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WEAK TYPES OF COMPACTNESS

G. N. MILIARAS*

American University of Athens, Athens, 11525, Greece

Abstract. In this paper, we examine two new topological properties which are weaker than $[k, k]-$compact.

Keywords: Set; cardinal; regular; singular; cofinality of a cardinal; cover; [a,b]-compact; $k-$pointwise compact; $k-$setwise compact; countably compact; complete accumulation point.

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1. Introduction

Throughout this paper, a, b, k and c denote cardinal numbers with a and b infinite and $a \leq b$. The set of all cardinals k such that $a \leq k \leq b$ is designated by $[a, b]$. The cardinality of a set X is denoted by $|X|$ and ordinal numbers are denoted by β, γ and δ.

The theory of $[a, b]-$compactness gives a unified approach to the important notions of compactness, the Lindelof property, countable compactness, and subsets having complete accumulation points. See, for example [1−4, 7−10, 12−15] and the references cited therein.

Before we proceed, we state the following definitions:

*Corresponding author

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Definition 1.1. The compactness number of a topological space X, denoted by $Cn(X)$, is the smallest cardinal k such that: every open cover $\mathcal{U}$ of X with $|\mathcal{U}| \geq k$ has a subcover of cardinality less than k.

Definition 1.2. A space is $[a,b]-$compact ($[a,b)-$compact) if every open cover of cardinality less than or equal to b (less than b) has a subcover of cardinality less than a.

Remark 1.1. In Definition 1.1, if $a = b$ then X is called $[a,a]-$compact. It is clear that X is $[a,b]-$compact if and only if X is $[k,k]-$compact for every k with $a \leq k \leq b$.

Definition 1.3. A space is $[a,b]^r-$compact if it is $[k,k]-$compact for every regular cardinal k with $a \leq k \leq b$.

(The readers may find the above definitions and some of their consequences in [4, 6, 15].)

Definition 1.4. Let X be a topological space and $\mathcal{U}$ an open cover of X. The star of any point x of X with respect to $\mathcal{U}$, denoted by $St(x,\mathcal{U})$ is the set $\cup\{U \in \mathcal{U} | x \in U\}$, and the order of x denoted by $ord(x,\mathcal{U})$ is the cardinal $|\{U \in \mathcal{U} | x \in U\}|$.

Definition 1.5. Let X be a topological space and $S \subseteq X$. A point s of X is a complete accumulation point of S, if for every open neighborhood U of s we have $|U \cap S| = |S|$.

In 1929, Alexandroff and Urysohn [2], established the following theorem:

Theorem 1.1 [2]**.** *Let k be a regular cardinal. A topological space X is $[k,k]-$compact, if and only if every subset S of X with $|S| = k$ has a complete accumulation point.*

Very recently, Miliaras [11], established the following theorem and corollary:

Theorem 1.2 [11]**.** *Let k be a cardinal, the following are equivalent:*

(i) k is regular.

(ii) For every set X with $|X| > k$, and for every cover U with $|\mathcal{U}| = k$, which has no subcover of cardinality less than k, there exists a subset S of X with $|S| = k$, than cannot be covered by less than k elements of U.

Corollary 1.1 [11]**.** *Let X be a set, $\mathcal{U}$ be a cover of X with $|\mathcal{U}| = k$ singular, with no subcover of cardinality less than k. Assume that every subset S of X with $|S| = k$ is* *covered by less than k elements of $\mathcal{U}$. Then there is a cardinal $\mu \geq cf(k)$, $\mu < k$ such that every subset of X of cardinality k is covered by at most μ elements of $\mathcal{U}$.*

In this paper, we examine two new topological properties which are weaker than $[k,k]-$compact.

3. Main results

The following lemmas are needed for the proof of the main results:

Lemma 2.1. *Let X be a topological space, $\mathcal{U}$ is an open cover of X such that $|\mathcal{U}| = a$, $\mathcal{U}$ has no subcover of smaller cardinality and for every $x \in X$, $ord(x,\mathcal{U}) \leq \lambda < a$. Then*

(i) *there exists an open cover $\mathcal{V}$ of X such that $|\mathcal{V}| = a$ with no subcover of smaller cardinality and a subset S of X with $|S| = a$ such that every $s \in S$ is contained in a unique element of $\mathcal{V}$.*

(ii) for every subset S' of S, $\overline{S'} \subseteq \cup\{St(s,\mathcal{V})|s \in S'\}$ and if X is T_1 then S' is closed in X.

(iii) *for every cardinal $c < a$ there exists an open cover $\mathcal{W}$ of X such that $|\mathcal{W}| = c$ with no subcover of smaller cardinality.*

Proof. Pick any point $x_0 \in X$ and set $V_0 = St(x_0,\mathcal{U})$. Since $ord(x_0,\mathcal{U}) \leq \lambda < a$ we can pick $x_1 \notin V_0$ and let $V_1 = St(x_1,\mathcal{U})$, clearly $x_0 \notin V_1$. Using tranfinite induction, for every ordinal $b < a$ we can construct a family $\mathcal{V}^b = \{V_\gamma|\gamma < b,\ V_\gamma = St(x_\gamma,\mathcal{U})\}$ such that if $\beta \neq \delta$, $x_\beta \notin V_\delta$ and $x_\delta \notin V_\beta$. We may continue this procedure until we construct a family that covers X. We can do this since $\mathcal{U}$ is a cover of X. Let $\mathcal{V}$ be this family, clearly $|\mathcal{V}| = a$, since every element of $\mathcal{U}$ is contained in at most one element of $\mathcal{V}$. Let $S = \{x_\gamma|\gamma < a\}$ be the set of points chosen for the construction of $\mathcal{V}$. Clearly x_γ belongs to a unique V_γ and $|S| = a$. The proof of Part (i) of lemma is complete.

Let $S' = \{x_\beta|\beta < \mu\}$ be a subset of S. Set $W = \cup\{V_\beta|\beta < \mu\}$ where $V_\beta = St(x_\beta,\mathcal{U})$. Choose $x \in \overline{S'}$ and $V \in \mathcal{V}$, where $\mathcal{V}$ is the family of sets of Part (i), such that $x \in V$, then $V \cap S' \neq \varnothing$, since every $x_\beta \in S'$ belongs to a unique element of $\mathcal{V}$, V has to be one of the V_β 's. Thus $x \in V \subseteq W$ which means that $\overline{S'} \subseteq W$.

Now let X be a T_1 space, $x \in \overline{S}$ and $x \neq x_\gamma$ for every $\gamma < a$. Let $V_\gamma \in \mathcal{V}$ with $x \in V_\gamma$. Then since each V_γ contains a unique element of S which is x_γ and is closed since X is T_1, let A be an open neighborhood of x that does not contain x_γ, then $A \cap V_\gamma$ does not intersect S. This contradicts to our assumption, so S is closed and since by its structure it is discrete, every subset of S is closed too. The proof of Part (ii) of lemma is complete.

Let $\mathcal{V}' = \{V_{\gamma_\theta} | \theta < c < a\}$ be a subcollection of $\mathcal{V}$. Then $X \setminus \cup \{V_\gamma | \gamma \neq \gamma_\theta\}$ is covered by all the elements of $\mathcal{V}'$. Therefore $\mathcal{B} = \mathcal{V}' \cup \{X \setminus \cup \{V_\gamma | \gamma \neq \gamma_\theta\}\}$ is a cover of X of cardinality c, since the set $\{x_{\gamma_\theta} | x_{\gamma_\theta} \in V\gamma_\theta,\ \theta < c\}$ is covered by exactly c elements of $\mathcal{V}'$ and it can not be intesected by $X \setminus \cup \{V_\gamma | \gamma \neq \gamma_\theta\}$, by the structure of V_γ 's. So, $\mathcal{B}$ is the desired cover. The proof of Part (iii) is complete.

The proof of the lemma is complete.

Lemma 2.2. *Let X be $[k, k]-$compact and $Cn(X) > k^+$. Then X contains no closed subset of compactness number k^+.*

Proof. Let $F \subseteq X$ closed with $Cn(F) = k^+$ and $\mathcal{U}$ be an open cover of F with $|\mathcal{U}| = k$. Since F is $[k, k]-$compac too, $\mathcal{U}$ has a subcover of smaller cardinality. This contradicts $Cn(F) = k^+$.

The proof of the lemma is complete.

Remark 2.1. Lemma 2.2 gave rise to the following property:

Definition 2.1. A topological space X is $k-$setwise compact if it contains no closed subset of compactness number k^+.

Remark 2.2. As we saw, every $[k, k]-$compact space is $k-$setwise compact. The inverse is not always true. To see this, consider the space $\omega_1 \times \omega_2$ with basic open sets the products of the line segments $[0, a) \times [0, b)$. It is obvious that this space is ω_1-setwise compact, but its subsets of the form $[0, a) \times \omega_2$, $a < \omega_1$, form an open cover of the space of cardinality ω_1 with no subcover of smaller cardinality. On the other hand, we notice that a T_1, ω_1-setwise compact space is countably compact ($[\omega, \omega]-$compact), since it has no countable, closed and discrete subset and therefore, every countable subset of this space has a complete accumulation point. So, by Theorem 1.1 the above described space is countably compact.One might ask: *Is every T_1, $k-$setwise compact space $[k,k]-$compact;* We have seen in [6−9] $k-$setwise compact spaces to be, under conditions, $[k,k]-$compact.

Theorem 2.1. *Let X be a $k-$setwise compact space and $Cn(X) > k^+$. Then X has no open cover $\mathcal{U}$ such that $|\mathcal{U}| = a > k$ with no subcover of smaller cardinality, for every $U \in \mathcal{U}$, $|U| \leq k$ and $ord(x,\mathcal{U}) \leq k$ for every $x \in X$ and $U \in \mathcal{U}$.*

Proof. Let $\mathcal{U}$ be an open cover of X with the properties mentined above. Then repeating the construction in the proof of Lemma 2.1, for $c = k$ we consider the collection $\mathcal{V}_k = \left\{V_{\gamma_\beta} = St(x_{\gamma_\beta},\mathcal{U})|\beta < k\right\}$. Then since $\left|St(x_{\gamma_\beta},\mathcal{U})\right| \leq k$ we have $|\cup\mathcal{V}_k| = k$. Let $S' = \left\{x_{\gamma_\beta}|\beta < k\right\}$. Then $\overline{S'} \subseteq \cup\mathcal{V}_k$ by Lamma 2.1, and since $S' \subseteq \overline{S'} \subseteq \cup\mathcal{V}_k$, we have $|S'| = k$. Thus $Cn(\overline{S'}) = Cn(S') = k^+$. This contradicts to our assumption.

The proof of the theorem is complete.

Theorem 2.2. *Let X be a T_1, $k-$setwise compact space such that $Cn(X) > k^+$. Then there is no open cover $\mathcal{U}$ of X such that $|\mathcal{U}| = a \geq k$ with no subcover of smaller cardinality, and $ord(x,\mathcal{U}) \leq \lambda < a$ for every $x \in X$.*

Proof. Assume that X has a cover $\mathcal{U}$ described as above. We repeat the construction of $\mathcal{V}$ and S in the proof of Lemma 2.1 with $|S'| = k$. Since S' is closed and discrete we have $Cn(S') = k^+$. This contradicts our assumption.

The proof of the theorem is complete.

Theorem 2.3. *Let X be a $[k,k]-$compact space and $Cn(X) > k^+$. Then there is no open cover $\mathcal{U}$ of X with $|\mathcal{U}| > k$ with no subcover of smaller cardinality, and $ord(x,\mathcal{U}) \leq \lambda < k$ for every $x \in X$.*

Proof. Assume that X has a cover $\mathcal{U}$ described as above. Then from Lemma 2.1 there exists an open cover $\mathcal{V}$ such that $|\mathcal{V}| = k$ with no subcover of smaller cardinality. This contradicts our assumption.

The proof of the theorem is complete.

Remark 2.3. Theorem 1.2 gives rise to the following property:

Definition 2.2. A topological space X is $k-$pointwise compact if for every open cover $\mathcal{U}$ with $|\mathcal{U}| = k$, every subset S of X with $|S| \leq k$ can be covered by less than k elements of $\mathcal{U}$.

Remark 2.4. It is clear from Theorem 1.2 that if k is a regular cardinal, $k-$pointwise compact is equivalent to $[k, k]-$compact. So, the above definition is of interest only if k is singular.

Theorem 2.4. *Let X be a $k-$pointwise compact space such that $Cn(X) > k^+$ and k singular. Then the following statements hold:*

(I) *X has no open cover $\mathcal{U}$ such that $|\mathcal{U}| \geq a > k$ with no subcover of smaller cardinality, and $ord(x,\mathcal{U}) \leq \lambda < a$ for every $x \in X$.*

(II) *If every point of X has an open neighborhood of cardinality at most k, then X is $k-$setwise compact.*

Proof. Assume that X has a cover $\mathcal{U}$ described as above. Then from Lemma 2.1 there exists an open cover $\mathcal{V}$ such that $|\mathcal{V}| = k$ and a subset S' of X with $|S'| = k$, that can not be covered by less than k elements of $\mathcal{V}$. This contradicts our assumption. The proof of Part (I) of theorem is complete.

Let $F \subseteq X$ closed with $Cn(F) = k^+$, clearly $|F| \geq k$. Let $\mathcal{U}$ be an open cover of F such that $|U| \leq k$ for every $U \in \mathcal{U}$. Since $Cn(F) = k^+$, $\mathcal{U}$ has an open subcover $\mathcal{U}'$ such that $|\mathcal{U}'| \leq k$. Then since $F \subseteq \cup\mathcal{U}'$ and $|\cup\mathcal{U}'| \leq k$ we have $|F| \leq k$. Thus F is covered by less than k elements of $\mathcal{U}'$ by $k-$pointwise compactness. Therefore $Cn(F) < k^+$. This contradicts our assumption. The proof of Part (II) of theorem is complete.

The proof of the theorem is complete.

Remark 2.5. Corollary 1.1 is leading us to the following theorem:

Theorem 2.5. *Let b be a singular cardinal, X be a $b-$pointwise compact space and for every open cover U of X with $|\mathcal{U}| = b$, every subset S of X with $|S| \leq b$ can be covered by less than $a < b$ elements of U. Then X is $[a, b]^r-$compact.*

Proof. Let k be a regular cardinal with $a \leq k < b$. Then X is $[k, k]-$compact (immediate from Theorem 1.2). Therefore X is $[a, b]^r-$compact. The proof of the theorem is complete.

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