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HEMI-SLANT WARPED PRODUCT SUBMANIFOLDS OF LP-SASAKIAN MANIFOLDS

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Abstract. In this paper we study the hemi-slant warped product submanifolds of an LP-Sasakian manifold. We obtain some basic results in this setting and finally proved an inequality for the squared norm of the second fundamental form in the terms of warping function and slant angle.

Keywords: warped product; hemi-slant; LP-Sasakian manifold.

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1. Introduction

The study of the slant immersions in the almost Hermitian manifolds was initiated by B. Y.Chen [5]. A. Lotta [1] extended the notion of the slant immersions in the setting of the almost contact metric manifolds. N. Papaghiuc [12] introduced a class of submanifolds in an almost Hermitian manifolds, called the semi-slant submanifolds, this class includes the class of the proper CR-submanifolds and the slant submanifolds. J. L. Cabrerizo et al. [8] initiated

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the study of the contact version of the semi-slant submanifolds and also defined Bi-slant submanifolds. A step forward, A. Carriazo [2] defined and study the Bi-slant immersions in the almost Hermitian manifolds and simultaneously gave the notion of Anti-slant submanifolds in the almost Hermitian manifolds. B. sahin [3] study these submanifolds with the name of hemi-slant submnaifolds for their warped product in the setting of Kaehler manifolds. Recently, M. A. Khan et al. [11] studied slant submnaifolds of LP-contact manifolds

R. L. Bishop and B. O' Neil [13] introduced the notion of the warped product manifolds. These manifolds are generalization of the Riemannian product manifolds and occur naturally. Recently, many important physical applications of the warped product manifolds have been discovered, giving motivation to the study of these spaces with differential geometric point of view. For instance, it has been accomplished that the warped product manifolds provide an excellent setting to the model space time near black hole or bodies with large gravitational fields (c.f., [7], [13], [15]). Recently, S. K. Hui et al [14] studied the warped product submanifolds of the LP-Sasakian manifolds and in particular they have showen the existence of the hemi-slant warped product submanifolds via some examples. In this continuation we study the hemi-slant warped product submanifolds of the LP-Sasakian manifolds.

2. Preliminaries

An n-dimensional smooth manifold $\bar{M}$ is said to be an LP-Sasakian manifold [6] if it admits a (1, 1) tensor field ϕ, a unit time like contravariant vector field ξ, a 1-form η, and a Lorentzian metric g, which satisfy

$$\eta(\xi) = -1, \quad g(X,\xi) = \eta(X), \quad \phi^2 X = X + \eta(X)\xi \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y), \quad \bar{\nabla}_X \xi = \phi X, \tag{2.2}$$

$$(\bar{\nabla}_X \phi)Y = g(X,Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi, \tag{2.3}$$

where $\bar{\nabla}$ denotes the operator of covariant derivetive with respect to the Lorentzian metric g. It can be easily seen that in a LP-Sasakian manifold, the following relations hold

$$\phi\xi = 0, \quad \eta(\phi X) = 0, \quad rank\phi = n-1 \tag{2.4}$$

for any vector field X,Y tangent to $\bar{M}$ the tensor field $\Omega(X,Y)$ is a symmetric (0, 2) tensor field [9]. Also, since the vector field η is closed in an LP-Sasakian manifold, we have [9]

$$(\bar{\nabla}_X\eta)Y = \Omega(X,Y), \quad \Omega(X,\xi) = 0,$$

for any vector fields X,Y tangent to $\bar{M}$.

Let M be a submanifold of an LP- contact metric manifold $\bar{M}$ with induced metric g and if ∇ and $\nabla^{\perp}$ are the induced connection on the tangent bundle TM and the normal bundle $T^{\perp}M$ of M, respectively then the Gauss and Weingarten formulae are given by the following equations

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{2.5}$$

$$\bar{\nabla}_X N = -A_N X + \nabla^{\perp}_X N, \tag{2.6}$$

for each $X,Y \in TM$ and $N \in T^{\perp}M$, where h and A_N are the second fundamental form and the shape operator respectively for the immersion of M into $\bar{M}$ and they are related as

$$g(h(X,Y),N) = g(A_N X,Y), \tag{2.7}$$

where g denotes the Riemannian metric on $\bar{M}$ as well as on M.

For any $X \in TM$, we write

$$\phi X = PX + FX, \tag{2.8}$$

where PX and FX are the tangential and normal components of ϕX respectively.

Similarly, for any $N \in T^{\perp}M$, we write

$$\phi N = tN + fN, \tag{2.9}$$

where tN and fN are the tangential and normal components of ϕN. The covariant derivatives of the tensor field P and F are defined as

$$(\bar{\nabla}_X P)Y = \nabla_X PY - P\nabla_X Y, \tag{2.10}$$

$$(\bar{\nabla}_X F)Y = \nabla^{\perp}_X FY - F\nabla_X Y. \tag{2.11}$$

From equations (2.3), (2.5), (2.6), (2.8) and (2.9) we have

$$(\bar{\nabla}_X P)Y = g(X,Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi + th(X,Y) + A_{FY}X \tag{2.12}$$

$$(\bar{\nabla}_X F)Y = fh(X,Y) - h(X,PY). \tag{2.13}$$

Throughout the paper, we consider ξ to be tangent to M. The submanifold M is said to be invariant if F is identically zero, on the other hand M is said to be anti-invariant if P is identically zero.

Definition 2.1 [11]. A submanifold M of an almost contact metric manifold $\bar{M}$ is said to be a slant submanifold if for any $x \in M$ and $X \in T_xM - \langle\xi\rangle$ the angle between X and ϕX is constant. The constant angle $\theta \in [0, \pi/2]$ is then called the slant angle of M in $\bar{M}$. If $\theta = 0$ the submanifold is an invariant submanifold, if $\theta = \pi/2$ then it is an anti-invariant submanifold, if $\theta \neq 0, \pi/2$ then it is a proper slant submanifold.

For the slant submanifolds of the LP-contact metric manifolds M. A. Khan et al. [11] proved the following Lemma

Lemma 2.1. Let M be a submanifold of an LP-contact metric manifold $\bar{M}$, such that $\xi \in TM$ then M is slant submanifold if and only if there exist a constant $\lambda \in [0,1]$ such that

$$P^2 = \lambda(I + \eta \otimes \xi), \tag{2.14}$$

where $\lambda = \cos^2\theta$.

Thus, one has the following consequences of above formulae

$$g(PX, PY) = \cos^2\theta[g(X,Y) + \eta(X)\eta(Y)], \tag{2.15}$$

$$g(FX,FY) = \sin^2\theta[g(X,Y) + \eta(X)\eta(Y)], \tag{2.16}$$

for any $X,Y \in TM$.

A submanifold M of an LP contact metric manifold $\bar{M}$ is said to be a hemi-slant submanifold, if there exist two orthogonal complementary distributions $D^{\perp}$ and D_{θ} on M such that

(i) $TM = D^{\perp} \oplus D_{\theta} \oplus \langle\xi\rangle$,

(ii) The distribution $D^{\perp}$ is anti-invariant i.e., $\phi D^{\perp} \subseteq T^{\perp}M$,

(iii) The distribution D_{θ} is slant with slant angle $\theta \neq \pi/2$.

It is straight forward to see that the semi-invariant submanifolds and slant submanifolds are the hemi-slant submanifolds with $\theta = 0$ and $D^{\perp} = \{0\}$ respectively.

If μ is an invariant subspace under ϕ of the normal bundle $T^{\perp}M$, then in the case of hemi-slant submanifold, the normal bundle $T^{\perp}M$ can be decomposed as

$$T^{\perp}M = \mu \oplus FD_{\theta} \oplus FD^{\perp}. \tag{2.17}$$

A hemi-slant submanifold M is called a hemi-slant product if the distributions $D^{\perp}$ and D_{θ} are involutive and parallel on M. In this case M is foliated by the leaves of these distributions.

As a generalization of the product manifolds and in particular of a hemi-slant product submanifold, one can consider warped product of the manifolds which are defined as

Definition 2.2 Let (B, g_B) and (F, g_F) be two Riemannian manifolds with the Riemannian metric g_B and g_F respectively and f be a positive differentiable function on B. The warped product of B and F is the Riemannian manifold $(B \times F, g)$, where

$$g = g_B + f^2 g_F.$$

For a warped product manifold $N_1 \times_f N_2$, we denote by D_1 and D_2 the distributions defined by the vectors tangent to the leaves and fibers respectively. In other words, D_1 is obtained by the tangent vectors of N_1 via the horizontal lift and D_2 is obtained by the tangent vectors of N_2 via vertical lift. In the case of the hemi-slant warped product submanifolds D_1 and D_2 are replaced by $D_\perp$ and D_θ respectively.

The warped product manifold $(B\times F, g)$ is denoted by $B\times_f F$. If X is the tangent vector field to the manifold $M = B\times_f F$ at (p,q) then

$$\|X\|^2 = \|d\pi_1 X\|^2 + f^2(p)\|d\pi_2 X\|^2.$$

R. L. Bishop and B. O'Neill [13] proved the following

Theorem 2.1 Let $M = B\times_f F$ be the warped product manifolds. If $X, Y \in TB$ and $V, W \in TF$ then

(i) $\nabla_X Y \in TB$,

(ii) $\nabla_X V = \nabla_V X = (\frac{Xf}{f})V$,

(iii) $\nabla_V W = \frac{-g(V,W)}{f}\nabla f$.

∇f is the gradient of f and is defined as

$$g(\nabla f, X) = Xf, \tag{2.18}$$

for all $X \in TM$.

Corollary 2.1 On a warped product manifold $M = N_1 \times_f N_2$, the following statements hold

(i) N_1 is totally geodesic in M,

(ii) N_2 is totally umbilical in M.

Throughout, we denote by $N_\perp$ and N_θ an anti- invariant and a slant submanifold respectively of an LP-contact metric manifold $\bar{M}$.

3. Main results

S. K. Hui et al. [14] proved that the warped product of the type $N_\perp \times_f N_\theta$ does not exist and they also proved that the warped product of the type $N_\theta \times_f N_\perp$ exist and ξ is tangential to

N_θ. Throughout this section we consider the warped product of the type $N_\theta \times_f N_\perp$ and called them hemi-Slant warped product submanifolds for these submanifolds by Theorem 2.1 we have

$$\nabla_X Z = \nabla_Z X = Zln f X \tag{3.1}$$

for any $X \in TN_\theta and Z \in TN_\perp$.

Now we start the section from the following some properties

Lemma 3.1. On a hemi- slant warped product submanifold $M = N_\theta \times_f N_\perp$ of an LP-Sasakian manifold $\bar{M}$, we have

(i) $g(h(X,Y),FZ) + g(h(X,Z),FY) = 0$
(ii) $g(h(X,Z),FZ) = 0$,
(iii) $g(h(Z,Z),FX) = PXlnf\|Z\|^2 - \eta(X)\|Z\|^2$,
(iv) $g(h(Z,Z),FPX) = \cos^2\theta Xlnf\|Z\|^2$,

for any $X,Y \in TN_\theta$ and $Z \in TN_\perp$.

Proof. For the warped product of the type $N_\theta \times_f N_\perp$, $(\bar{\nabla}_X P)Y \in TN_\theta$, for any $X,Y \in TN_\theta$, then from (2.12), we have

$$(\nabla_X P)Y = g(X,Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi + th(X,Y) + A_{FY}X$$

taking the inner product with $Z \in TN_\perp$, we get $g(h(X,Y),FZ) + g(h(X,Z),FY) = 0$ which is the part (i).

Now for any $X \in TN_\theta$ and $Z \in TN_\perp$ from (2.10) and (2.12)

$$-P\nabla_X Z = th(X,Z) + A_{FZ}X,$$

applying (3.1), the above equation gives that $g(h(X,Z),FZ) = 0$, which is the part (ii).

Finally, for any $Z \in TN_\perp$ by (2.12), we have

$$-P\nabla_Z Z = g(Z,Z)\xi + th(Z,Z) + A_{FZ}Z$$

taking the inner product by $X \in TN_\theta$ and using (3.1) and the part (ii), the above equation gives

$$g(th(Z,Z),X)+\|Z\|^2\eta(X)=PXlnf\|Z\|^2,$$

or

$$g(h(Z,Z),FX)=PXlnf\|Z\|^2-\eta(X)\|Z\|^2,$$

this is the part (iii) of the lemma and by replacing X by PX and using the Lemma 2.1, we get the part (iv).

Let us denote by $D_\theta \oplus \langle\xi\rangle$ and $D_\perp$ the tangent bundles on N_θ and $N_\perp$ respectively and let $\{X_0=\xi, X_1, \dots X_q, X_{q+1}=PX_1, \dots, X_{2q}=PX_q\}$ and $\{Z_1,\dots,Z_p\}$ be the local orthonormal frames of vector fields on N_θ and $N_\perp$ respectively with $2q+1$ and p real dimensions, then

$$\|h\|^2=\sum_{i,j=1}^{2q+1} g(h(X_i,X_j),h(X_i,X_j))+\sum_{i=1}^{2q+1}\sum_{r=1}^{p} g(h(X_i,Z_r),h(X_i,Z_r))$$

$$+\sum_{r,s=1}^{p} g(h(Z_r,Z_s),h(Z_r,Z_s)). \tag{3.2}$$

Theorem 3.2. Let $M=N_\theta \times_f N_\perp$ be a hemi-slant warped product submanifold of an LP-Sasakian manifold $\bar{M}$ such that N_θ and $N_\perp$ are the slant and anti-invariant submanifolds respectively of $\bar{M}$. If $2PXlnf\eta(X)\geq(\eta(X))^2$, then the squared norm of the second fundamental form h satisfies

$$\|h\|^2 \geq 2pK\csc^2\theta(1+\cos^2\theta)\|\nabla lnf\|^2), \tag{3.3}$$

where ∇lnf is the gradient of lnf and $K=\sum(1+\eta(X))$.

Proof. In view of the decomposition (2.17), we may write

$$h(U,V)=h_{FD_\theta}(U,V)+h_{FD^\perp}(U,V)+h_\mu(U,V), \tag{3.4}$$

for each $U,V\in TM$, where $h_{FD_\theta}(U,V)\in FD_\theta$, $h_{FD_\perp}(U,V)\in FD^\perp$ and $h_\mu(U,V)\in\mu$. It is evident that the projections $h_{FD_\theta}(U,V)$ and $h_{FD_\perp}(U,V)$ are of the following types

$$h_{FD_\theta}(U,V)=\sum_{r=1}^{2q} h^r(U,V)FX_r \tag{3.5}$$

$$h_{FD_\perp}(U,V) = \sum_{i=1}^{p} h^i(U,V)FZ_i \tag{3.6}$$

where

$$h^r(U,V) = \csc^2\theta g(h(U,V),FX_r) \tag{3.7}$$

$$h^i(U,V) = g(h(U,V),FZ_i) \tag{3.8}$$

for each $U,V \in TM$. In view of the above formulae we have

$$g(h_{FD_\theta}(Z_i,Z_i),h_{FD_\theta}(Z_i,Z_i)) = g(h^r(Z_i,Z_i)FX_r,h^r(Z_i,Z_i)FX_r))$$

$$+\sum_{s\neq i} g(h^s(Z_i,Z_i)FX_r,h^s(Z_i,Z_i)FX_r)).$$

Now using (2.16), the above equation becomes

$$g(h_{FD_\theta}(Z_i,Z_i),h_{FD_\theta}(Z_i,Z_i)) = (h^r(Z_i,Z_i))^2\sin^2\theta(1+(\eta(X_r))^2)$$

$$+\sum_{s\neq i}(h^s(Z_i,Z_i))^2\sin^2\theta(1+(\eta(X_s))^2)$$

In view of (3.7), we get

$$g(h_{FD_\theta}(Z_i,Z_i),h_{FD_\theta}(Z_i,Z_i)) = \csc^2\theta(1+(\eta(X_r))^2)(g(h(Z_i,Z_i),FX_r))^2$$

$$+\csc^2\theta(1+(\eta(X_s))^2)(g(h(Z_i,Z_i),FX_s))^2.$$

From the part (iii) of the Lemma 3.1.

$$g(h_{FD_\theta}(Z_i,Z_i),h_{FD_\theta}(Z_i,Z_i)) = \csc^2\theta(1+(\eta(X_r))^2)(PX_r lnf-\eta(X_r))^2$$

$$+\csc^2\theta(1+(\eta(X_s))^2)((PX_s lnf-\eta(X_s))^2.$$

Summing over $r,s = 0,1,\dots q,q+1,\dots 2q$, $i = 1,\dots p$ and applying the assumption and the Lemma 2.1, we get

$$\sum_{i=0}^{p} g(h_{FD_\theta}(Z_i,Z_i),h_{FD_\theta}(Z_i,Z_i)) \geq 2pK\csc^2\theta(1+\cos^2\theta)\|\nabla lnf\|^2), \tag{3.9}$$

Similarly using the part (ii) of the Lemma 3.1 with the help of (3.6) and (3.8), we can prove the following

$$\sum_{r=0}^{2q}\sum_{i=0}^{p} g(h_{FD_\perp}(X_r,Z_i),h_{FD_\perp}(X_r,Z_i)) = 0. \tag{3.10}$$

The results follows from (3.2), (3.9) and (3.10).

Conflict of Interests

The authors declare that there is no conflict of interests.

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