Available online at http://scik.org
J. Math. Comput. Sci. 9 (2019), No. 2, 131-145
https://doi.org/10.28919/jmcs/3855
ISSN: 1927-5307

L-FUZZY CLOSURE OPERATORS AND L-FUZZY COTOPOLOGIES

JU-MOK OH, YONG CHAN KIM*

Department of Mathematics, Gangneung-Wonju National University, Gangneung 210-702, Korea

Abstract. In this paper, we investigate L-fuzzy closure operators and L-fuzzy cotopologies in a complete residuated lattice. Also, we study the categorical relationship between L-fuzzy closure spaces and L-fuzzy cotopological space. Finally, we give their examples.

Keywords: complete residuated lattice; L-fuzzy closure space; L-fuzzy cotopological space.

2000 AMS Subject Classification: 54A40, 03E72, 03G10, 06A15.

1. Introduction

Hájek [7] introduced a complete residuated lattice which is an algebraic structure for many valued logic. Bělohlávek [2] investigated information systems and decision rules in complete residuated lattices. Höhle [8]introduced L-fuzzy topological structure with algebraic structure L(cqm, quantales, MV-algebra).

* Corresponding author
E-mail address: yck@gwnu.ac.kr
Received August 20, 2018

Fuzzy topological structures were extended and applied in many directions [3-6, 8-12]. Fang and Yue [6] studied the relationship between L-fuzzy closure systems and L-fuzzy topological spaces from a category viewpoint for a complete residuated lattice L.

We investigate L-fuzzy closure operators and L-fuzzy cotopologies in a complete residuated lattice. Also, we study the categorical relationship between L-fuzzy closure spaces and L-fuzzy cotopological space. Moreover, there exists the Galois correspondence between L-fuzzy cotopological spaces and L-fuzzy closure spaces. In particular, we give their examples.

2. Preliminaries

Definition 2.1. [2,7,8] An algebra $(L,\wedge,\vee,\odot,\to,\bot,\top)$ is called a complete residuated lattice if it satisfies the following conditions:

(L1) $(L,\leq,\vee,\wedge,\bot,\top)$ is a complete lattice with the greatest element $\top$ and the least element $\bot$;

(L2) $(L,\odot,\top)$ is a commutative monoid;

(L3) $x\odot y\leq z$ iff $x\leq y\to z$ for $x,y,z\in L$.

In this paper, we assume that $L=(L,\vee,\wedge,\odot,\to,\bot,\top)$ be a complete residuated lattice which is defined by $x\oplus y=x^*\to y,\ x^*=x\to 0$.

Lemma 2.2. [2,7,8] For each $x,y,z,x_i,y_i,w\in L$, we have the following properties.

(1) $1\to x=x$, $0\odot x=0$ and $x\to 0=x^*$,

(2) If $y\leq z$, then $x\odot y\leq x\odot z$, $x\oplus y\leq x\oplus z$, $x\to y\leq x\to z$ and $z\to x\leq y\to x$.

(3) $x\odot y\leq x\wedge y\leq x\vee y\leq x\oplus y$,

(4) $(\bigvee_i y_i)^*=\bigwedge_i y_i^*$,

(5) $x\odot(\bigwedge_i y_i)\leq\bigwedge_i(x\odot y_i)$,

(6) $x\oplus(\bigwedge_i y_i)=\bigwedge_i(x\oplus y_i)$,

(7) $x\to(\bigwedge_i y_i)=\bigwedge_i(x\to y_i)$,

(8) $(\bigvee_i x_i)\to y=\bigwedge_i(x_i\to y)$,

(9) $x\to(\bigvee_i y_i)\geq\bigvee_i(x\to y_i)$,

(10) $(\bigwedge_i x_i)\to y\geq\bigvee_i(x_i\to y)$,

(11) $(x \odot y) \to z = x \to (y \to z) = y \to (x \to z)$,

(12) $x \odot (x \to y) \leq y$ and $x \to y \leq (y \to z) \to (x \to z)$,

(13) $x \leq x^{**}$ and $x \to y \leq y^* \to x^*$,

(14) $(x \to y) \odot (z \to w) \leq (x \odot z) \to (y \odot w)$,

(15) $x \to y \leq (x \odot z) \to (y \odot z)$ and $(x \to y) \odot (y \to z) \leq x \to z$,

For $\alpha \in L, \lambda \in L^X$, we denote $(\alpha \to \lambda), (\alpha \odot \lambda), \alpha_X \in L^X$ as $(\alpha \to \lambda)(x) = \alpha \to \lambda(x)$, $(\alpha \odot \lambda)(x) = \alpha \odot \lambda(x)$, $\alpha_X(x) = \alpha$.

Definition 2.3.[2,7,8] Let X be a set. A mapping $R : X \times X \to L$ is called an *L-partial order* if it satisfies the following conditions:

(E1) reflexive if $R(x,x) = \top$ for all $x \in X$,

(E2) transitive if $R(x,y) \odot R(y,z) \leq R(x,z)$, for all $x,y,z \in X$,

(E3) antisymmetric if $R(x,y) = R(y,x) = \top$, then $x = y$.

Lemma 2.4. [2,5,10] For a given set X, define a binary mapping $S : L^X \times L^X \to L$ by

$$S(\lambda,\mu) = \bigwedge_{x \in X} (\lambda(x) \to \mu(x)).$$

Then, for each $\lambda,\mu,\rho,\nu \in L^X$, and $\alpha \in L$, the following properties hold.

(1) S is an L-partial order on L^X.

(2) $\lambda \leq \mu$ iff $S(\lambda,\mu) \geq \top$,

(3) If $\lambda \leq \mu$, then $S(\rho,\lambda) \leq S(\rho,\mu)$ and $S(\lambda,\rho) \geq S(\mu,\rho)$,

(4) $S(\lambda,\mu) \odot S(\nu,\rho) \leq S(\lambda \odot \nu, \mu \odot \rho)$ and $S(\lambda,\mu) \odot S(\nu,\rho) \leq S(\lambda \oplus \nu, \mu \oplus \rho)$,

(5) $S(\mu,\rho) \leq S(\lambda,\mu) \to S(\lambda,\rho)$ and $S(\mu,\rho) \leq S(\rho,\lambda) \to S(\mu,\lambda)$,

(6) $\alpha \odot S(\mu,\rho) \leq S(\alpha \to \mu, \rho)$,

(7) $\bigvee_{\mu \in L^X} (S(\mu,\rho) \odot S(\lambda,\mu)) = S(\lambda,\rho)$.

(8) If $\phi : X \to Y$ is a map, then for $\lambda,\mu \in L^X$ and $\rho,\nu \in L^Y$,

$$S(\lambda,\mu) \leq S(\phi^{\to}(\lambda), \phi^{\to}(\mu)),$$

$$S(\rho,\nu) \leq S(\phi^{\leftarrow}(\rho), \phi^{\leftarrow}(\nu)),$$

and the equalities hold if ϕ is bijective.

Definition 2.5. [6,8] A map $\mathscr{F} : L^X \to L$ is called an L-fuzzy cotopology on X if it satisfies the following conditions:

(F1) $\mathscr{F}(\perp_X) = \mathscr{F}(\top_X) = \top$,

(F2) $\mathscr{F}(\lambda \oplus \mu) \geq \mathscr{F}(\lambda) \odot \mathscr{F}(\mu)$, $\forall\, \lambda, \mu \in L^X$,

(F3) $\mathscr{F}(\bigwedge_i \lambda_i) \geq \bigwedge_i \mathscr{F}(\lambda_i)$, $\forall\, \{\lambda_i\}_{i\in\Gamma} \subseteq L^X$.

The pair $(X, \mathscr{F})$ is called an L-fuzzy cotopological space. An L-fuzzy cotopological space is called enriched if

(R) $\mathscr{F}(\alpha \to \lambda) \geq \mathscr{F}(\lambda)$ for all $\lambda \in L^X$ and $\alpha \in L$.

Let $(X, \mathscr{F}_1)$ and $(Y, \mathscr{F}_2)$ be two L-fuzzy cotopological spaces. A mapping $\phi : X \to Y$ is said to be L-fuzzy continuous iff for each $\lambda \in L^Y$, $\mathscr{F}_2(\lambda) \leq \mathscr{F}_1(\phi^{\leftarrow}(\lambda))$.

Remark 2.6. A set $\Im \subset L^X$ is called an L-cotopology on X if (t1) $\perp_X, \top_X \in \Im$, (t2) $(\lambda \oplus \mu) \in \Im$, for each $\lambda, \mu \in \Im$, (t3) $\bigwedge_i \lambda_i \in \Im$, for all $\lambda_i \in \Im$. An L-cotopology $\Im$ is called enriched if $\alpha \to \lambda \in \Im$, for all $\lambda \in \Im$ and $\alpha \in L$.

3. L-fuzzy closure spaces and L-fuzzy cotopological spaces

Lemma 3.1. Let L be a complete residuated lattice. Define $x \oplus y = x^* \to y$.

(1) $(x \to y) \oplus (z \to w) \leq (x \odot z) \to (y \oplus w)$.

(2) $(x \to y) \odot (z \to w) \leq (x \oplus z) \to (y \oplus w)$.

Proof. (1) Since $(x \odot y^*)^* = x \to y^{**} \geq x \to y$, then $x \odot y^* \leq (x \to y)^*$.

$$
\begin{aligned}
&[(x \to y) \oplus (z \to w)] \odot (x \odot z) \odot y^* \\
&= [(x \to y)^* \to (z \to w)] \odot (x \odot y^*) \odot z \\
&\leq [(x \to y)^* \to (z \to w)] \odot (x \to y)^* \odot z \\
&\leq (z \to w) \odot z \leq w
\end{aligned}
$$

Hence $[(x \to y) \oplus (z \to w)] \odot (x \odot z) \odot y^* \leq w$ iff $[(x \to y) \oplus (z \to w)] \odot (x \odot z) \leq y \oplus w$ iff $[(x \to y) \oplus (z \to w)] \leq (x \odot z) \to (y \oplus w)$.

(2)

$$
\begin{aligned}
&[(x \to y) \odot (z \to w)] \odot (x \oplus z) \odot y^* \\
&\leq [(y^* \to x^*) \odot (z \to w)] \odot (x^* \to z) \odot y^* \\
&\leq x^* \odot (z \to w) \odot (x^* \to z) \leq (z \to w) \odot z \leq w
\end{aligned}
$$

Hence $[(x \to y) \odot (z \to w)] \odot (x \oplus z) \odot y^* \leq w$ iff $[(x \to y) \odot (z \to w)] \odot (x \oplus z) \leq y \oplus w$ iff $[(x \to y) \odot (z \to w)] \leq (x \oplus z) \to (y \oplus w)$.

Definition 3.1. A map $\mathscr{C} : L^X \to L^X$ is called an L-fuzzy closure operator if it satisfies the following conditions:

(C1) $\mathscr{C}(\perp_X) = \perp_X$,

(C2) for $\lambda \in L^X$, $\lambda \leq \mathscr{C}(\lambda)$,

(C3) if $\lambda \leq \mu$, $\mathscr{C}(\lambda) \leq \mathscr{C}(\mu)$,

(C4) for all $\lambda, \mu \in L^X$, $\mathscr{C}(\lambda \oplus \mu) \leq \mathscr{C}(\lambda) \oplus \mathscr{C}(\mu)$.

The pair $(X, \mathscr{C})$ is called an L-fuzzy closure space.

An L-fuzzy closure space is called stratified if

(R) $\mathscr{C}(\alpha \to \lambda) \leq \alpha \to \mathscr{C}(\lambda)$ for all $\lambda \in L^X$ and $\alpha \in L$.

Let $(X, \mathscr{C}_1)$ and $(Y, \mathscr{C}_2)$ be two L-fuzzy closure spaces. A mapping $\phi : X \to Y$ is said to be C-map if $\phi^{\to}(\mathscr{C}_1(\lambda)) \leq \mathscr{C}_2(\phi^{\to}(\lambda))$ for each $\lambda \in L^X$.

Lemma 3.2. Let $\mathscr{C} : L^X \to L^X$ a map. The following statement are equivalent.

(1) For all $\lambda, \mu \in L^X$, $S(\lambda, \mu) \leq S(\mathscr{C}(\lambda), \mathscr{C}(\mu))$.

(2) If $\lambda \leq \mu$, then $\mathscr{C}(\lambda) \leq \mathscr{C}(\mu)$ and $\mathscr{C}(\alpha \odot \rho) \geq \alpha \odot \mathscr{C}(\rho)$ for all $\lambda \in L^X$ and $\alpha \in L$.

(3) If $\lambda \leq \mu$, then $\mathscr{C}(\lambda) \leq \mathscr{C}(\mu)$ and $\mathscr{C}(\alpha \to \rho) \leq \alpha \to \mathscr{C}(\rho)$ for all $\lambda \in L^X$ and $\alpha \in L$.

Proof. (1) ($\Rightarrow$) (2). If $\lambda \leq \mu$, then $\top = S(\lambda, \mu) \leq S(\mathscr{C}(\lambda), \mathscr{C}(\mu))$. Hence $\mathscr{C}(\lambda) \leq \mathscr{C}(\mu)$. Put $\mu = \alpha \odot \lambda$. Then $S(\lambda, \alpha \odot \lambda) = \alpha \leq S(\mathscr{C}(\lambda), \mathscr{C}(\alpha \odot \lambda))$. Hence $\alpha \odot \mathscr{C}(\lambda) \leq \mathscr{C}(\alpha \odot \lambda)$.

(2) ($\Rightarrow$) (3). Since $\alpha \odot \mathscr{C}(\alpha \to \lambda) \leq \mathscr{C}(\alpha \odot (\alpha \to \lambda)) \leq \mathscr{C}(\lambda)$, $\mathscr{C}(\alpha \to \lambda) \leq \alpha \to \mathscr{C}(\lambda)$.

(3) (⇒) (1). Since $S(\lambda,\mu)\odot\lambda \le \mu$ iff $\lambda \le S(\lambda,\mu)\to\mu$, $\mathscr{C}(\lambda)\le\mathscr{C}(S(\lambda,\mu)\to\mu)\le S(\lambda,\mu)\to\mathscr{C}(\mu)$. Hence $S(\lambda,\mu)\le S(\mathscr{C}(\lambda),\mathscr{C}(\mu))$.

Theorem 3.3. Let $(X,\mathscr{F})$ be an L-fuzzy cotopological space. Define a map $\mathscr{C}_{\mathscr{F}} : L^X \longrightarrow L^X$ as follows:

$$\mathscr{C}_{\mathscr{F}}(\lambda) = \bigwedge_{\mu\in L^X} (\mathscr{F}(\mu)\to(S(\lambda,\mu)\to\mu))$$

Then $(X,\mathscr{C}_{\mathscr{F}})$ is a stratified L-fuzzy closure space.

Proof. (C1) $\mathscr{C}_{\mathscr{F}}(\perp_X) = \bigwedge_{\mu\in L^X}(\mathscr{F}(\mu)\to(S(\perp_X,\mu)\to\mu)) \le \mathscr{F}(\perp_X)\to(S(\perp_X,\perp_X)\to\perp_X) = \perp_X$.

(C2), we have $\mathscr{C}_{\mathscr{F}}(\lambda)\ge\lambda$ for each $\lambda\in L^X$ from:

$$\begin{aligned}
S(\lambda,\mathscr{C}_{\mathscr{F}}(\lambda)) &= \bigwedge_{x\in X}(\lambda(x)\to\mathscr{C}_{\mathscr{F}}(\lambda)(x))\\
&= \bigwedge_{x\in X}\Big(\lambda(x)\to\bigwedge_{\mu\in L^X}(\mathscr{F}(\mu)\to(S(\lambda,\mu)\to\mu(x)))\Big)\\
&= \bigwedge_{x\in X}\bigwedge_{\mu\in L^X}\Big((\mathscr{F}(\mu)\odot S(\lambda,\mu)\odot\lambda(x))\longrightarrow\mu(x))\Big)\\
&= \bigwedge_{x\in X}\bigwedge_{\mu\in L^X}\Big((\mathscr{F}(\mu)\odot S(\lambda,\mu)\longrightarrow(\lambda(x)\longrightarrow\mu(x))\Big)\\
&= \bigwedge_{\mu\in L^X}\Big((\mathscr{F}(\mu)\odot S(\lambda,\mu))\to\bigwedge_{x\in X}(\lambda(x)\longrightarrow\mu(x))\Big)\\
&= \bigwedge_{\mu\in L^X}\Big((\mathscr{F}(\mu)\odot S(\lambda,\mu))\to S(\lambda,\mu)\Big)\ge\top.
\end{aligned}$$

(C3) and, by Lemma 3.2, $\mathscr{C}_{\mathscr{F}}$ is stratified from

$$S(\mathscr{C}_{\mathscr{F}}(\lambda),\mathscr{C}_{\mathscr{F}}(\mu)) = \bigwedge_{x\in X}(\mathscr{C}_{\mathscr{F}}(\lambda)(x)\to\mathscr{C}_{\mathscr{F}}(\mu)(x))$$

$$= \bigwedge_{x\in X}\Big(\bigwedge_{\rho\in L^X}(\mathscr{F}(\rho)\odot S(\lambda,\rho)\to\rho(x))\to\bigwedge_{\nu\in L^X}(\mathscr{F}(\nu)\odot S(\mu,\nu)\to\nu(x))\Big)$$

$$\geq \bigwedge_{x\in X}\bigwedge_{\rho\in L^X}\Big((\mathscr{F}(\rho)\odot S(\lambda,\rho)\to\rho(x))\to((\mathscr{F}(\rho)\odot S(\mu,\rho)\to\rho(x))\Big)$$

$$\geq \bigwedge_{x\in X}\bigwedge_{\rho\in L^X}\Big(\mathscr{F}(\rho)\odot S(\mu,\rho)\to(\mathscr{F}(\rho)\odot S(\lambda,\rho)\Big)\text{ (by Lemma 2.2 (12))}$$

$$\geq \bigwedge_{x\in X}\bigwedge_{\rho\in L^X}\Big(S(\mu,\rho)\to S(\lambda,\rho)\Big)\geq S(\lambda,\mu).\text{(by Lemma 2.4(5))}$$

(C4)

$$\mathscr{C}_{\mathscr{F}}(\lambda)\oplus\mathscr{C}_{\mathscr{F}}(\mu)$$

$$=\bigwedge_{\rho\in L^X}(\mathscr{F}(\rho)\to(S(\lambda,\rho)\to\rho))\oplus\bigwedge_{\nu\in L^X}(\mathscr{F}(\nu)\to(S(\mu,\nu)\to\nu))$$

(by Lemma 3.1(1))

$$=\bigwedge_{\rho\in L^X}\bigwedge_{\nu\in L^X}\Big((\mathscr{F}(\rho)\odot S(\lambda,\rho)\to\rho)\oplus(\mathscr{F}(\nu)\odot S(\mu,\nu)\to\nu)\Big)$$

$$=\bigwedge_{\rho,\nu\in L^X}\Big(\mathscr{F}(\rho)\odot\mathscr{F}(\nu)\odot S(\lambda,\rho)\odot S(\mu,\nu)\to(\rho\oplus\nu)\Big)$$

$$\geq\bigwedge_{\rho,\nu\in L^X}\Big(\mathscr{F}(\rho\odot\nu)\odot S(\lambda\oplus\mu,(\rho\oplus\nu))\to(\rho\oplus\nu)\Big)$$

(by Lemma 2.4(4))

$$\geq\mathscr{C}_{\mathscr{F}}(\lambda\oplus\mu).$$

Remark 3.4. Let $(X,\Im)$ be an L-cotopological space. Define a map $\mathscr{C}_{\Im}:L^X\to L^X$ as follows:

$$\mathscr{C}_{\Im}(\lambda)=\bigwedge\{\mu\in L^X\mid\lambda\leq\mu,\mu\in\Im\}.$$

Then $(X,\mathscr{C}_{\Im})$ is an L-fuzzy closure space. Moreover, if $(X,\Im)$ is enriched, $(X,\mathscr{C}_{\Im})$ is stratified.

Corollary 3.4. Let $(X,\Im)$ be an L-cotopological space. Define $\mathscr{C}_{\Im}: L^X \to L^X$ by

$$\mathscr{C}_{\Im}(\lambda) = \bigwedge_{\rho\in\Im} (S(\lambda,\rho) \to \rho).$$

Then the following properties hold.

(1) $(X,\mathscr{C}_{\Im})$ is an L-fuzzy closure space.

(2) If $(X,\Im)$ is enriched, then $(X,\mathscr{C}_{\Im})$ is a stratified L-fuzzy closure space.

(3) $\mathscr{C}_{\Im}(\lambda) \geq \bigwedge\{\mu \mid \lambda \leq \mu, \mu \in \Im\}$,

(4) If $(X,\Im)$ is enriched, then the equality in (3) holds.

Theorem 3.5. Let $(X,\mathscr{C})$ be an L-fuzzy closure space. Define a map $\mathscr{F}_{\mathscr{C}}: L^X \to L$ by:

$$\mathscr{F}_{\mathscr{C}}(\lambda) = S(\mathscr{C}(\lambda),\lambda).$$

Then,

(1) $\mathscr{F}_{\mathscr{C}}$ is an L-fuzzy cotopology on X.

(2) If $\mathscr{C}$ is stratified, then $\mathscr{F}_{\mathscr{C}}$ is an enriched L-fuzzy cotopology.

(3) $\mathscr{C}_{\mathscr{F}_{\mathscr{C}}} \geq \mathscr{C}$.

(4) If $\mathscr{C}(\mathscr{C}(\lambda)) = \mathscr{C}(\lambda)$ for all $\lambda \in L^X$, then $\mathscr{F}_{\mathscr{C}}(\mathscr{C}(\lambda)) = \top$ and $\mathscr{C}_{\mathscr{F}_{\mathscr{C}}} \leq \mathscr{C}$.

(5) If $\mathscr{F}$ is an L-fuzzy cotopology on X, then $\mathscr{F}_{\mathscr{C}_{\mathscr{F}}} \geq \mathscr{F}$.

Proof. (F1)

$$\mathscr{F}_{\mathscr{C}}(\top_X) = \bigwedge_{x\in X} (\mathscr{C}(\top_X) \to \top_X(x)) = \bigwedge_{x\in X} (\bot_X(x) \to \bot_X(x)) = \top,$$

$$\mathscr{F}_{\mathscr{C}}(\bot_X) = \bigwedge_{x\in X} (\mathscr{C}(\bot_X) \to \bot_X(x)) = \bigwedge_{x\in X} (\top_X(x) \to \top_X(x)) = \top.$$

(F2) By Lemma 3.1(1), we havre

$$\begin{aligned}
\mathscr{F}_{\mathscr{C}}(\lambda\oplus\mu) &= \bigwedge_{x\in X} (\mathscr{C}((\lambda\oplus\mu))(x) \to (\lambda\oplus\mu)(x)) \\
&\geq \bigwedge_{x\in X} (\mathscr{C}(\lambda)(x)\oplus\mathscr{C}(\mu)(x) \to \lambda(x)\oplus\mu(x)) \\
&\geq \bigwedge_{x\in X} (\mathscr{C}(\lambda)(x) \to \lambda(x)) \odot \bigwedge_{x\in X} (\mathscr{C}(\mu)(x) \to \mu(x)) \\
&= \mathscr{F}_{\mathscr{C}}(\lambda)\odot\mathscr{F}_{\mathscr{C}}(\mu).
\end{aligned}$$

(F3) By Lemma 2.2(8),we have

$$\begin{aligned}\mathscr{F}_{\mathscr{C}}(\bigwedge_i \lambda_i) &= \bigwedge_{x\in X}(\mathscr{C}(\bigwedge_i \lambda_i)(x) \to \bigwedge_i \lambda_i(x))\\ &\geq \bigwedge_{x\in X}(\bigwedge_i \mathscr{C}(\lambda_i)(x) \to \bigwedge_i \lambda_i(x))\\ &\geq \bigwedge_i \bigwedge_{x\in X}(\mathscr{C}(\lambda_i)(x) \to \lambda_i(x)) = \bigwedge_i \mathscr{F}_{\mathscr{C}}(\lambda_i).\end{aligned}$$

(2) By Lemma 2.2 (12), we have

$$\mathscr{F}_{\mathscr{C}}(\alpha \to \lambda) = \bigwedge_{x\in X}(\mathscr{C}(\alpha \to \lambda)(x) \to (\alpha \to \lambda)(x))$$

$$\geq \bigwedge_{x\in X}((\alpha \to \mathscr{C}(\lambda)(x)) \to (\alpha \to \lambda(x)))$$

$$\geq \bigwedge_{x\in X}(\mathscr{C}(\lambda)(x) \to \lambda(x)) = \mathscr{F}_{\mathscr{C}}(\lambda).$$

(3)

$$\mathscr{C}_{\mathscr{F}_{\mathscr{C}}}(\lambda)(x) = \bigwedge_{\mu\in L^X}(\mathscr{F}_{\mathscr{C}}(\mu) \to (S(\lambda,\mu) \to \mu(x)))$$

$$= \bigwedge_{\mu\in L^X}(S(\mathscr{C}(\mu),\mu) \odot S(\lambda,\mu) \to \mu(x))$$

(by the definition of $\mathscr{F}_{\mathscr{C}}$)

$$\geq \bigwedge_{\mu\in L^X}(S(\mathscr{C}(\mu),\mu) \odot S(\mathscr{C}(\lambda),\mathscr{C}(\mu)) \to \mu(x))$$

$$\geq \bigwedge_{\mu\in L^X}(S(\mathscr{C}(\lambda),\mu) \to \mu(x)) \text{ (by Lemma 2.4(3))}$$

$$\geq \mathscr{C}(\lambda)(x).$$

(4) By (C2), $\mathscr{C}(\mathscr{C}(\lambda)) \geq \mathscr{C}(\lambda)$.

Hence $\mathscr{F}(\mathscr{C}(\lambda)) = \mathscr{C}(\lambda)$. Thus, $\mathscr{F}_{\mathscr{C}}(\mathscr{C}(\lambda)) = \top$. Moreover,

$$\begin{aligned}
\mathscr{C}_{\mathscr{F}_{\mathscr{C}}}(\lambda)(x) &= \bigwedge_{\mu \in L^X} (\mathscr{F}_{\mathscr{C}}(\mu) \to (S(\lambda,\mu) \to \mu(x))) \\
&\leq \mathscr{F}_{\mathscr{C}}(\mathscr{C}(\lambda)) \to (S(\lambda, \mathscr{C}(\lambda)) \to \mathscr{C}(\lambda)(x)) \\
&\quad \text{(by the definition of } \mathscr{F}_{\mathscr{C}}) \\
&= \mathscr{C}(\lambda)(x).
\end{aligned}$$

(5)

$$\begin{aligned}
\mathscr{F}_{\mathscr{C}_{\mathscr{F}}}(\lambda) &= S\Big(\mathscr{C}_{\mathscr{F}}(\lambda), \lambda\Big) \\
&= \bigwedge_{x \in X} \Big(\bigwedge_{\mu \in L^X} (\mathscr{F}(\mu) \to (S(\lambda,\mu) \to \mu)) \to \lambda(x)\Big) \\
&\geq \bigwedge_{x \in X} \Big((\mathscr{F}(\lambda) \to (S(\lambda,\lambda) \to \lambda)) \to \lambda(x)\Big) \\
&= \bigwedge_{x \in X} \Big((\mathscr{F}(\lambda) \to \lambda(x)) \to \lambda(x)\Big) \\
&\geq \mathscr{F}(\lambda).
\end{aligned}$$

Remark 3.6. Let $(X, \mathscr{C})$ be an L-fuzzy closure space. Define a subset $\mathfrak{I}_{\mathscr{C}} \subset L^X$ by:

$$\mathfrak{I}_{\mathscr{C}} = \{\lambda \in L^X \mid \mathscr{C}(\lambda) = \lambda\}.$$

Then, $\mathfrak{I}_{\mathscr{C}}$ is an L-cotopology on X with $\mathscr{C}_{\mathfrak{I}_{\mathscr{C}}} \geq \mathscr{C}$. If $\mathscr{C}$ is stratified, then $\mathfrak{I}_{\mathscr{C}}$ is an enriched L-cotopology.

Theorem 3.7 Let $(X, \mathscr{F}_X)$ and $(Y, \mathscr{F}_Y)$ be L-fuzzy cotopological spaces and $\phi : X \to Y$ be a map. Then

(1) For each $\lambda \in L^X$,

$$\bigwedge_{\nu \in L^Y} (\mathscr{F}_Y(\nu) \to (\mathscr{F}_X(\phi^{\leftarrow}(\nu)) \leq S(\phi^{\rightarrow}(\mathscr{C}_{\mathscr{F}_X}(\lambda)), \mathscr{C}_{\mathscr{F}_Y}(\phi^{\rightarrow}(\lambda)))$$

(2) If a mapping $\phi : (X, \mathscr{F}_X) \to (Y, \mathscr{F}_Y)$ is continuous, then $\phi : (X, \mathscr{C}_{\mathscr{F}_X}) \to (Y, \mathscr{C}_{\mathscr{F}_Y})$ is a C-map.

Proof. (1)

$$\begin{aligned}
& S(\phi^{\rightarrow}(\mathscr{C}_{\mathscr{F}_X}(\lambda)), \mathscr{C}_{\mathscr{F}_Y}(\phi^{\rightarrow}(\lambda))) \\
& = \bigwedge_{y \in Y} (\phi^{\rightarrow}(\mathscr{C}_{\mathscr{F}_X}(\lambda))(y) \to \mathscr{C}_{\mathscr{F}_Y}(\phi^{\rightarrow}(\lambda))(y))
\end{aligned}$$

$$\begin{aligned}
& = \bigwedge_{x \in X} (\phi^{\rightarrow}(\mathscr{C}_{\mathscr{F}_X}(\lambda))(\phi(x)) \to \mathscr{C}_{\mathscr{F}_Y}(\phi^{\rightarrow}(\lambda))(\phi(x))) \\
& = \bigwedge_{x \in X} (\mathscr{C}_{\mathscr{F}_X}(\lambda)(x) \to \mathscr{C}_{\mathscr{F}_Y}(\phi^{\rightarrow}(\lambda))(\phi(x))) \\
& = \bigwedge_{x \in X} (\bigwedge_{\rho \in L^X} ((\mathscr{F}_X(\rho) \odot S(\lambda, \rho) \to \rho(x)) \\
& \to \bigwedge_{\nu \in L^Y} ((\mathscr{F}_Y(\nu) \odot S(\phi^{\rightarrow}(\lambda), \nu) \to \nu(\phi(x))))) \\
& = \bigwedge_{x \in X} (\bigwedge_{\nu \in L^Y} ((\mathscr{F}_X(\phi^{\leftarrow}(\nu)) \odot S(\lambda, \phi^{\leftarrow}(\nu)) \to \phi^{\leftarrow}(\nu)(x)) \\
& \to \bigwedge_{\nu \in L^Y} ((\mathscr{F}_Y(\nu) \odot S(\phi^{\rightarrow}(\lambda), \nu) \to \nu(\phi(x))))) \\
& = \bigwedge_{x \in X} (\bigwedge_{\nu \in L^Y} \Big((\mathscr{F}_X(\phi^{\leftarrow}(\nu)) \odot S(\lambda, \phi^{\leftarrow}(\nu)) \to \phi^{\leftarrow}(\nu)(x)) \\
& \to ((\mathscr{F}_Y(\nu) \odot S(\phi^{\rightarrow}(\lambda), \nu) \to \nu(\phi(x))))\Big) \\
& = \bigwedge_{x \in X} (\bigwedge_{\nu \in L^Y} \Big((\mathscr{F}_Y(\nu) \odot S(\phi^{\rightarrow}(\lambda), \nu) \to (\mathscr{F}_X(\phi^{\leftarrow}(\nu)) \odot S(\lambda, \phi^{\leftarrow}(\nu))\Big) \\
& = \bigwedge_{\nu \in L^Y} (\mathscr{F}_Y(\nu) \to \mathscr{F}_X(\phi^{\leftarrow}(\nu)))
\end{aligned}$$

(2) Since $\mathscr{F}_Y(\nu) \leq \mathscr{F}_X(\phi^{\leftarrow}(\nu))$, by (1), $\phi^{\rightarrow}(\mathscr{C}_{\mathscr{F}_X}(\lambda)) \leq \mathscr{C}_{\mathscr{F}_Y}(\phi^{\rightarrow}(\lambda))$.

Theorem 3.8 Let $(X,\mathscr{C}_X)$ and $(Y,\mathscr{C}_Y)$ be fuzzy closure spaces and $\phi : X \to Y$ be a map. Then

(1) $S(\mathscr{C}_X(\phi^{\leftarrow}(\lambda)),\phi^{\leftarrow}(\mathscr{C}_Y(\lambda))) \leq \mathscr{F}_{\mathscr{C}_Y}(\lambda) \to \mathscr{F}_{\mathscr{C}_X}(\phi^{\leftarrow}(\lambda))$ for each $\lambda \in L^Y$.

(2) If a mapping $\phi : (X,\mathscr{C}_X) \to (Y,\mathscr{C}_Y)$ is an C-map, then $\phi : (X,\mathscr{F}_{\mathscr{C}_X}) \to (Y,\mathscr{F}_{\mathscr{C}_Y})$ is continuous.

Proof. (1) By Lemma 2.2, we have

$$\begin{aligned}
&\mathscr{F}_{\mathscr{C}_Y}(\lambda) \to \mathscr{F}_{\mathscr{C}_X}(\phi^{\leftarrow}(\lambda)) \\
&= \bigwedge_{y\in Y}\Big(\mathscr{C}_Y(\lambda)(y) \to \lambda(y)\Big) \to \bigwedge_{x\in X}\Big(\mathscr{C}_X(\phi^{\leftarrow}(\lambda))(x) \to \phi^{\leftarrow}(\lambda)(x)\Big) \\
&\geq \bigwedge_{x\in X}\Big(\phi^{\leftarrow}(\mathscr{C}_Y(\lambda))(x) \to \phi^{\leftarrow}(\lambda)(x)\Big) \to \bigwedge_{x\in X}\Big(\mathscr{C}_X(\phi^{\leftarrow}(\lambda))(x) \to \phi^{\leftarrow}(\lambda)(x)\Big) \\
&\geq \bigwedge_{x\in X}\Big(\mathscr{C}_X(\phi^{\leftarrow}(\lambda))(x) \to \phi^{\leftarrow}(\mathscr{C}_Y(\lambda))(x)\Big)
\end{aligned}$$

(2) Let $\phi^{\rightarrow}(\mathscr{C}_X(\lambda)) \leq \mathscr{C}_Y(\phi^{\rightarrow}(\lambda))$. Then, put $\lambda = \phi^{\leftarrow}(\mu)$,

$$\mathscr{C}_X(\phi^{\leftarrow}(\mu)) \leq \phi^{\leftarrow}(\phi^{\rightarrow}(\mathscr{C}_X(\phi^{\leftarrow}(\mu)))) \leq \phi^{\leftarrow}(\mathscr{C}_Y(\phi^{\rightarrow}(\phi^{\leftarrow}(\mu)))) \leq \phi^{\leftarrow}(\mathscr{C}_Y(\mu)).$$

Thus, by (1), if $\mathscr{C}_X(\phi^{\leftarrow}(\lambda)) \leq \phi^{\leftarrow}(\mathscr{C}_Y(\lambda))$, then $\mathscr{F}_{\mathscr{C}_Y}(\lambda) \leq \mathscr{F}_{\mathscr{C}_X}(\phi^{\leftarrow}(\lambda))$.

Example 3.9. Let $(L = [0,1], \odot, \to)$ be a complete residuated lattice defined by

$$x \odot y = (x+y-1) \vee 0,\ \ x \to y = (1-x+y) \wedge 1.$$

$$x \oplus y = (x+y) \wedge 1,\ \ x^* = 1-x.$$

Let $X = \{x,y,z\}$ be a set and $\rho, \rho\oplus\rho \in L^X$ such that

$$\rho(x) = 0.4, \rho(y) = 0.8, \rho(z) = 0.7,$$

$$\rho\oplus\rho(x) = 0.8, \rho\oplus\rho(y) = 1, \rho\oplus\rho(z) = 1.$$

(1) We define an L-fuzzy cotopology $\mathscr{F}: L^X \to L$ as follows

$$\mathscr{F}(\lambda) = \begin{cases} 1, & \text{if } \lambda = 1_X,\ \lambda = 0_X, \\ 0.6, & \text{if } \lambda = \rho, \\ 0.3, & \text{if } \lambda = \rho \oplus \rho, \\ 0, & \text{otherwise.} \end{cases}$$

From Theorem 3.3, we obtain an L-fuzzy closure operator $\mathscr{C}_{\mathscr{F}}: L^X \to L^X$ as follows

$$\mathscr{C}_{\mathscr{F}}(\lambda) = \bigwedge_{\mu \in L^X} (\mathscr{F}(\mu) \to (S(\lambda,\mu) \to \mu))$$
$$= (S(\lambda,0_X) \to 0_X) \wedge (0.6 \to (S(\lambda,\rho) \to \rho)) \wedge (0.3 \to (S(\lambda,\rho \oplus \rho) \to \rho \oplus \rho)).$$

For $\lambda_1 = (0.9, 0.4, 0.2)$,

$$\begin{aligned} \mathscr{C}_{\mathscr{T}}(\lambda_1) &= (S(\lambda,0_X) \to 0_X) \wedge (0.6 \to (S(\lambda,\rho) \to \rho)) \\ &\wedge (0.3 \to (S(\lambda,\rho \oplus \rho) \to \rho \oplus \rho)) = (0.9, 0.9, 0.9) \end{aligned}$$

$$\mathscr{F}_{\mathscr{C}_{\mathscr{F}}}(\lambda_1) = S(\mathscr{C}_{\mathscr{T}}(\lambda_1), \lambda_1) = 0.3 \geq \mathscr{T}(\lambda_1) = 0.$$

(2) We define an L-fuzzy closure operator $\mathscr{C}: L^X \to L^X$ as follows

$$\mathscr{C}(\lambda) = \begin{cases} 0_X, & \text{if } \lambda = 0_X, \\ \rho, & \text{if } 0_X \neq \lambda \leq \rho, \\ \rho \oplus \rho, & \text{if } \rho \not\geq \lambda \leq \rho \oplus \rho, \\ 1_X, & \text{otherwise.} \end{cases}$$

$\mathscr{C}$ is not stratified because

$$\mathscr{C}(0.9 \to \rho) = \mathscr{C}((0.5, 0.9, 0.8)) = (0.8, 1, 1) \not\leq 0.9 \to \mathscr{C}(\rho) = (0.5, 0.9, 0.8).$$

From Theorem 3.5, we obtain an L-fuzzy cotopology $\mathscr{F}_{\mathscr{C}}: L^X \to L$ as follows

$$\mathscr{F}_{\mathscr{C}}(\lambda) = \begin{cases} 1_X, & \text{if } \lambda = 1_X, \\ S(\rho,\lambda), & \text{if } 0_X \neq \lambda \leq \rho,, \\ S(\rho \oplus \rho,\lambda), & \text{if } \rho \not\geq \lambda \leq \rho \oplus \rho, \\ S(1_X,\lambda), & \text{otherwise.} \end{cases}$$

Definition 3.10. [1] Suppose that $F:\mathscr{D}\to\mathscr{C}$, $G:\mathscr{C}\to\mathscr{D}$ are concrete functors. The pair (F,G) is called a *Galois correspondence* between $\mathscr{C}$ and $\mathscr{D}$ if for each $Y\in\mathscr{C}$, $id_Y: F\circ G(Y)\to Y$ is a $\mathscr{C}$-morphism, and for each $X\in\mathscr{D}$, $id_X: X\to G\circ F(X)$ is a $\mathscr{D}$-morphism.

If (F,G) is a Galois correspondence, then it is easy to check that F is a left adjoint of G, or equivalently that G is a right adjoint of F.

Let **FC** be denote the category of L-fuzzy closure spaces and C-maps for morphisms.

Let **FCTS** be denote the category of L-fuzzy cotopological spaces and continuous mappings for morphisms.

Theorem 3.12. (1) $\mathbf{F}:\mathbf{FC}\to\mathbf{FCTS}$ defined as $\mathbf{F}(X,C_X)=(X,\mathscr{F}_{C_X})$ is a functor.

(2) $\mathbf{G}:\mathbf{FCTS}\to\mathbf{FC}$ defined as $\mathbf{G}(X,\mathscr{F}_X)=(X,C_{\mathscr{F}_X})$ is a functor.

(3) The pair $(\mathbf{F},\mathbf{G})$ is a Galois correspondence between **FC** and **FCTS**.

Proof. (1) and (2) are follows from Theorems 3.8(2) and 3.7(2), respectively.

(3) By Theorem 3.5(5), if $(X,\mathscr{F}_X)$ is an L-fuzzy cotopology, then $\mathbf{F}(\mathbf{G}(X,\mathscr{F}_X)=(X,\mathscr{F}_{C_{\mathscr{F}_X}})\geq (X,\mathscr{F}_X)$. Hence, the identity map $id_X:(X,\mathscr{F}_{C_{\mathscr{F}_X}})=\mathbf{F}(\mathbf{G}(X,\mathscr{F}_X))\to(X,\mathscr{F}_X)$ is a continuous map. Moreover, if (X,C_X) is an L-fuzzy closure space, by Theorem 3.5(3), $\mathbf{G}(\mathbf{F}(X,C_X)=(X,C_{\mathscr{F}_{C_X}})\geq(X,C_X)$. Hence the identity map $id_X:(X,C_X)\to\mathbf{G}(\mathbf{F}(X,C_X))=\mathbf{G}(\mathbf{F}(X,C_X)$ is a C-map. Therefore $(\mathbf{F},\mathbf{G})$ is a Galois correspondence.

Conflict of Interests

The authors declare that there is no conflict of interests.

REFERENCES

[1] J. Adámek, H. Herrlich, G.E. Strecker, *Abstract and Concrete Categories,* Wiley, New York, 1990.

[2] R. Bělohlávek, *Fuzzy Relational Systems*, Kluwer Academic Publishers, New York, 2002.

[3] J. Fang, The relationship between L-ordered convergence structures and strong L-topologies, *Fuzzy Sets Syst.*,**161** (2010), 2923-2944.

[4] J. Fang, I-fuzzy Alexandrov topologies and specialization orders, *Fuzzy Sets Syst.,* 128 (2007) 2359-2374.

[5] J. Fang, The relationship between L-ordered convergence structures and strong L-topologies, *Fuzzy Sets Syst.,* 161 (2010) 2923-2944.

[6] J. Fang, Y. Yue, L-fuzzy closure systems *Fuzzy Sets Syst.,* 161 (2010) 1242-1252.

[7] P. Hájek, *Metamathematices of Fuzzy Logic*, Kluwer Academic Publishers, Dordrecht, 1998.

[8] U. Höhle, S.E. Rodabaugh, *Mathematics of Fuzzy Sets: Logic, topology, and Measure Theory*, The Handbooks of Fuzzy Sets Series 3, Kluwer Academic Publishers, Boston, 1999.

[9] T. Kubiak, *On fuzzy topologies*, Ph.D. Thesis, Adam Mickiewicz Uniformity, Poznan, Poland, 1985.

[10] H. Lai, D. Zhang, Fuzzy preorder and fuzzy topology, *Fuzzy Sets Syst.,* 157(2006) 1865-1885.

[11] S.E. Rodabaugh, E.P. Klement, *Topological and Algebraic Structures in Fuzzy Sets*, The Handbook of Recent Developments in the Mathematics of Fuzzy Sets, Kluwer Academic Publishers, Boston, Dordrecht, London, 2003.

[12] A. Šostak, On a fuzzy topological structure, *Suppl. Rend Circ. Matem. Palermo*, Ser. II [11] (1985), 125-186.