} G(C_i), H')$.

Let $n' = n-(2k+1)(r-1) = (n-r+1) - 2k(r-1) = (|V(H')|$. Since H' is K_{2k+1}-free graph then, using proof 2 of Turáns theorem, H' is $T_{n',2k}$ and the vertices of H' can be partitioned into $2k$ equivalent classes $H'_1, H'_2, \ldots, H_{2k}$, where $|V(H'_i)| = \lceil \frac{n'}{2k} \rceil$ or $\lfloor \frac{n'}{2k} \rfloor$. Note that vertices of H'_i are non-adjacent for all $i = 1, \ldots, 2k$, but vertices of H'_i are adjacent to all vertices of H'_j. In Figure 1, let

$$C_1 = v_{11} \ldots v_{1(2k+1)} v_{11}$$

$$\vdots$$

$$C_{r-1} = v_{(r-1)1} \ldots v_{(r-1)(2k+1)} v_{(r-1)1}$$

Note that $|H'_i| = \left\lceil \frac{n-(2k+1)(r-1)}{2} \right\rceil$ or $\left\lfloor \frac{n-(2k+1)(r-1)}{2} \right\rfloor$, so that

$$\begin{aligned} \mathscr{E}(v_{ij}, H') &\leqslant \sum_{i=1}^{2k} |H'_i| = n-(2k+1)(r-1) \\ &= (n-r+1) - 2k(r-1) \end{aligned}$$

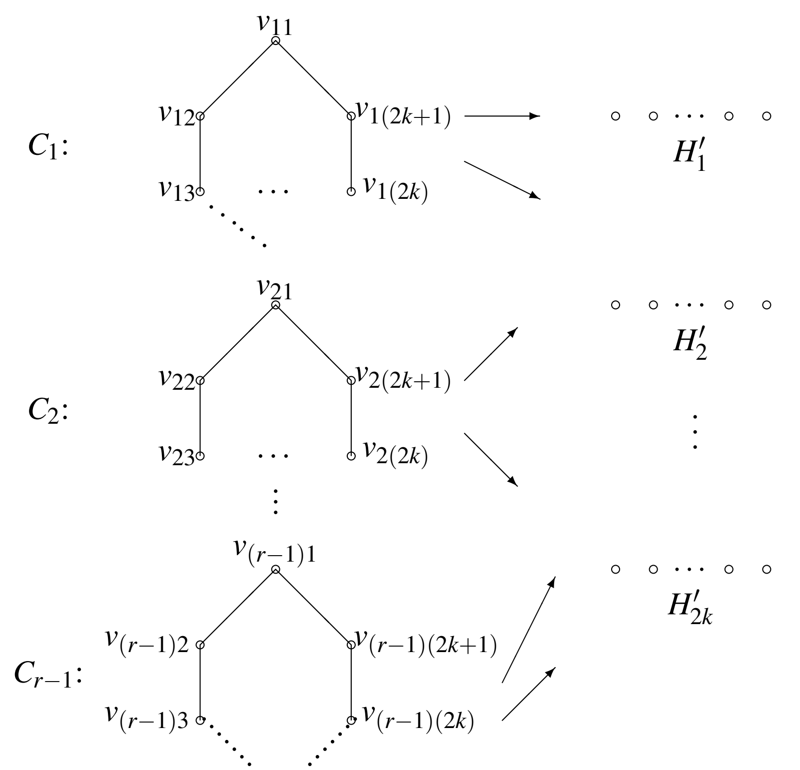

Figure 1

In Figure 1, if $v_{ij} \in V(C_i)$ is adjacent to a vertex $x \in V(H'_l)$ and to a vertex $y \in V(H'_j)$ then we can construct a cycle of length $2k+1$, $C'_i = v_{ij}x\ldots yv_{ij}$ since each vertex in H'_t is adjacent to every vertex in H'_m, for $t \neq m$. Now if we take another vertex $w_{ij} \in V(C_i)$ and assume that its adjacent to $x' \in V(H_t)$ and to $y' \in V(H_l)$ then we can construct another disjoint cycle, C''_i of length $2k+1$. If we replace C_i with C'_i and C''_i then we have r vertex-disjoint cycles in G, a contradiction. This implies that if a vertex in $V(C_i)$ is adjacent to more that one component of $V(H') = V(H)$ then the other vertices of C_i cannot be adjacent to more than one component of $V(H')$. It follows that

$$\begin{aligned}
\mathscr{E}(G(C_i),H) &= \mathscr{E}(G(C_i),H') \\
&\leqslant (n-r+1)-2k(r-1)+2k\left(\frac{1}{2k}((n-r+1)-2k(r-1))\right) \\
&= (n-r+1)-2k(r-1)+(n-r+1)-2k(r-1) \\
&= 2(n-r+1)-4k(r-1).
\end{aligned}$$

Therefore

$$\mathscr{E}(\cup_{i=1}^{r-1} G(C_i), H) \leqslant (r-1)(2(n-r+1) - 4k(r-1))$$
$$= 2(n-r+1)(r-1) - 4k(r-1)^2.$$

Now, since $|V(\cup_{i=1}^{r-1} G(C_i)| = (2k+1)(r-1)$ then

$$\mathscr{E}(\bigcup_{i=1}^{r-1} G(C_i)) \leqslant \frac{(2k+1)(r-1)\big((2k+1)(r-1)-1\big)}{2}$$
$$\leqslant \frac{(2k+1)(r-1)\big((2k+2)(r-1)\big)}{2}$$
$$= (2k+1)(k+1)(r-1)^2.$$

□

The following lemma is needed for the proof of Theorem 3.6.

3.5. **Lemma.** *Let n, r, k be three positive integers such that* $r \geqslant 2$ *and* $n \geqslant 6k(r-1)$. *Then*

$$(2-k)(r-1)(n-r+1) + (3k^2-k+1)(r-1)^2 < (r-1)(n-r+1).$$

Proof. Suppose not. Then

$$(2-k)(r-1)(n-r+1) + (3k^2-k+1)(r-1)^2 \geqslant (r-1)(n-r+1),$$

so that

$$(2-k)(n-r+1) + (3k^2-k+1)(r-1) \geqslant (n-r+1).$$

This implies that

$$n-r+1 \leqslant \frac{(3k^2-k+1)(r-1)}{k-1},$$

so that

$$n \leqslant (r-1)\left(\frac{3k^2-k+1}{k-1}+1\right)$$
$$= (r-1)\left(\frac{3k^2}{k-1}\right)$$
$$\leqslant (r-1)(3k^2)\left(\frac{2}{k}\right) = 6k(r-1),$$

a contradiction to the fact that $n > 6k(r-1)$. Therefore Lemma 3.5 follows. □

3.6. **Theorem.** *Let k be a positive integer and $G \in \mathscr{S}(n, r, 2k+1)$. Then for $n > 6k(r-1)$:*

$$\mathscr{E}(G) \leqslant \left\lfloor \frac{(n-r+1)^2}{2} \right\rfloor + (r-1)(n-r+1).$$

Furthermore, equality holds if and only if $G = \Omega(n, r)$.

Proof. We prove the theorem using induction on r. for $r = 2$ the theorem holds by Theorem 3.1.

Assume that the result is true for $r-1$. We need to show that the result is true for $r \geqslant 3$. Let $G \in \mathscr{S}(n; r, 2k+1)$. If G contains no $r-1$ vertex disjoint cycles of length $2k+1$, then by induction

$$\begin{aligned}
\mathscr{E}(G) &\leqslant \left\lfloor \frac{(n-(r-1)+1)^2}{4} \right\rfloor + ((r-1)-1)(n-(r-1)+1) \\
&= \left\lfloor \frac{(n-r+2)^2}{4} \right\rfloor + (r-2)(n-r+2) \\
&\leqslant \frac{(n-r+1)^2 + 2(n-r+1) + 1 + 4((r-1)-1)(n-(r-1)+1))}{4} + 1 \\
&= \frac{(n-r+1)^2}{4} + \frac{2(n-r+1) + 4(r-1)(n-r+1) + 4(r-1) - 4(n-r+1) - 4}{4} + 1 \\
&= \frac{(n-r+1)^2}{4} + (r-1)(n-r+1) - \frac{1}{2}(n-r+1)(r-1) - 1 + 1 \\
&\leqslant \left\lfloor \frac{(n-r+1)^2}{4} \right\rfloor + (r-1)(n-r+1), \quad \text{for } n \geqslant 3r-3.
\end{aligned}$$

Assume that G has $r-1$ vertex-disjoint cycles each of length $2k+1$ and has no r vertex-disjoint cycles of length $2k+1$. Let $C_1, C_2, \ldots, C_{r-1}$ be such cycles in G. Let $H = G - \cup_{i=1}^{r-1} G(C_i)$, so that H has no cycle of length $2k+1$ since, otherwise, G will have r vertex-disjoint cycles of length $2k+1$. Since $|V(H)| = n' = n-(r-1)(2k+1)$ then, using Lemma 2.5, we have

$$\begin{aligned}
\mathscr{E}(H) &\leqslant \lfloor \frac{n'^2}{4} \rfloor = \lfloor \frac{((n-r+1) - 2k(r-1))^2}{4} \rfloor \\
&\leqslant \lfloor \frac{(n-r+1)^2}{4} \rfloor - k(r-1)(n-r+1) + k^2(r-1)^2.
\end{aligned}$$

From Theorem 3.4 we have:

$$\mathscr{E}(\cup_{i-1}^{r-1} G(C_i), H) \leqslant 2(n-r+1)(r-1) - 4k(r-1)^2$$

and

$$\mathscr{E}(\bigcup_{i=1}^{r-1} G(C_i)) \leqslant (2k+1)(k+1)(r-1)^2.$$

It follows that:

$$\begin{aligned}
\mathscr{E}(G) &= \mathscr{E}(H) + \mathscr{E}(\cup_{i=1}^{r-1} G(C_i), H) + \mathscr{E}(\cup_{i=1}^{r-1} G(C_i)) \\
&\leqslant \left\lfloor \frac{(n-r+1)^2}{4} \right\rfloor - k(r-1)(n-r+1) + k^2(r-1)^2 \\
&\qquad + 2(n-r+1)(r-1) - 4k(r-1)^2 + (2k+1)(k+1)(r-1)^2 \\
&= \left\lfloor \frac{(n-r+1)^2}{4} \right\rfloor + (2-k)(r-1)(n-r+1) + (k^2 - 4k + 2k^2 + 3k + 1)(r-1)^2 \\
&= \left\lfloor \frac{(n-r+1)^2}{4} \right\rfloor + (2-k)(r-1)(n-r+1) + (3k^2 - k + 1)(r-1)^2 \\
&\leqslant \left\lfloor \frac{(n-r+1)^2}{4} \right\rfloor + (r-1)(n-r+1) \qquad \text{(using Lemma 3.5)}
\end{aligned}$$

Furthermore, equality holds for $\Omega(n,r) = K_{r-1,\lfloor \frac{n-r+1}{2} \rfloor, \lceil \frac{n-r+1}{2} \rceil}$ since

$$\begin{aligned}
\mathscr{E}(\Omega(n,r)) &= (r-1)\left\lfloor \frac{n-r+1}{2} \right\rfloor + (r-1)\left\lceil \frac{n-r+1}{2} \right\rceil + \left\lceil \frac{n-r+1}{2} \right\rceil \left\lfloor \frac{n-r+1}{2} \right\rfloor \\
&= (r-1)\lfloor n-r+1 \rfloor + \left\lfloor \frac{(n-r+1)^2}{4} \right\rfloor \\
&= \left\lfloor \frac{(n-r+1)^2}{4} \right\rfloor + (r-1)(n-r+1).
\end{aligned}$$

□

Conflict of Interests

The authors declare that there is no conflict of interests.

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