Available online at http://scik.org
J. Math. Comput. Sci. 9 (2019), No. 1, 46-59
https://doi.org/10.28919/jmcs/3863
ISSN: 1927-5307

ON THE AVERAGE DEGREE EIGENVALUES AND AVERAGE DEGREE ENERGY OF GRAPHS

S. C. PATEKAR*, S. A. BARDE AND M. M. SHIKARE

Department of Mathematics, Savitribai Phule Pune University, Pune 411007, India

Abstract. Given a graph G with n vertices $v_1, v_2, \ldots, v_n$ and the vertex degrees $d_1, d_2, ..., d_n$ respectively. We associate to G an average degree matrix $A_v(G)$ whose $(i,j)^{th}$ entry is $\frac{d_i+d_j}{2}$. We explore some properties of the eigenvalues and energy of $A_v(G)$.

Keywords: real symmetric matrix; eigenvalues; rank; average degree of graph.

2010 AMS Subject Classification: 05C50.

1. Introduction

Let G be a graph with $V(G) = \{1, ..., n\}$ and $E(G) = \{e_1, ..., e_n\}$. The adjacency matrix of G, denoted by $A(G)$, is the $n \times n$ matrix defined as follows. The rows and the columns of $A(G)$ are indexed by $V(G)$. If $i \neq j$ then the (i, j)-entry of $A(G)$ is 0 for vertices i and j non-adjacent, and the (i, j)-entry is 1 for i and j adjacent. If G is simple, the (i,i)-entry of $A(G)$ is 0 for $i = 1, \ldots, n$. We often denoted $A(G)$ simply by A. The eigenvalues of a matrix A are called as

*Corresponding author
E-mail address: shri82patekar@gmail.com
Received August 23, 2018

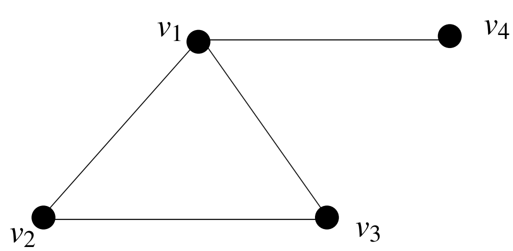
the eigenvalues of the graph G. The spectrum of a finite graph G is its set of eigenvalues together with their multiplicities. Several properties of eigenvalues of graphs and their applications have been explored in $[2,3]$.

We define a new matrix, called the average degree matrix of a graph, in the following way.

Definition 1.1. Let G be a graph with n vertices $v_1, v_2, \ldots, v_n$ and the vertex degrees $d_1, d_2, ..., d_n$ respectively. Let $A_v(G) = (a_{ij})$ be an $n \times n$ square matrix where, $a_{ij} = \frac{d_i+d_j}{2}$. We say that $A_v(G)$ is the average degree matrix of the graph G.

We observe that $A_v(G)$ is a real and symmetric matrix. Therefore its eigenvalues are real. We call the eigenvalues of $A_v(G)$ as average degree eigenvalues of G. We call the set of average degree eigenvalues of G together with their multiplicities as the average degree spectrum of G.

Example 1.2. Consider the graph G as shown in the following figure.



G

Then the average degree matrix of G is

$$A_v(G) = \begin{bmatrix} 3 & 5/2 & 5/2 & 2 \\ 5/2 & 2 & 2 & 3/2 \\ 5/2 & 2 & 2 & 3/2 \\ 2 & 3/2 & 3/2 & 1 \end{bmatrix}$$

The characteristic polynomial of $A_v(G)$ is

$det(xI - A_v(G)) = x^2(x-4-3\sqrt{2})(x-4+3\sqrt{2})$

Thus the average degree spectrum of G is $4+3\sqrt{2}$, $4-3\sqrt{2}$, 0, 0. The eigenvalues of G are 2.170, 0.311, -1, and -1.481. In this paper, we explore various properties of the average degree eigenvalues of the graphs. For terminology in graph theory, we refer $[2,6]$ and for matrix theory, we refer $[4]$.

2. Average Degree Eigenvalues of Some Graphs

In the following proposition, we investigate average degree eigenvalues of the complete graph K_n.

Proposition 2.1. For any positive integer n, the average degree eigenvalues of complete graph K_n are $n(n-1)$ with multiplicity 1 and 0 with multiplicity $n-1$.

Proof. Consider J_n, the $n \times n$ matrix of all entries ones. It is a symmetric, rank 1 matrix, and hence it has only one non-zero eigenvalue, which must equal the trace.

Thus, the eigenvalues of J_n are n with multiplicity 1 and 0 with multiplicity $n-1$. Now,

$$A_v\ (K_n) = \begin{bmatrix} (n-1) & (n-1) & \dots & (n-1) \\ (n-1) & (n-1) & \dots & (n-1) \\ \vdots & \vdots & \ddots & \vdots \\ (n-1) & (n-1) & \dots & (n-1) \end{bmatrix}$$

= $(n-1)\ J_n$

Therefore the eigenvalues of $A_v\ (K_n)$ are $n(n-1)$ with multiplicity 1 and 0 with multiplicity $n-1$.

In the following theorem, we explore average degree eigenvalues of the complete bipartite graph $K_{m,n}$.

Theorem 2.2. For any positive integer m, n, the average degree eigenvalues of complete bipartite graph $K_{m,n}$ are $\frac{2mn+(m+n)\sqrt{mn}}{2}$, $\frac{2mn-(m+n)\sqrt{mn}}{2}$ and 0 with multiplicity $m+n-2$.

Proof. The average degree matrix of $K_{m,n}$ is

$$A_v(K_{m,n}) = \begin{bmatrix} nJ_{m\times m} & (\frac{m+n}{2})J_{m\times n} \\ & \\ (\frac{m+n}{2})J_{n\times m} & mJ_{n\times n} \end{bmatrix}$$

The characteristic polynomial is $C_{A_v}(x) = x^{m+n} - 2mnx^{m+n-1} - \frac{mn(m-n)^2}{4}x^{m+n-2}$.

The roots of this polynomial are $\frac{2mn+(m+n)\sqrt{mn}}{2}$, $\frac{2mn-(m+n)\sqrt{mn}}{2}$ and 0 with multiplicity $m+n-2$.

Corollary 2.3. For any positive integer m, the average degree eigenvalues of the complete bipartite graph $K_{m,m}$ are $2m^2$ with multiplicity 1 and 0 with multiplicity $2m-1$.

The following theorem determines the average degree eigenvalues of regular graphs.

Theorem 2.4. The average degree eigenvalues of a k-regular graph on n vertices are kn with multiplicity 1 and 0 with multiplicity $n-1$.

Proof. As G is a k-regular graph, the degrees of it's all vertices are k.

The average degree matrix is

$$A_v(G) = \begin{bmatrix} k & k & \dots & k \\ k & k & \dots & k \\ \vdots & \vdots & \ddots & \vdots \\ k & k & \dots & k \end{bmatrix}$$

$= k\, J_n$

The eigenvalues of J_n are n with multiplicity 1 and 0 with multiplicity $n-1$.

Therefore the average degree eigenvalues of G are kn with multiplicity 1 and 0 with multiplicity $n-1$.

When specialized to a cycle, theorem 2.4 gives us the following corollary.

Corollary 2.5. The average degree eigenvalues of cycle graph C_n are $2n$ with multiplicity 1 and 0 with multiplicity $n-1$.

Proof. As C_n is $2-$ regular, therefore by theorem 2.4, the average degree eigenvalues of C_n are $2n$ with multiplicity 1 and 0 with multiplicity $n-1$.

In the following proposition, we explore average degree eigenvalues of the path graph.

Proposition 2.6. The average degree eigenvalues of the path graph P_n $(n \geq 3)$ are $\frac{2(n-1)+\sqrt{n^2+3n(n-2)}}{2}$ with multiplicity 1, $\frac{2(n-1)-\sqrt{n^2+3n(n-2)}}{2}$ with multiplicity 1 and 0 with multiplicity $n-2$.

Proof. The average degree matrix of path graph P_n is

$$A_v\ (P_n) = \begin{bmatrix} 1 & \frac{3}{2} & \dots & \frac{3}{2} & 1 \\ \frac{3}{2} & 2 & \dots & 2 & \frac{3}{2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{3}{2} & 2 & \dots & 2 & \frac{3}{2} \\ 1 & \frac{3}{2} & \dots & \frac{3}{2} & 1 \end{bmatrix}$$

The characteristic polynomial is $C_{A_v}(x) = x^{n-2}(x^2 - 2(n-1)x - \frac{(n-2)}{2})$.

Therefore the average degree eigenvalues of the path graph P_n $(n \geq 3)$ are $\frac{2(n-1)+\sqrt{n^2+3n(n-2)}}{2}$, $\frac{2(n-1)-\sqrt{n^2+3n(n-2)}}{2}$ and 0 with multiplicity $n-2$.

3. Properties of Average Degree Matrix and Average Degree Eigenvalues of Graphs

The following theorem gives the rank of average degree matrix.

Theorem 3.1. Let G be a graph with n vertices with atleast one edge and A_v (G) be its average degree matrix. Then rank of A_v (G) is 1 or 2.

Proof. Let G be a graph with n vertices and m edges. Then the average degree matrix of G is

$$A_v(G) = \begin{bmatrix} d_1 & \frac{d_1+d_2}{2} & \frac{d_1+d_3}{2} & \dots & \frac{d_1+d_n}{2} \\ \frac{d_2+d_1}{2} & d_2 & \frac{d_2+d_3}{2} & \dots & \frac{d_2+d_n}{2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{d_{n-1}+d_1}{2} & \frac{d_{n-1}+d_2}{2} & \frac{d_{n-1}+d_3}{2} & \ddots & \frac{d_{n-1}+d_n}{2} \\ \frac{d_n+d_1}{2} & \frac{d_n+d_2}{2} & \frac{d_n+d_3}{2} & \dots & d_n \end{bmatrix}$$

Here we have to show that the rank of $A_v(G)$ is 1 or 2. We show that any minor of $A_v(G)$ of order 3 is zero then we are through. We can write $A_v(G)$ in form

$$A_v(G)=\begin{bmatrix} \frac{d_1}{2}+\frac{d_1}{2} & \frac{d_1}{2}+\frac{d_2}{2} & \frac{d_1}{2}+\frac{d_3}{2} & \dots & \frac{d_1}{2}+\frac{d_n}{2} \\ \frac{d_2}{2}+\frac{d_1}{2} & \frac{d_2}{2}+\frac{d_2}{2} & \frac{d_2}{2}+\frac{d_3}{2} & \dots & \frac{d_2}{2}+\frac{d_n}{2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{d_{n-1}}{2}+\frac{d_1}{2} & \frac{d_{n-1}}{2}+\frac{d_2}{2} & \frac{d_{n-1}}{2}+\frac{d_3}{2} & \ddots & \frac{d_{n-1}}{2}+\frac{d_n}{2} \\ \frac{d_n}{2}+\frac{d_1}{2} & \frac{d_n}{2}+\frac{d_2}{2} & \frac{d_n}{2}+\frac{d_3}{2} & \dots & \frac{d_n}{2}+\frac{d_n}{2} \end{bmatrix}$$

Suppose we consider a minor of order 3 of $A_v(G)$ corresponding to the vertices v_i, v_j and v_k with respect to there degrees d_i, d_j and d_k.

$$=\begin{vmatrix} \frac{d_i}{2}+\frac{d_i}{2} & \frac{d_i}{2}+\frac{d_j}{2} & \frac{d_i}{2}+\frac{d_k}{2} \\ \frac{d_j}{2}+\frac{d_i}{2} & \frac{d_j}{2}+\frac{d_j}{2} & \frac{d_j}{2}+\frac{d_k}{2} \\ \frac{d_k}{2}+\frac{d_i}{2} & \frac{d_k}{2}+\frac{d_j}{2} & \frac{d_k}{2}+\frac{d_k}{2} \end{vmatrix}$$

We can write given minor in the form using the property of determinant

$$=\begin{vmatrix} \frac{d_i}{2} & \frac{d_i}{2} & \frac{d_i}{2} \\ \frac{d_j}{2}+\frac{d_i}{2} & \frac{d_j}{2}+\frac{d_j}{2} & \frac{d_j}{2}+\frac{d_k}{2} \\ \frac{d_k}{2}+\frac{d_i}{2} & \frac{d_k}{2}+\frac{d_j}{2} & \frac{d_k}{2}+\frac{d_k}{2} \end{vmatrix}+\begin{vmatrix} \frac{d_i}{2} & \frac{d_j}{2} & \frac{d_k}{2} \\ \frac{d_j}{2}+\frac{d_i}{2} & \frac{d_j}{2}+\frac{d_j}{2} & \frac{d_j}{2}+\frac{d_k}{2} \\ \frac{d_k}{2}+\frac{d_i}{2} & \frac{d_k}{2}+\frac{d_j}{2} & \frac{d_k}{2}+\frac{d_k}{2} \end{vmatrix}$$

$$=\begin{vmatrix} \frac{d_i}{2} & \frac{d_i}{2} & \frac{d_i}{2} \\ \frac{d_j}{2} & \frac{d_j}{2} & \frac{d_j}{2} \\ \frac{d_k}{2}+\frac{d_i}{2} & \frac{d_k}{2}+\frac{d_j}{2} & \frac{d_k}{2}+\frac{d_k}{2} \end{vmatrix}+\begin{vmatrix} \frac{d_i}{2} & \frac{d_i}{2} & \frac{d_i}{2} \\ \frac{d_i}{2} & \frac{d_j}{2} & \frac{d_k}{2} \\ \frac{d_k}{2}+\frac{d_i}{2} & \frac{d_k}{2}+\frac{d_j}{2} & \frac{d_k}{2}+\frac{d_k}{2} \end{vmatrix}$$

$$+\begin{vmatrix} \frac{d_i}{2} & \frac{d_j}{2} & \frac{d_k}{2} \\ \frac{d_i}{2} & \frac{d_j}{2} & \frac{d_k}{2} \\ \frac{d_k}{2}+\frac{d_i}{2} & \frac{d_k}{2}+\frac{d_j}{2} & \frac{d_k}{2}+\frac{d_k}{2} \end{vmatrix}+\begin{vmatrix} \frac{d_i}{2} & \frac{d_j}{2} & \frac{d_k}{2} \\ \frac{d_j}{2} & \frac{d_j}{2} & \frac{d_j}{2} \\ \frac{d_k}{2}+\frac{d_i}{2} & \frac{d_k}{2}+\frac{d_j}{2} & \frac{d_k}{2}+\frac{d_k}{2} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{d_i}{2} & \frac{d_i}{2} & \frac{d_i}{2} \\ \frac{d_j}{2} & \frac{d_j}{2} & \frac{d_j}{2} \\ \frac{d_k}{2} & \frac{d_k}{2} & \frac{d_k}{2} \end{vmatrix} + \begin{vmatrix} \frac{d_i}{2} & \frac{d_i}{2} & \frac{d_i}{2} \\ \frac{d_j}{2} & \frac{d_j}{2} & \frac{d_j}{2} \\ \frac{d_i}{2} & \frac{d_j}{2} & \frac{d_k}{2} \end{vmatrix} + \begin{vmatrix} \frac{d_i}{2} & \frac{d_i}{2} & \frac{d_i}{2} \\ \frac{d_i}{2} & \frac{d_j}{2} & \frac{d_k}{2} \\ \frac{d_i}{2} & \frac{d_j}{2} & \frac{d_k}{2} \end{vmatrix} + \begin{vmatrix} \frac{d_i}{2} & \frac{d_i}{2} & \frac{d_i}{2} \\ \frac{d_i}{2} & \frac{d_j}{2} & \frac{d_k}{2} \\ \frac{d_k}{2} & \frac{d_k}{2} & \frac{d_k}{2} \end{vmatrix}$$

$$+ \begin{vmatrix} \frac{d_i}{2} & \frac{d_j}{2} & \frac{d_k}{2} \\ \frac{d_i}{2} & \frac{d_j}{2} & \frac{d_k}{2} \\ \frac{d_k}{2} & \frac{d_k}{2} & \frac{d_k}{2} \end{vmatrix} + \begin{vmatrix} \frac{d_i}{2} & \frac{d_j}{2} & \frac{d_k}{2} \\ \frac{d_i}{2} & \frac{d_j}{2} & \frac{d_k}{2} \\ \frac{d_i}{2} & \frac{d_j}{2} & \frac{d_k}{2} \end{vmatrix} + \begin{vmatrix} \frac{d_i}{2} & \frac{d_j}{2} & \frac{d_k}{2} \\ \frac{d_j}{2} & \frac{d_j}{2} & \frac{d_j}{2} \\ \frac{d_i}{2} & \frac{d_j}{2} & \frac{d_k}{2} \end{vmatrix} + \begin{vmatrix} \frac{d_i}{2} & \frac{d_j}{2} & \frac{d_k}{2} \\ \frac{d_j}{2} & \frac{d_j}{2} & \frac{d_j}{2} \\ \frac{d_k}{2} & \frac{d_k}{2} & \frac{d_k}{2} \end{vmatrix} \tag{1}$$

Observe that in equation (1) all determinants are zero. Hence any minor of $A_v(G)$ of order 3 is zero. Hence the rank of $A_v(G)$ is less than 3, means the rank is either 1 or 2.

Corollary 3.2. Let G be a graph with $n \geq 3$ vertices with atleast one edge then 0 is one of the average degree eigenvalue of G.

Proof. By theorem 3.1, every average degree matrix of order n has rank either 1 or 2. This implies that it has only either 1 or 2 eigenvalues are nonzero and remaining eigenvalues are 0. Hence atleast one average degree eigenvalue of G must be 0.

Corollary 3.3. Every average degree matrix of a graph G with $n \geq 3$ vertices is a singular matrix i.e. $\det A_v(G) = 0$.

Proof. Let G be a graph with n vertices then it has n average degree eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$. We know that the $\det A_v(G) = \prod_{i=1}^{n} \lambda_i$. From corollary 3.2, atleast one average degree eigenvalue of $A_v(G)$ must be 0. This implies that $\det A_v(G) = 0$.

Definition 3.4. Let $A \in M_n$. The spectral radius of A is $\rho(A)$ = max $\{|\lambda|: \lambda \in \sigma(A)\}$, where $\sigma(A)$ is the spectrum of A.

Following theorem is well known [4].

Theorem 3.5. Let $A, B \in M_n$ and suppose that B is non-negative. If $|A| \leqslant B$, then $\rho(A) \leqslant \rho(|A|) \leqslant \rho(B)$.

A directed graph is said to be strongly connected if every vertex is reachable from every other vertex.

One can associate with a non-negative matrix A a certain directed graph G_A. It has exactly n vertices, where n is size of A, and there is an edge from vertex i to vertex j precisely when

$a_{ij} > 0$. Then the matrix A is irreducible if and only if its associated graph G_A is strongly connected.

Let G be a graph without isolated vertices and A_v be its average degree matrix. Then associated digraph of A_v is strongly connected. Hence we have the following proposition.

Proposition 3.6. The average degree matrix of a connected graph is irreducible.

The following theorem is known result of Perron-Frobenius theory.

Theorem 3.7. If $A \geq B$ are non-negative matrices and A is irreducible, then $\rho(A) > \rho(B)$.

For a matrix $A = (a_{ij})$ and $B = (b_{ij})$, $A \geq B$ denotes $a_{ij} \geq b_{ij}$ for all i, j. If $A \geq B$ and $a_{ij} > b_{ij}$ for atleast one i, j, then we write $A > B$.

Proposition 3.8. Let G be a graph in which the vertices v_i and v_j are not adjacent. Let $G + e_{ij}$ be the graph obtained from G by connecting its vertices v_i and v_j. Then $A_v(G + e_{ij}) > A_v(G)$.

Proof. The transformation $G \longrightarrow G + e_{ij}$ either increases or leaves unchanged the vertex degree of any vertex. So $A_v(G + e_{ij}) > A_v(G)$. In addition, $a_{ij}(G + e_{ij}) = a_{ij}(G) + 1$.

Combining Theorem 3.7 and Proposition 3.8 , we directly arrive at:

Theorem 3.9. Suppose G be a graph without isolated vertices. Let $G + e_{ij}$ be the graph as specified in proposition 3.8. Then $\rho(G + e_{ij}) > \rho(G)$.

Proposition 3.10. Let G be a graph with n vertices and without isolated vertices. Then $n \leq \rho(G) \leq n(n-1)$. The left inequality holds if and only if G is a forest of K_2 (with n even) and the right equality holds if and only if $G = K_n$.

Proof. We prove the first inequality. Consider the forest G_1 on n vertices, where each component is K_2. Then $J_n = A_v(G_1) < A_v(G)$. By Perron-Frobenius Theorem, $n = \rho(G_1) < \rho(G)$. Now if $G = G_1$, then $\rho(G) = n$ and the left equality holds. Suppose the left equality $\rho(G) = n$ holds and $G \neq G_1$. Thus, G and G_1 have n vertices and $G \neq G_1$. There is at least one pair of vertices say v_i and v_j which is adjacent in G but it is not adjacent in G_1. Then $A_v(G_1) < A_v(G)$ and $n = \rho(G_1) < \rho(G) = n$. This is a contradiction. Thus $G \neq G_1$.

We prove the second inequality. By repeated applications of the theorem 3.9, $\rho(G) < \rho(K_n) = n(n-1)$. If $G = K_n$, then $\rho(G) = n(n-1)$ and the right equality holds. Now, suppose the right equality $\rho(G) = n(n-1)$ holds and $G \neq K_n$. Thus, G has n vertices, K_n has n vertices and

$G \neq K_n$. There is at least one pair of vertices say v_i and v_j in G which is not adjacent. The consider $G+e_{ij}$. By repeated applications of the theorem 3.9, $n(n-1) = \rho(G) < \rho(G+e_{ij}) \leq \rho(K_n) = n(n-1)$. This is a contradiction. Thus $G = K_n$.

In the following theorem, we investigate relation between the largest average degree eigenvalue $\lambda_1(G)$ and the largest degree $\Delta(G)$ of graph G.

Theorem 3.11. For a graph G, $\Delta(G) \leq \lambda_1(G)$.

Proof. Let A_v be the average degree matrix of G and x be eigenvector of G corresponding to the eigenvalue $\lambda_1(G)$.

Then $A_v\, x = \lambda_1(G)\, x$

$\lambda_1 x_i = a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n$

But all $a_{ij} \geq 1$ and we have $\forall\, x_j > 0$

Therefore, $\lambda_1 x_i \geq x_1 + x_2 + \cdots + x_n$

From the i^{th} equation of this vector equation we get

$$\lambda_1(G)x_i \geq \sum_{j=1}^{n} x_j, i = 1, 2, \cdots, n. \tag{2}$$

$\lambda_1 x_1 + \lambda_1 x_2 + \cdots + \lambda_1 x_n \geq n\sum_{j=1}^{n} x_j$

$\lambda_1\, \sum_{i=1}^{n} x_i \geq n\sum_{j=1}^{n} x_j$

$\lambda_1 \geq n \geq n-1$

As G is a simple graph the maximum degree of any vertex in a G with n vertices is atmost $n-1$.

Let $x_k > 0$ be the maximum co-ordinate of x from Eq. (2) $\lambda_1(G)\, x_k \geq \Delta(G)\, x_k$.

In the following theorem, we give an upper bound to the sum of squares of average degree eigenvalues.

Theorem 3.12. Let G be a graph with n vertices, m edges and let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the average degree eigenvalues of G. Then $\sum\limits_{i=1}^{n} \lambda_i^2 \leq 2m^2(n+1)$.

Proof. Let G be a graph with n vertices, m edges and $d_1, d_2, \ldots, d_n$ be the degrees of the vertices $v_1, v_2, \ldots, v_n$, respectively. We know that

$\sum_{i=1}^{n} \lambda_i =$ trace of the matrix.

This implies that

$$\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} d_i \tag{3}$$

We have to find $\sum_{i=1}^{n}\lambda_i^2$. Average degree matrix is

$$A_v = \begin{bmatrix} d_1 & \frac{d_1+d_2}{2} & \frac{d_1+d_3}{2} & \dots & \frac{d_1+d_n}{2} \\ \frac{d_2+d_1}{2} & d_2 & \frac{d_2+d_3}{2} & \dots & \frac{d_2+d_n}{2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{d_{n-1}+d_1}{2} & \frac{d_{n-1}+d_2}{2} & \frac{d_{n-1}+d_3}{2} & \ddots & \frac{d_{n-1}+d_n}{2} \\ \frac{d_n+d_1}{2} & \frac{d_n+d_2}{2} & \frac{d_n+d_3}{2} & \dots & d_n \end{bmatrix}$$

Squaring both sides we can write $(A_v)^2$ in the following form

$$(A_v)^2 = \begin{bmatrix} (d_1)^2+\sum_{i=2}^{n}(\frac{d_1+d_i}{2})^2 & & & \\ & (d_2)^2+(\frac{d_2+d_1}{2})^2+\sum_{i=3}^{n}(\frac{d_2+d_i}{2})^2 & & \\ & & \ddots & \\ & & & (d_n)^2+\sum_{i=1}^{n-1}(\frac{d_n+d_i}{2})^2 \end{bmatrix}$$

We know that,

$$(\sum_{i=1}^{n} d_i)^2 = \sum_{i=1}^{n} d_i^2 + 2(d_1d_2+d_1d_3+\cdots+d_1d_n+d_2d_3+\cdots+d_2d_n+\cdots+d_{n-2}d_n+d_{n-1}d_n) \tag{4}$$

But, from the diagonal entries of $(A_v)^2$ and from equation (3) we have,

$$\begin{aligned} \sum_{i=1}^{n}\lambda_i^2 &= d_1^2+(\frac{d_1+d_2}{2})^2+\cdots+(\frac{d_1+d_n}{2})^2 \\ &\quad + d_2^2 + (\frac{d_2+d_1}{2})^2+\cdots+(\frac{d_2+d_n}{2})^2 \\ &\quad +\cdots+ d_n^2 + (\frac{d_n+d_1}{2})^2+\cdots+(\frac{d_n+d_{n-1}}{2})^2, \end{aligned}$$

$$\begin{aligned} = \; & d_1^2+\frac{d_1^2+2d_1d_2+d_2^2}{4}+\cdots+\frac{d_1^2+2d_1d_n+d_n^2}{4} \\ & + d_2^2 + \frac{d_2^2+2d_1d_2+d_1^2}{4}+\cdots+\frac{d_2^2+2d_2d_n+d_n^2}{4} \\ & +\cdots+d_n^2+\frac{d_n^2+2d_nd_1+d_1^2}{4}+\cdots+\frac{d_n^2+2d_nd_{n-1}+d_{n-1}^2}{4}, \end{aligned}$$

After simplification of above terms we get,

$$\begin{aligned} \sum_{i=1}^{n}\lambda_i^2 = \; & \sum_{i=1}^{n} d_i^2 + (\frac{n-1}{2})\sum_{i=1}^{n} d_i^2 \\ & +(d_1d_2+\cdots+d_1d_n+d_2d_3+\cdots+d_2d_n+\cdots+d_{n-2}d_n+d_{n-1}d_n) \end{aligned}$$

$$= \quad (\frac{n+1}{2})\sum_{i=1}^{n} d_i^2 + (d_1d_2 + \cdots + d_1d_n + d_2d_3 + \cdots + d_2d_n + \cdots + d_{n-2}d_n + d_{n-1}d_n)$$

However, in this equation $(d_1d_2 + d_1d_3 + \cdots + d_1d_n + d_2d_3 + \cdots + d_2d_n + \cdots + d_{n-2}d_n + d_{n-1}d_n) > 0$ because $d_i > 0, \forall i$.

Now adding the term $(d_1d_2 + d_1d_3 + d_1d_4 + \cdots + d_1d_{n-1} + d_1d_n + d_2d_3 + \cdots + d_2d_n + \cdots + d_{n-2}d_n + d_{n-1}d_n)$ n times to the right side of above equation we get,

$$\begin{aligned}
\sum_{i=1}^{n} \lambda_i^2 \quad &\leq \quad \frac{1}{2}[(n+1)\sum_{i=1}^{n} d_i^2 + 2(d_1d_2 + \cdots + d_1d_n + d_2d_3 + \cdots + d_{n-2}d_n + d_{n-1}d_n)] \\
&+ \quad n(d_1d_2 + \cdots + d_1d_n + d_2d_3 + \cdots + d_{n-2}d_n + d_{n-1}d_n)
\end{aligned}$$

$$\begin{aligned}
&\leq \quad \frac{1}{2}[(n+1)\sum_{i=1}^{n} d_i^2 + 2(d_1d_2 + \cdots + d_1d_n + d_2d_3 + \cdots + d_{n-2}d_n + d_{n-1}d_n) \\
&+ \quad 2n(d_1d_2 + \cdots + d_1d_n + d_2d_3 + \cdots + d_{n-2}d_n + d_{n-1}d_n)]
\end{aligned}$$

$$\leq \quad \frac{1}{2}[(n+1)\sum_{i=1}^{n} d_i^2 + 2(n+1)(d_1d_2 + \cdots + d_1d_n + d_2d_3 + \cdots + d_{n-2}d_n + d_{n-1}d_n)]$$

$$\leq \quad \frac{n+1}{2}[\sum_{i=1}^{n} d_i^2 + 2(d_1d_2 + \cdots + d_1d_n + d_2d_3 + \cdots + d_{n-2}d_n + d_{n-1}d_n)]$$

From equation (4), we have $\sum_{i=1}^{n} \lambda_i^2 \leq \frac{n+1}{2}[(d_1 + d_2 + \cdots + d_n)^2]$
Using the Hand shaking Lemma, we obtain
$\sum_{i=1}^{n} \lambda_i^2 \leq \frac{n+1}{2}[(2m)^2] = \frac{n+1}{2}[4m^2]$. This implies that

$$\sum_{i=1}^{n} \lambda_i^2 \leq 2m^2(n+1). \tag{5}$$

Corollary 3.13. Let G be a graph with n vertices, m edges and let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the average degree eigenvalues of G. Then $\lambda_n \leq m\sqrt{\frac{2(n+1)}{n}}$.

Proof. If $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ are the average degree eigenvalues of G then from equation (5) of the proof of theorem 3.12, we obtain

$n\lambda_n^2 \leq 2m^2(n+1)$, where λ_n is a smallest eigenvalue of G.

Thus, $\lambda_n \leq m\sqrt{\frac{2(n+1)}{n}}$.

4. Average Degree Energy of Graphs

The ordinary energy, $E(G)$, of a graph G is defined to be the sum of the absolute values of the ordinary eigenvalues of G [1,7].

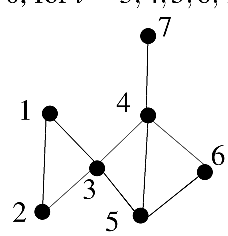
Definition 4.1. Let G be a graph on n vertices. The average degree energy, $AE(G)$ is defined as the sum of the absolute values of the average degree eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$ of G.

In the following table, we explore the average degree energy of some classes of graphs which have two or three distinct average degree eigenvalues denoted by λ_1, λ_2 and λ_3.

Graphs	λ_1	λ_2	λ_3	Average Degree Energy
K_n	$n(n-1)$	0	-	$n(n-1)$
C_n	$2n$	0	-	$2n$
Q_n	$n2^n$	0	-	$n2^n$
P_n	$\frac{2(n-1)+\sqrt{n^2+3n(n-2)}}{2}$	$\frac{2(n-1)-\sqrt{n^2+3n(n-2)}}{2}$	0	$\sqrt{n^2+3n(n-2)}$
Petersen Graph	30	0	-	30

Remark 4.2. If the rank of $A_v(G)$ is 1, then $A_v(G)$ has only one non-zero average degree eigenvalue $\rho(G)$ and thus $AE(G) = \rho(G)$.

Remark 4.3. If the rank of $A_v(G)$ is 2, then $A_v(G)$ has two non-zero average degree eigenvalues say λ_1 and λ_2. If λ_1 is much larger than λ_2, then as a reasonably accurate approximation, we have $AE(G) \approx \rho(G)$. As an example of this kind, consider the graph depicted in the following figure for which $\lambda_1 = 18.721, \lambda_2 = -0.721$ and $\lambda_i = 0$, for $i = 3,4,5,6,7$.



The following proposition shows that every positive even integer is an average degree energy of some graph.

Proposition 4.4. Every positive even integer is an average degree energy of some graph.

Proof. Let $m = 2k$, (where $k \geq 1$) be a positive even integer. Then consider the cycle C_k on k vertices. The cycle has k edges and its average degree energy is $2k = m$.

Proposition 4.5. Let G be a graph on n vertices and m edges. Then $AE(G) \geq 2m$ and equality holds if G is a regular graph.

Proof. We know, $\sum_i \lambda_i = 2m$ and $\sum_i |\lambda_i| \geq \sum_i \lambda_i$. This implies that $AE(G) = \sum_i |\lambda_i| \geq 2m$. Clearly equality holds if G is a regular graph.

Proposition 4.6. Let G be a graph with at least one edge. Then $AE(G) \geq \text{rank } A_v(G)$.

Proof. By Theorem 3.1, rank of $A_v(G)$ is either 1 or 2 and by Proposition 4.5, $AE(G) \geq 2m$. Thus, $AE(G) \geq \text{rank } A_v(G)$.

Theorem 4.7. Let G be a graph in which the vertices v_i and v_j are not adjacent. Let $G + e_{ij}$ be the graph obtained from G by connecting its vertices v_i and v_j. Then $AE(G + e_{ij}) > AE(G)$.

Proof. By Theorem 3.9, $\rho(G + e_{ij}) > \rho(G)$ and also we know $AE(G) \approx \rho(G)$. This implies that $AE(G + e_{ij}) > AE(G)$.

We have the following immediate corollaries of theorem 4.7 .

Corollary 4.8. Let G be a connected graph on n vertices, and let $H \neq G$ be a spanning, connected subgraph of G. Then $AE(G) > AE(H)$.

Corollary 4.9. Let G be a connected graph on n vertices and let H be a vertex induced subgraph of G on n' vertices where $n' < n$. Then $AE(G) > AE(H)$.

Conclusion. In the present paper, the concepts of average degree matrix, average degree eigenvalues and average degree energy of a graph are given and studied. In the literature the degree sum matrix [5] of a graph, in which diagonal entries are zero, exists, whereas in average degree matrix introduced here, the diagonal entries are not necessarily zeros. Hence the results obtained in this paper for average degree matrix are not overlapping with the results of degree sum matrix.

Conflict of Interests

The authors declare that there is no conflict of interests.

References

[1] I. Gutman, The energy of a graph: old and new results, in: A. Betten, A. Kohnert, R. Laue, A. Wassermann (Eds.), Al- gebraic Combinatorics and Applications, Springer-Verlag, Berlin, 2001, pp. 196–211.

[2] K. Lowel W. Beineke, Robin J. Wilson, Topics in Algebraic Graph Theory, Cambridge University Press, 2004.

[3] Brouwer A. E., Haemers W. E.,Spectra of Graphs, Springer, New York, 2010.

[4] Roger A. Horn, Charles R. Johnson, Matrix Analysis, Cambridge University Press, 1985.

[5] H.S.Ramane, D.S.Revankar and J.B.Patil, Bounds for the degree sum eigenvalue and degree sum energy of a graph, Int. J. Pure Appl. Math. Sci. 2(2013), 161–167.

[6] Douglas B.West, Introduction to Graph theory, Prentice-Hall, U.S.A, 2001.

[7] X. Li, Y. Shi, I. Gutman, Graph Energy, Springer, Berlin, 2012.