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COMMON FIXED POINTS OF GERAGHTY-BERINDE TYPE CONTRACTION MAPS IN PARTIAL METRIC SPACES

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Abstract. In this paper, we introduce Geraghty-Berinde type contraction maps for a pair of maps in partial metric spaces and prove the existence of common fixed points in which the pair is weakly compatible and restricting the completeness of X to its subspace. Also, we extend the same for two pairs of maps. We provide examples in support of our results.

Keywords: common fixed points; partial metric space; weakly compatible maps; Geraghty-Berinde type contraction maps.

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1. Introduction

The development of fixed point theory is based on the generalization of contraction conditions in one direction or/and generalization of ambient space of the operator under consideration on the other. Banach contraction principle plays an important role in solving nonlinear equations,

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and it is one the most useful result in fixed point theory. In 1994, Matthews [18] introduced the notion of a partial metric in which the concept of self distance need not be equal to zero.

2. Preliminaries

The Banach fixed point theorem in the context of partial metric spaces due to Matthews [18] is the following:

Theorem 2.1. [18] Let (X,p) be a complete partial metric space, and let $T : X \to X$ be a mapping such that there exists $k \in [0,1)$, satisfying $p(Tx,Ty) \leq kp(x,y)$ for all $x,y \in X$. Then T has a unique fixed point in X.

In 1996, Neill [21] defined the notion of the dualistic partial metric, later Oltra and Valero [20] proved Banach fixed point theorem on complete dualistic partial metric spaces. Further, Valero [22] established a fixed point theorem using a nonlinear contractive condition instead of a Banach contraction condition. The notation of almost contractions was introduced by Berinde ([9], [10]) as a generalization of contraction maps. For further works in this direction, we refer ([8], [11], [12], [13]). In 2012, Altun and Acar [2] characterized this concept in the context of partial metric spaces and proved some fixed point results. For more works on fixed point results and common fixed point results in partial metric spaces, we refer ([1]-[6]).

In 1973, Geraghty [17] proved a fixed point theorem, generalizing Banach contraction principle. Several authors proved later various results using Geraghty-type conditions.
Recently, Dukić, Kadelburg and Radenović [14] proved a fixed point theorem using Geraghty-type contraction in partial metric spaces as follows.

We denote $\mathfrak{F} = \{\beta : [0,\infty) \to [0,1) / \beta(t_n) \to 1 \text{ as } n \to \infty \Rightarrow t_n \to 0 \text{ as } n \to \infty\}$.

Theorem 2.2. [14] Let (X,d) be a complete partial metric space and let $T : X \to X$ be a selfmapping. Suppose that there exists $\mathfrak{F} \in \beta$ such that $p(Tx,Ty) \leq \beta(p(x,y))p(x,y)$ holds for all $x,y \in X$. Then T has a unique fixed point $u \in X$ and for each $x \in X$ the Picard sequence $\{T^n x\}$ converges to u when $n \to \infty$.

Definition 2.1. [18] Let X be a nonempty set. A mapping $p : X \times X \to R^+,\ R^+ = [0,\infty)$ is said to be a partial metric, if it satisfies the following conditions:

For any $x,y,z \in X$

$(P1)$ $x = y \Leftrightarrow p(x,x) = p(x,y) = p(y,y)$,

$(P2)$ $p(x,x) \leq p(x,y),\ p(y,y) \leq p(x,y)$,

$(P3)$ $p(x,y) = p(y,x)$,

$(P4)$ $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$. The pair (X,p) is called a partial metric space.

If p is a partial metric on X, then the functions $p^s, p^w : X \times X \to R^+,\ R^+ = [0,\infty)$ defined by $p^s(x,y) = 2p(x,y) - p(x,x) - p(y,y)$ and $p^w(x,y) = p(x,y) - \min\{p(x,x), p(y,y)\}$ are ordinary metrics on X.

Example 2.1. [18] Consider $X = [0,\infty)$ with $p(x,y) = \max\{x,y\}$. Then (X,p) is a partial metric space. It is clear that p is not a (usual) metric.

Note that in this case, $p^s(x,y) = |x-y|$.

Example 2.2. [15] Let $X = \{[a,b] : a,b \in R, a \leq b\}$ and define $p([a,b],[c,d]) = \max\{b,d\} - \min\{a,c\}$. Then (X,p) is a partial metric space.

Each partial metric p on X generates τ_0 topology τ_p on X, which has a base the family of open p-balls $\{B_p(x,\varepsilon) : x \in X, \varepsilon > 0\}$, where $B_p(x,\varepsilon) = \{y \in X / p(x,y) < p(x,x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

Clearly, a limit of a sequence in a partial metric space need not be unique. Moreover, the function p need not be continuous.

Example 2.3. [19] Consider $X = [0,\infty)$ with $p(x,y) = \max\{x,y\}$. For $\{x_n\} = \{1\}$, $p(x_n,x) = x = p(x,x)$ for each $x \geq 1$.

Definition 2.2. [18] Let (X,p) be a partial metric space. A sequence $\{x_n\}$ is converges to x if and only if $p(x,x) = \lim\limits_{n\to\infty} p(x,x_n)$.

Definition 2.3. [18] Let (X,p) be a partial metric space. A sequence $\{x_n\}$ is said to be a Cauchy sequence if $\lim\limits_{n,m\to\infty} p(x_n,x_m)$ exists and is finite.

Definition 2.4. [18] A partial metric space (X,p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges with respect to τ_p, to a point $x \in X$, such that $p(x,x) = \lim\limits_{n,m\to\infty} p(x_n,x_m)$.

The following lemmas in a partial metric space are useful in proving our main results.

Lemma 2.1.[18] Let (X,p) be a partial metric space. Then the sequence $\{x_n\}$ is a Cauchy sequence in X if and only if it is a Cauchy sequence in the metric space (X,p^s).

Lemma 2.2. [18] A partial metric space (X,p) is complete if and only if the metric space

(X,p^s) is complete. Moreover, $\lim_{n\to\infty} p^s(x,x_n)=0 \Leftrightarrow p(x,x)=\lim_{n\to\infty} p(x,x_n)=\lim_{n,m\to\infty} p(x_n,x_m)$.

Definition 2.5. [16] Let X be a nonempty set. Let $f:X\to X$ and $g:X\to X$ be two selfmaps. If $fx=gx$ implies that $fgx=gfx$ for x in X, then we say that the pair (f,g) is weakly compatible.

The following theorem is due to Dinarvand [19].

Theorem 2.3. [19] Let (X,p) be a complete partial metric space and let $T:X\to X$ be a selfmap. Suppose that there exist $\beta\in\mathfrak{F}$ and $L\geq 0$ such that
$p(Tx,Ty)\leq \beta(M(x,y))M(x,y)+LN(x,y)$ holds for all $x,y\in X$, where
$M(x,y)=\max\{p(x,y),p(x,Tx),p(y,Ty),\frac{1}{2}[p(x,Ty)+p(y,Tx)]\}$ and
$N(x,y)=\min\{p^w(x,Tx),p^w(y,Ty),p^w(x,Ty),p^w(y,Tx)\}$.
Then T has a unique fixed point $u\in X$. Moreover, $P(u,u)=0$.

In the following, we introduce Geraghty-Berinde type contraction map for a pair of maps.

Definition 2.6. Let (X,p) be a partial metric space, and let f and g be selfmaps of X. If there exist $\beta\in\mathfrak{F}$ and $L\geq 0$ such that

$$p(fx,fy)\leq \beta(M(x,y))M(x,y)+LN(x,y) \tag{2.1}$$

for all $x,y\in X$, where $M(x,y)=\max\{p(gx,gy),p(gx,fx),p(gy,fy),\frac{1}{2}[p(gx,fy)+p(gy,fx)]\}$ and $N(x,y)=\min\{p^w(gx,fx),p^w(gx,fy),p^w(gy,fx)\}$,
then we call the pair (f,g) is a Geraghty-Berinde type contraction maps.

If $L=0$ in (2.1) then we say that the pair (f,g) is a generalized Geraghty type contraction maps.

Example 2.4. Let $X=[0,1]$. We define $p(x,y)=\max\{x,y\}$ for all $x,y\in X$. Then (X,p) is a partial metric space. We define selfmaps $f,\ g:X\to X$ by $f(x)=\frac{x^2}{2},\ g(x)=x^2$ and

$$\beta(t)=\begin{cases} 0 & \text{if } t=0 \\ \frac{1}{1+t} & \text{if } t>0 \end{cases} \quad \text{with } L\geq 0.$$

Then clearly the pair (f,g) is a Geraghty-Berinde type contraction map.

The following proposition is useful to prove our main results.

Proposition 2.1.[7] Let (X,p^s) be a metric space with $\lim_{n\to\infty} p^s(y_n,y_{n+1})=0$. Assume that $\{y_{2n}\}$ is a Cauchy sequence in (X,p^s) then $\{y_n\}$ is also Cauchy in (X,p^s).

In Section 3, we extend Theorem 2.3 to a pair of maps in which we prove the existence of common fixed points of Geraghty-Berinde type contraction maps in which the pair is weakly compatible in partial metric spaces and by restricting the completeness of X (Theorem 3.1). Also, we extend the same for two pairs of selfmaps. In Section 4, we draw some corollaries from our main results and provide examples in support of our results.

3. Main results

Theorem 3.1. *Let (X,p) be a partial metric space and let the pair (f,g) be Geraghty-Berinde type contraction maps. If $f(X) \subseteq g(X)$, the pair (f,g) is weakly compatible and $g(X)$ is a complete subspace of X then f and g have a unique common fixed point in X.*

Proof. Let x_0 be arbitrary point in X. Since $f(X) \subseteq g(X)$ there exists $x_1 \in X$ such that

$fx_0 = gx_1 = y_0$ (say).

In general we have there exists $x_n \in X$ satisfying $fx_n = gx_{n+1} = y_n$ (say), $n = 0,1,2,\dots$.

Case (i): Assume that $p(y_n,y_{n+1}) > 0$ for some n.

We show that $p(y_n,y_{n+1}) \leq p(y_{n-1},y_n), n = 1,2,3,\dots$.

We consider

$$\begin{aligned} p(y_n,y_{n+1}) &= p(fx_n,fx_{n+1}) \\ &\leq \beta(M(x_n,x_{n+1}))M(x_n,x_{n+1}) + LN(x_n,x_{n+1}) \end{aligned} \tag{3.1}$$

Since $\frac{1}{2}[p(y_{n-1},y_{n+1}) + p(y_n,y_n)] \leq \frac{1}{2}[p(y_{n-1},y_n) + p(y_n,y_{n+1})] \leq \max\{p(y_{n-1},y_n), p(y_n,y_{n+1})\}$

Now

$$\begin{aligned} M(x_n,x_{n+1}) &= \max\{p(gx_n,gx_{n+1}), p(gx_n,fx_n), p(gx_{n+1},fx_{n+1}), \\ &\qquad \tfrac{1}{2}[p(gx_n,fx_{n+1}) + p(gx_{n+1},fx_n)]\} \\ &= \max\{p(y_{n-1},y_n), p(y_{n-1},y_n), p(y_n,y_{n+1}), \tfrac{1}{2}[p(y_{n-1},y_{n+1}) + p(y_n,y_n)]\} \end{aligned}$$

Therefore $M(x_n,x_{n+1}) = \max\{p(y_{n-1},y_n), p(y_n,y_{n+1})\}$ and

$$\begin{aligned} N(x_n,x_{n+1}) &= \min\{p^w(gx_n,fx_n), p^w(gx_n,fx_{n+1}), p^w(gx_{n+1},fx_n)\} \\ &= \min\{p^w(y_{n-1},y_n), p^w(y_{n-1},y_{n+1}), p^w(y_n,y_n)\}. \end{aligned}$$

As $p^w(y_n,y_n) = 0$, it follows that $N(x_n,x_{n+1}) = 0$.

If $M(x_n,x_{n+1}) = p(y_n,y_{n+1})$ then from (3.1), we have

$$p(y_n,y_{n+1}) \leq \beta(p(y_n,y_{n+1}))p(y_n,y_{n+1}) < p(y_n,y_{n+1}),$$

which is a contradiction.

Hence, $M(x_n,x_{n+1}) = p(y_{n-1},y_n)$.

Therefore from (3.1), we obtain

$$p(y_n,y_{n+1}) \leq \beta(p(y_{n-1},y_n))p(y_{n-1},y_n) < p(y_{n-1},y_n).$$

Hence, $\{p(y_n,y_{n+1})\}$ is a decreasing sequence of nonnegative reals and bounded below by 0.

So, there exists $r \geq 0$ such that $\lim\limits_{n\to\infty} p(y_n,y_{n+1}) = r.$ (3.2)

We claim that $r = 0$. On the contrary suppose $r > 0$.

On letting $n \to \infty$ in (3.1), and using (3.2), we get

$r \leq \beta(r)r < r,$

a contradiction.

Hence, $\lim\limits_{n\to\infty} p(y_n,y_{n+1}) = 0.$ (3.3)

Thus from (P_2), we get that $\lim\limits_{n\to\infty} p(y_n,y_n) = 0.$ (3.4)

By the definition of p^s, (3.3) and (3.4), we get $\lim\limits_{n\to\infty} p^s(y_n,y_{n+1}) = 0.$ (3.5)

Next, we prove that $\{y_n\}$ is Cauchy in (X,p^s). On the contrary suppose that $\{y_n\}$ is not Cauchy. There exist $\varepsilon > 0$ and monotone increasing sequence of natural numbers $\{m_k\}$ and $\{n_k\}$ such that $n_k > m_k$ with

$$p^s(y_{m_k},y_{n_k}) \geq \varepsilon \text{ and } p^s(y_{m_k},y_{n_k-1}) < \varepsilon. \tag{3.6}$$

Now we prove that (i) $\lim\limits_{k\to\infty} p(y_{m_k},y_{n_k}) = \frac{\varepsilon}{2}$.

Since $\varepsilon \leq p^s(y_{m_k},y_{n_k})$ for all k, we have

$$\varepsilon \leq \liminf_{k\to\infty} p^s(y_{m_k},y_{n_k}). \tag{3.7}$$

Now for each positive integer k, by the triangular inequality, we get

$$p^s(y_{m_k},y_{n_k}) \leq p^s(y_{m_k},y_{n_k-1}) + p^s(y_{n_k-1},y_{n_k})$$

On taking limit superior as $k \to \infty$, from (3.5) and (3.6), we have

$$\limsup_{k\to\infty} p^s(y_{m_k},y_{n_k}) \leq \varepsilon. \tag{3.8}$$

Hence, from (3.7) and (3.8), we get $\lim\limits_{k\to\infty} p^s(y_{m_k},y_{n_k})$ exists and $\lim\limits_{k\to\infty} p^s(y_{m_k},y_{n_k}) = \varepsilon$.

Hence, from the definition of p^s and (3.4), we have $\lim\limits_{k\to\infty} p(y_{m_k},y_{n_k}) = \frac{\varepsilon}{2}$.

In similar way, it is easy to see that

(ii) $\lim\limits_{k\to\infty} p(y_{n_k+1},y_{m_k}) = \frac{\varepsilon}{2}$; (iii) $\lim\limits_{k\to\infty} p(y_{n_k},y_{m_k-1}) = \frac{\varepsilon}{2}$ and
(iv) $\lim\limits_{k\to\infty} p(y_{m_k-1},y_{n_k+1}) = \frac{\varepsilon}{2}$.

We now consider

$$p(y_{n_k+1},y_{m_k}) = p(fx_{n_k+1},fx_{m_k})$$

$$\leq \beta(M(x_{n_k+1},x_{m_k}))M(x_{n_k+1},x_{m_k})+LN(x_{n_k+1},x_{m_k}), \tag{3.9}$$

where

$$\begin{aligned} M(x_{n_k+1},x_{m_k}) &= \max\{p(gx_{n_k+1},gx_{m_k}),p(gx_{n_k+1},fx_{n_k+1}),p(gx_{m_k},fx_{m_k}), \\ &\qquad \tfrac{1}{2}[p(gx_{n_k+1},fx_{m_k})+p(gx_{m_k},fx_{n_k+1})]\} \\ &= \max\{p(y_{n_k},y_{m_k-1}),p(y_{n_k},y_{n_k+1}),p(y_{m_k-1},y_{m_k}), \\ &\qquad \tfrac{1}{2}[p(y_{n_k},y_{m_k})+p(y_{m_k-1},y_{n_k+1})]\} \end{aligned}$$

On letting $k\to\infty$ and using (3.3), (i), (ii), (iii) and (iv), we get

$$\lim_{k\to\infty} M(x_{n_k+1},x_{m_k}) = \max\{\tfrac{\varepsilon}{2},0,0,\tfrac{1}{2}[\tfrac{\varepsilon}{2}+\tfrac{\varepsilon}{2}]\} = \tfrac{\varepsilon}{2} \text{ and}$$

$$\begin{aligned} N(x_{n_k+1},x_{m_k}) &= \min\{p^w(gx_{n_k+1},fx_{n_k+1}),p^w(gx_{n_k+1},fx_{m_k}),p^w(gx_{m_k},fx_{n_k+1})\} \\ &= \min\{p^w(y_{n_k},y_{n_k+1}),p^w(y_{n_k},y_{m_k}),p^w(y_{m_k-1},y_{n_k+1})\}. \end{aligned}$$

On letting $k\to\infty$ and using (3.3), (3.4), (i) and (iv), it follows that $N(x_{n_k+1},x_{m_k})=0$.

On letting $k\to\infty$ in (3.9), we obtain

$$\tfrac{\varepsilon}{2} \leq \beta(\tfrac{\varepsilon}{2})\tfrac{\varepsilon}{2} < \tfrac{\varepsilon}{2},$$

a contradiction.

Hence, $\{y_n\}$ is a Cauchy sequence in (X,p^s).

Case (ii): Assume that $y_n = y_{n+1}$ for some n.

If $p(y_{n+1},y_{n+2})>0$. We have

$$\begin{aligned} M(x_{n+1},x_{n+2}) &= \max\{p(gx_{n+1},gx_{n+2}),p(gx_{n+1},fx_{n+1}),p(gx_{n+2},fx_{n+2}), \\ &\qquad \tfrac{1}{2}[p(gx_{n+1},fx_{n+2})+p(gx_{n+2},fx_{n+1})]\} \\ &= \max\{p(y_n,y_{n+1}),p(y_n,y_{n+1}),p(y_{n+1},y_{n+2}), \\ &\qquad \tfrac{1}{2}[p(y_n,y_{n+2})+p(y_{n+1},y_{n+1})]\} \end{aligned}$$

However, $p(y_n,y_{n+1})=p(y_{n+1},y_{n+1})\leq p(y_{n+1},y_{n+2})$, from (P_2) and

$\frac{1}{2}[p(y_{n+1},y_{n+1})+p(y_n,y_{n+2})] \leq \frac{1}{2}[p(y_n,y_{n+1})+p(y_{n+1},y_{n+2})] \leq p(y_{n+1},y_{n+2})$,

$M(x_{n+1},x_{n+2})=p(y_{n+1},y_{n+2})$ and $N(x_{n+1},x_{n+2})=0$.

From the inequality (2.1), we have

$$\begin{aligned} p(y_{n+1},y_{n+2}) &= p(fx_{n+1},fx_{n+2}) \\ &\leq \beta(M(x_{n+1},x_{n+2}))M(x_{n+1},x_{n+2})+LN(x_{n+1},x_{n+2}) \\ &= \beta(p(y_{n+1},y_{n+2}))p(y_{n+1},y_{n+2}) < p(y_{n+1},y_{n+2}), \end{aligned}$$

which is a contradiction.

Hence, $y_{n+1} = y_{n+2}$.

Continuing in this way, we can conclude that $y_n = y_{n+k}$ for all $k \geq 0$.

Thus, $\{y_n\}$ is a Cauchy sequence in (X, p^s).

From the Lemma 2.1, it follows that $\{y_n\}$ is a Cauchy sequence in (X, p).

Therefore $\lim_{n,m\to\infty} p(y_n, y_m) = 0. \qquad (3.10)$

Suppose $g(X)$ is complete.

Since $y_n = fx_n = gx_{n+1}$, it follows that $\{y_n\} \subseteq g(X)$ is a Cauchy sequence in the complete metric space $(g(X), p^s)$, it follows that $\{y_n\}$ converges in $(g(X), p^s)$.

Thus, $\lim_{n\to\infty} p^s(y_n, u) = 0$ for some $u \in g(X)$. i.e., $\lim_{n\to\infty} y_n = u = gt \in g(X)$ for some $t \in X$.

Since $\{y_n\}$ is a Cauchy sequence in X and $y_n \to u$, it follows that $y_{n+1} \to u$ as $n \to \infty$.

From the Lemma 2.2, we have

$p(u,u) = \lim_{n\to\infty} p(y_{n+1}, u) = \lim_{n\to\infty} p(y_n, u) = \lim_{n,m\to\infty} p(y_n, y_m)$.

From (3.10), we have

$$p(u,u) = \lim_{n\to\infty} p(y_{n+1}, u) = \lim_{n\to\infty} p(y_n, u) = \lim_{n,m\to\infty} p(y_n, y_m) = 0.$$

We now show that $ft = u$.

Suppose $p(ft, u) > 0$.

We now consider

$$p(ft, y_{n+1}) = p(ft, fx_{n+1}) \leq \beta(M(t, x_{n+1}))M(t, x_{n+1}) + LN(t, x_{n+1}) \qquad (3.11)$$

where

$$\begin{aligned} M(t, x_{n+1}) &= \max\{p(gt, gx_{n+1}), p(gt, ft), p(gx_{n+1}, fx_{n+1}), \\ &\qquad \tfrac{1}{2}[p(gt, fx_{n+1}) + p(gx_{n+1}, ft)]\} \\ &= \max\{p(u, y_n), p(u, ft), p(y_n, y_{n+1}), \tfrac{1}{2}[p(u, y_{n+1}) + p(y_n, ft)]\}. \end{aligned}$$

On letting $n \to \infty$, we get

$\lim_{n\to\infty} M(t, x_{n+1}) = \max\{0, p(u, ft), 0, \frac{1}{2}(p(u, ft))\} = p(u, ft)$ and

$$\begin{aligned} N(t, x_{n+1}) &= \min\{p^w(gt, ft), p^w(gt, fx_{n+1}), p^w(gx_{n+1}, ft)\} \\ &= \min\{p^w(u, ft), p^w(u, y_{n+1}), p^w(y_n, ft)\}. \end{aligned}$$

On letting $n \to \infty$, we get $\lim_{n\to\infty} N(t, x_{n+1}) = \min\{p(u, ft), 0, p(u, ft)\} = 0$.

On letting $n \to \infty$ in (3.11), we obtain

$p(ft, u) \leq \beta(p(ft, u))p(ft, u) < p(ft, u)$, a contradiction.

Hence, $ft = gt = u$.

Since the pair (f,g) is weakly compatible and $ft = gt = u$, we have $fu = gu$

We now prove that $fu = u$.

On the contrary, suppose that $p(fu,u) > 0$. From the inequality (2.1), we have

$$p(fu,y_{n+1}) = p(fu,fx_{n+1}) \leq \beta(M(u,x_{n+1}))M(u,x_{n+1}) + LN(u,x_{n+1}) \tag{3.12}$$

where

$$\begin{aligned} M(u,x_{n+1}) &= \max\{p(gu,gx_{n+1}),p(gu,fu),p(gx_{n+1},fx_{n+1}), \\ &\qquad \tfrac{1}{2}[p(gu,fx_{n+1})+p(gx_{n+1},fu)]\} \\ &= \max\{p(fu,y_n),p(fu,fu),p(y_n,y_{n+1}),\tfrac{1}{2}[p(fu,y_{n+1})+p(y_n,fu)]\}. \end{aligned}$$

On letting $n \to \infty$, we get

$\lim\limits_{n\to\infty} M(u,x_{n+1}) = \max\{p(fu,u),p(fu,fu),0,\frac{1}{2}[p(u,ft)+p(u,fu)]\} = p(fu,u)$ and

$$\begin{aligned} N(u,x_{n+1}) &= \min\{p^w(gu,fu),p^w(gu,fx_{n+1}),p^w(gx_{n+1},fu)\} \\ &= \min\{p^w(fu,fu),p^w(fu,y_{n+1}),p^w(y_n,fu)\} \end{aligned}$$

On letting $n \to \infty$, we get $\lim\limits_{n\to\infty} N(u,x_{n+1}) = \min\{0,p(u,fu),p(u,fu)\} = 0$.

On letting $n \to \infty$ in (3.12), we obtain

$p(fu,u) \leq \beta(p(fu,u))p(fu,u) < p(fu,u)$,

a contradiction. Hence, $fu = gu = u$. Therefore u is a common fixed point of f and g.

Uniqueness of a common fixed point follows from the inequality (2.1).

Proposition 3.2. *Let (X,p) be a partial metric space, and let A,B,S and T be selfmaps of X. Assume that there exist $\beta \in \mathfrak{F}$ and $L \geq 0$ such that*

$$p(Ax,By) \leq \beta(M(x,y))M(x,y) + LN(x,y) \tag{3.13}$$

holds for all $x,y \in X$, where $M(x,y) = \max\{p(Sx,Ty),p(Sx,Ax),p(Ty,By),\frac{1}{2}[p(Sx,By)+p(Ty,Ax)]\}$ and $N = \min\{p^w(Sx,Ax),p^w(Ty,By),p^w(Sx,By),p^w(Ty,Ax)\}$. Then the following hold.

(i) If $A(X) \subseteq T(X)$ and the pair (B,T) is weakly compatible, and if z is a common fixed point of A and S then z is a common fixed point of A,B,S and T and it is unique.

(ii) If $B(X) \subseteq S(X)$ and the pair (A,S) is weakly compatible, and if z is a common fixed point of B and T then z is a common fixed point of A,B,S and T and it is unique.

Proof. First, we assume that (i) holds. Let z be a common fixed point of A and S.

Then $Az = Sz = z$.

Since $A(X) \subseteq T(X)$, there exists $u \in X$ such that $Tu = z$.

$Az = Sz = Tu = z$.

We now prove that $Tu = Bu$. Suppose that $Tu \neq Bu$.

From the inequality (3.13), we have

$$\begin{aligned}
p(Tu,Bu) &= p(Az,Bu) \\
&\leq \beta(M(z,u))M(z,u) \\
&= \beta(\max\{p(Sz,Tu), p(Sz,Az), p(Tu,Bu), \tfrac{1}{2}[p(Sz,Bu)+p(Tu,Az)]\}) \\
&\quad \max\{p(Sz,Tu), p(Sz,Az), p(Tu,Bu), \tfrac{1}{2}[p(Sz,Bu)+p(Tu,Az)]\} \\
&\quad + L\min\{p^w(Sz,Az), p^w(Tu,Bu), p^w(Sz,Bu), p^w(Tu,Az)\} \\
&= \beta(p(Tu,Bu))p(Tu,Bu) < p(Tu,Bu),
\end{aligned}$$

which is a contradiction.

Hence, $Bu = Tu = z$.

Since the pair (B,T) is weakly compatible, it follows that $BTu = TBu$ i.e, $Bz = Tz$.

Suppose $Bz \neq z$.

From the inequality (3.13), we have

$$\begin{aligned}
p(z,Bz) &= p(Az,Bz) \\
&\leq \beta(M(z,z))M(z,z) \\
&= \beta(\max\{p(Sz,Tz), p(Sz,Az), p(Tz,Bz), \tfrac{1}{2}[p(Sz,Bz)+p(Tz,Az)]\}) \\
&\quad \max\{p(Sz,Tz), p(Sz,Az), p(Tz,Bz), \tfrac{1}{2}[p(Sz,Bz)+p(Tz,Az)]\} \\
&\quad + L\min\{p^w(Sz,Az), p^w(Tz,Bz), p^w(Sz,Bz), p^w(Tz,Az)\} \\
&= \beta(p(z,Bz))p(z,Bz) < p(z,Bz),
\end{aligned}$$

which is a contradiction.

Thus, $Bz = Tz = z$. Hence, $Az = Bz = Sz = Tz = z$. Hence, z is a common fixed point of A, B, S and T. Let z' be another common fixed point of A, B, S and T.

From the inequality (3.13), we have

$$\begin{aligned}
p(z,z') &= p(Az,Bz') \\
&\leq \beta(M(z,z'))M(z,z') \\
&= \beta(\max\{p(Sz,Tz'), p(Sz,Az), p(Tz',Bz'), \tfrac{1}{2}[p(Sz,Bz')+p(Tz',Az)]\})
\end{aligned}$$

$$\max\{p(Sz,Tz'),p(Sz,Az),p(Tz',Bz'),\tfrac{1}{2}[p(Sz,Bz')+p(Tz',Az)]\}$$
$$+L\min\{p^w(Sz,Az),p^w(Tz',Bz'),p^w(Sz,Bz'),p^w(Tz',Az)\}$$
$$=\beta(p(z,z'))p(z,z')<p(z,z'),$$

which is a contradiction.

Hence, $z=z'$. Thus z is a unique common fixed point of A,B,S and T.

The proof of (ii) is similar to (i) and hence is omitted.

Theorem 3.3. *Let A,B,S and T be selfmaps of a partial metric space (X,p) and satisfy $A(X)\subseteq T(X),B(X)\subseteq S(X)$ and the inequality* (3.13). *If the pairs (A,S) and (B,T) are weakly compatible and $S(X)$ or $T(X)$ is a complete subspace of X then A,B,S and T have a unique common fixed point in X.*

Proof. Let $x_0\in X$ arbitrary point in X.

Since $A(X)\subseteq T(X)$ and $B(X)\subseteq S(X)$, there exist sequences of $\{x_n\}$ and $\{y_n\}$ in X, such that

$y_{2n}=Ax_{2n}=Tx_{2n+1}$ and

$y_{2n+1}=Bx_{2n+1}=Sx_{2n+2}$ for $n=0,1,2,\dots$.

Assume that $y_n=y_{n+1}$ for some n.

Case (i): n even. We write $n=2m,m=1,2,3,\dots$.

Now we consider

$$\begin{aligned}p(y_{n+1},y_{n+2})&=p(y_{2m+1},y_{2m+2})\\&=p(Ax_{2m+2},Bx_{2m+1})\\&\leq\beta(M(x_{2m+2},x_{2m+1}))M(x_{2m+2},x_{2m+1})+LN(x_{2m+2},x_{2m+1}),\end{aligned}\tag{3.14}$$

where

$$\begin{aligned}M(x_{2m+2},x_{2m+1})&=\max\{p(Sx_{2m+2},Tx_{2m+1}),p(Sx_{2m+2},Ax_{2m+2}),p(Tx_{2m+1},Bx_{2m+1}),\\&\qquad\tfrac{1}{2}[p(Sx_{2m+2},Bx_{2m+1})+p(Tx_{2m+1},Ax_{2m+2})]\}\\&=\max\{p(y_{2m+1},y_{2m}),p(y_{2m+1},y_{2m+2}),p(y_{2m},y_{2m+1}),\\&\qquad\tfrac{1}{2}[p(y_{2m+1},y_{2m+1})+p(y_{2m},y_{2m+2})]\}\\&=\max\{p(y_{2m+1},y_{2m}),p(y_{2m+1},y_{2m+2})\}=p(y_{2m+1},y_{2m+2})\text{ and}\end{aligned}$$

$$\begin{aligned}N(x_{2m+2},x_{2m+1})&=\min\{p^w(Sx_{2m+2},Ax_{2m+2}),p^w(Tx_{2m+1},Bx_{2m+1}),p^w(Sx_{2m+2},Bx_{2m+1}),\\&\qquad p^w(Tx_{2m+1},Ax_{2m+2})\}\\&=\min\{p^w(y_{2m+1},y_{2m+2}),p^w(y_{2m},y_{2m+1}),p^w(y_{2m},y_{2m+1}),\end{aligned}$$

$$p^w(y_{2m},y_{2m+2})\} = 0 \text{ as } y_{2m} = y_{2m+1} \text{ and } p^w(y_{2m},y_{2m}) = 0.$$

From the inequality (3.14), we have

$$\begin{aligned} p(y_{2m+1},y_{2m+2}) &\leq \beta(p(y_{2m+1},y_{2m+2}))p(y_{2m+1},y_{2m+2}) \\ &< p(y_{2m+1},y_{2m+2}), \end{aligned}$$

a contradiction if $p(y_{2m+1},y_{2m+2}) \neq 0$.

Therefore, $p(y_{2m+1},y_{2m+2}) = 0$ which implies that $y_{2m+2} = y_{2m+1} = y_{2m}$.

In general, we have $y_{2m+k} = y_{2m}$ for $k = 0,1,2,\ldots$.

Case (ii): n odd. We write $n = 2m+1$ for some $m = 1,2,3\ldots$.

We consider

$$\begin{aligned} p(y_{n+1},y_{n+2}) &= p(y_{2m+2},y_{2m+3}) \\ &= p(Ax_{2m+2},Bx_{2m+3}) \\ &\leq \beta(M(x_{2m+2},x_{2m+3}))M(x_{2m+2},x_{2m+3}) + LN(x_{2m+2},x_{2m+3}), \end{aligned} \tag{3.15}$$

where

$$\begin{aligned} M(x_{2m+2},x_{2m+3}) &= \max\{p(Sx_{2m+2},Tx_{2m+3}), p(Sx_{2m+2},Ax_{2m+2}), p(Tx_{2m+3},Bx_{2m+3}), \\ &\qquad \tfrac{1}{2}[p(Sx_{2m+2},Bx_{2m+3}) + p(Tx_{2m+3},Ax_{2m+2})]\} \\ &= \max\{p(y_{2m+1},y_{2m+2}), p(y_{2m+1},y_{2m+2}), p(y_{2m+2},y_{2m+3}), \\ &\qquad \tfrac{1}{2}[p(y_{2m+1},y_{2m+3}) + p(y_{2m+1},y_{2m+2})]\} \\ &= \max\{p(y_{2m+1},y_{2m+2}), p(y_{2m+2},y_{2m+3})\} = p(y_{2m+2},y_{2m+3}) \text{ and} \\ N(x_{2m+2},x_{2m+3}) &= \min\{p^w(Sx_{2m+2},Ax_{2m+2}), p^w(Tx_{2m+3},Bx_{2m+3}), p^w(Sx_{2m+2},Bx_{2m+3}), \\ &\qquad p^w(Tx_{2m+3},Ax_{2m+2})\} \\ &= \min\{p^w(y_{2m+1},y_{2m+2}), p^w(y_{2m+2},y_{2m+3}), p^w(y_{2m+1},y_{2m+3}), \\ &\qquad p^w(y_{2m+2},y_{2m+2})\} = 0 \text{ as } p^w(y_{2m+2},y_{2m+2}) = 0. \end{aligned}$$

From the inequality (3.15), we have

$$p(y_{2m+2},y_{2m+3}) \leq \beta(p(y_{2m+2},y_{2m+3}))p(y_{2m+2},y_{2m+3}) < p(y_{2m+2},y_{2m+3}),$$

a contradiction if $p(y_{2m+2},y_{2m+3}) > 0$.

Therefore, $p(y_{2m+2},y_{2m+3}) = 0$ which implies that $y_{2m+3} = y_{2m+2} = y_{2m+1}$.

In general, we have $y_{2m+k} = y_{2m+1}$ for $k = 1,2,3,\ldots$.

From Case (i) and Case (ii), we have $y_{n+k} = y_n$ for $k = 0,1,2,\ldots$. Hence, $\{y_{n+k}\}$ is a constant sequence and hence $\{y_n\}$ is Cauchy.

Now we assume that $y_n \neq y_{n+1}$ for all $n = 1,2,3,\ldots$.

If n is odd, then we write $n = 2m+1$ for some $m = 1,2,3,\ldots$.

We now consider

$$\begin{aligned} p(y_n, y_{n+1}) &= p(y_{2m+1}, y_{2m+2}) \\ &= p(Ax_{2m+2}, Bx_{2m+1}) \\ &\leq \beta(M(x_{2m+2}, x_{2m+1}))M(x_{2m+2}, x_{2m+1}) + LN(x_{2m+2}, x_{2m+1}) \end{aligned} \tag{3.16}$$

where

$$\begin{aligned} M(x_{2m+2}, x_{2m+1}) &= \max\{p(Sx_{2m+2}, Tx_{2m+1}), p(Sx_{2m+2}, Ax_{2m+2}), p(Tx_{2m+1}, Bx_{2m+1}), \\ &\qquad \tfrac{1}{2}[p(Sx_{2m+2}, Bx_{2m+1}) + p(Tx_{2m+1}, Ax_{2m+2})]\} \\ &= \max\{p(y_{2m+1}, y_{2m}), p(y_{2m+1}, y_{2m+2}), p(y_{2m}, y_{2m+1}), \\ &\qquad \tfrac{1}{2}[p(y_{2m+1}, y_{2m+1}) + p(y_{2m}, y_{2m+2})]\} \\ &= \max\{p(y_{2m+1}, y_{2m}), p(y_{2m+1}, y_{2m+2})\} \text{ and} \end{aligned}$$

$$\begin{aligned} N(x_{2m+2}, x_{2m+1}) &= \min\{p^w(Sx_{2m+2}, Ax_{2m+2}), p^w(Tx_{2m+1}, Bx_{2m+1}), p^w(Sx_{2m+2}, Bx_{2m+1}), \\ &\qquad p^w(Tx_{2m+1}, Ax_{2m+2})\} \\ &= \min\{p^w(y_{2m+1}, y_{2m+2}), p^w(y_{2m}, y_{2m+1}), p^w(y_{2m+1}, y_{2m+1}), \\ &\qquad p^w(y_{2m}, y_{2m+2})\} = 0. \end{aligned}$$

If $p(y_{2m+1}, y_{2m+2})$ is maximum then from (3.16), we have

$$\begin{aligned} p(y_{2m+1}, y_{2m+2}) &\leq \beta(p(y_{2m+1}, y_{2m+2}))p(y_{2m+1}, y_{2m+2}) \\ &< p(y_{2m+1}, y_{2m+2}), \end{aligned}$$

a contradiction.

Hence $p(y_{2m}, y_{2m+1})$ is maximum.

Therefore

$$p(y_{2m+1}, y_{2m+2}) \leq \beta(p(y_{2m}, y_{2m+1}))p(y_{2m}, y_{2m+1}) < p(y_{2m}, y_{2m+1}) \tag{3.17}$$

Similarly we can show that $p(y_{n-1}, y_n) \leq p(y_{n-2}, y_{n-1})$ for $n \geq 1$.

Thus, we have $p(y_n, y_{n+1}) \leq p(y_{n-1}, y_n)$ for all $n = 1,2,3,\ldots$.

Therefore $\{p(y_n, y_{n+1})\}$ is a decreasing sequence of nonnegative real numbers and hence converges to real $r \geq 0$(say).

Suppose $r > 0$.

On letting $n \to \infty$ in (3.17), we get

$r \leq \beta(r)r < r,$

a contradiction.

Hence, $r = 0$. Thus, $p(y_n, y_{n+1}) \to 0$ as $n \to \infty$.

On the similar lines, if n is even, it follows that $p(y_n, y_{n+1}) \to 0$ as $n \to \infty$.

Therefore $\lim_{n\to\infty} p(y_n, y_{n+1}) = 0. \quad (3.18)$

Therefore from $(P2)$, we get that $\lim_{n\to\infty} p(y_n, y_n) = 0$ and $\lim_{n\to\infty} p(y_{n+1}, y_{n+1}) = 0. \quad (3.19)$

By the definition of p^s, using (3.18) and (3.19), we get that $\lim_{n\to\infty} p^s(y_n, y_{n+1}) = 0. \quad (3.20)$

Now, we prove that $\{y_{2n}\}$ is a Cauchy sequence in (X, p^s).

On the contrary, suppose that $\{y_{2n}\}$ is not Cauchy. Then there exist an $\varepsilon > 0$ and monotone sequences of natural numbers $\{2m_k\}$ and $\{2n_k\}$such that $2n_k > 2m_k > k,$ with

$$p^s(y_{2m_k}, y_{2n_k}) \geq \varepsilon \text{ and } p^s(y_{2m_k}, y_{2n_k-2}) < \varepsilon \quad (3.21)$$

Now we prove that (i) $\lim_{k\to\infty} p(y_{2m_k}, y_{2n_k}) = \frac{\varepsilon}{2}$.

Since $\varepsilon \leq p^s(y_{2m_k}, y_{2n_k})$ for all k, we have

$$\varepsilon \leq \liminf_{k\to\infty} p^s(y_{2m_k}, y_{2n_k}). \quad (3.22)$$

Now for each positive integer k, by the triangular inequality, we get

$$p^s(y_{2m_k}, y_{2n_k}) \leq p^s(y_{2m_k}, y_{2n_k-2}) + p^s(y_{2n_k-2}, y_{2n_k-1}) + p^s(y_{2n_k-1}, y_{2n_k})$$

On taking limit superior as $k \to \infty$ and using (3.19) and (3.20), we have

$$\limsup_{k\to\infty} p^s(y_{2m_k}, y_{2n_k}) \leq \varepsilon. \quad (3.23)$$

Hence from (3.22) and (3.23), we get $\lim_{k\to\infty} p^s(y_{2m_k}, y_{2n_k})$ exists and $\lim_{k\to\infty} p^s(y_{2m_k}, y_{2n_k}) = \varepsilon$.

Hence, from the definition of p^s and (3.19), we have $\lim_{k\to\infty} p(y_{2m_k}, y_{2n_k}) = \frac{\varepsilon}{2}$.

In similar way, it is easy to see that

(ii) $\lim_{k\to\infty} p(y_{2n_k+1}, y_{2m_k}) = \frac{\varepsilon}{2}$; (iii) $\lim_{k\to\infty} p(y_{2n_k}, y_{2m_k-1}) = \frac{\varepsilon}{2}$ and

(iv) $\lim_{k\to\infty} p(y_{2m_k-1}, y_{2n_k+1}) = \frac{\varepsilon}{2}$.

From the inequality (3.13), we have

$$\begin{aligned} p(y_{2m_k}, y_{2n_k+1}) &= p(Ax_{2m_k}, Bx_{2n_k+1}) \\ &\leq \beta(M(x_{2m_k}, x_{2n_k+1}))M(x_{2m_k}, x_{2n_k+1}) + LN(x_{2m_k}, x_{2n_k+1}), \end{aligned} \quad (3.24)$$

where

$$\begin{aligned} M(x_{2m_k}, x_{2n_k+1}) = \max\{&p(Sx_{2m_k}, Tx_{2n_k+1}), p(Sx_{2m_k}, Ax_{2m_k}), p(Tx_{2n_k+1}, Bx_{2n_k+1}), \\ &\tfrac{1}{2}[p(Sx_{2m_k}, Bx_{2n_k+1}) + p(Tx_{2n_k+1}, Ax_{2m_k})]\} \end{aligned}$$

$$= \max\{p(y_{2m_k-1}, y_{2n_k}), p(y_{2m_k-1}, y_{2m_k}), p(y_{2n_k}, y_{2n_k+1}), \frac{1}{2}[p(y_{2m_k-1}, y_{2n_k+1}) + p(y_{2n_k}, y_{2m_k})]\} \text{ and}$$

$$N(x_{2m_k}, x_{2n_k+1}) = \min\{p^w(Sx_{2m_k}, Ax_{2m_k}), p^w(Tx_{2n_k+1}, Bx_{2n_k+1}), p^w(Sx_{2m_k}, Bx_{2n_k+1}), p^w(Tx_{2n_k+1}, Ax_{2m_k})\}$$

$$= \min\{p^w(y_{2m_k-1}, y_{2m_k}), p^w(y_{2n_k}, y_{2n_k+1}), p^w(y_{2m_k-1}, y_{2n_k+1}), p^w(y_{2n_k}, y_{2m_k})\}$$

Now, by the definition of p^w and using (3.19), (i), (ii), (iii), and (iv), we have

$\lim_{k\to\infty} M(x_{2m_k}, x_{2n_k+1}) = \max\{\frac{\varepsilon}{2}, 0, 0, \frac{1}{2}[\frac{\varepsilon}{2} + \frac{\varepsilon}{2}]\} = \frac{\varepsilon}{2}$

and $\lim_{k\to\infty} N(x_{2m_k}, x_{2n_k+1}) = \min\{0, 0, \frac{\varepsilon}{2}, \frac{\varepsilon}{2}\} = 0.$

Hence, on letting $k \to \infty$ in (3.24), we get $\frac{\varepsilon}{2} \leq \beta(\frac{\varepsilon}{2})\frac{\varepsilon}{2} < \frac{\varepsilon}{2}$, which is a contradiction.

Therefore $\{y_{2n}\}$ is Cauchy. Thus by Proposition 2.1, we have $\{y_n\}$ is a Cauchy sequence in (X, p^s).

Therefore $\lim_{n,m\to\infty} p^s(y_n, y_m) = 0.$

Now, from Lemma 2.1, it follows that $\{y_n\}$ is a Cauchy sequence in (X, p).

Suppose $T(X)$ is complete.

Since $y_{2n} = Ax_{2n} = Tx_{2n+1}$, it follows that $\{y_{2n}\} \subseteq T(X)$ is a Cauchy sequence in the complete metric space $(T(X), p^s)$, it follows that $\{y_{2n}\}$ converges in $(T(X), p^s)$, and $\{y_{2n}\}$ converges to u(say) in $T(X)$.

Thus, $\lim_{n\to\infty} p^s(y_{2n}, u) = 0$ for some $u \in T(X)$. i.e., $y_{2n} \to u = Tt \in T(X)$ for some $t \in X$.

Since $\{y_n\}$ is Cauchy in X and $y_{2n} \to u$, it follows that $y_{2n+1} \to u$ as $n \to \infty$.

From Lemma 2.2, we get

$$p(u, u) = \lim_{n\to\infty} p(y_{2n+1}, u) = \lim_{n\to\infty} p(y_{2n}, u) = \lim_{n,m\to\infty} p(y_{2n}, y_{2m}) = 0. \tag{3.25}$$

Now, we consider

$$p(Ax_{2n}, Bt) \leq \beta(M(x_{2n}, t))M(x_{2n}, t) + LN(x_{2n}, t), \tag{3.26}$$

where

$M(x_{2n}, t) = \max\{p(Sx_{2n}, Tt), p(Sx_{2n}, Ax_{2n}), p(Tt, Bt), \frac{1}{2}[p(Sx_{2n}, Bt) + p(Tt, Ax_{2n})]\}$ and

$$N(x_{2n}, t) = \min\{p^w(Sx_{2n}, Ax_{2n}), p^w(Tt, Bt), p^w(Sx_{2n}, Bt), p^w(Tt, Ax_{2n})\}. \tag{3.27}$$

On letting $n \to \infty$ in (3.27), using (3.18), (3.25) and $\lim_{n\to\infty} Sx_{2n} = \lim_{n\to\infty} Ax_{2n} = u$, we get

$\lim_{n\to\infty} M(x_{2n}, t) = \max\{0, 0, p(u, Bt), \frac{1}{2}[p(u, Bt) + 0]\} = p(u, Bt)$ and

$\lim_{n\to\infty} N(x_{2n}, t) = \min\{p^w(u, u), p^w(u, Bt), p^w(u, Bt), p^w(u, u)\} = 0$ as $p^w(u, u) = 0.$

Letting $n \to \infty$ in (3.26), we obtain

$p(u,Bt) \leq \beta(p(u,Bt))p(u,Bt) < p(u,Bt)$,

a contradiction.

Hence $Bt = Tt = u$.

Since the pair (B,T) is weakly compatible, it follows that $Bu = BTt = TBt = Tu$.

Suppose $p(u,Bu) \neq 0$. By the inequality (3.13), we have

$$p(Ax_{2n},Bu) \leq \beta(M(x_{2n},u))M(x_{2n},u) + LN(x_{2n},u), \tag{3.28}$$

where

$M(x_{2n},u) = \max\{p(Sx_{2n},Tu), p(Sx_{2n},Ax_{2n}), p(Tu,Bu), \frac{1}{2}[p(Sx_{2n},Bu) + p(Tu,Ax_{2n})]\}$ and

$$N(x_{2n},u) = \min\{p^w(Sx_{2n},Ax_{2n}), p^w(Tu,Bu), p^w(Sx_{2n},Bu), p^w(Tu,Ax_{2n})\}. \tag{3.29}$$

On letting $n \to \infty$ in (3.29), using (3.18), (3.25) and $\lim_{n\to\infty} Sx_{2n} = \lim_{n\to\infty} Ax_{2n} = u$, we get

$\lim_{n\to\infty} M(x_{2n},u) = \max\{p(u,Bu), 0, p(Bu,Bu), \frac{1}{2}[p(u,Bu) + p(Bu,u)]\} = p(u,Bu)$ and

$\lim_{n\to\infty} N(x_{2n},u) = \min\{p^w(u,u), p^w(Bu,Bu), p^w(u,Bu), p^w(Bu,u)\} = 0$ as $p^w(u,u) = 0$.

Letting $n \to \infty$ in (3.28), we obtain

$p(u,Bu) \leq \beta(p(u,Bu))p(u,Bu) < p(u,Bu)$,

a contradiction.

Therefore $p(u,Bu) = 0$ implies that $Bu = Tu = u$. Thus, u is a common fixed point of B and T.

By Proposition 3.2, we get that u is a unique common fixed point of A,B,S and T.

In a similar way, under the assumption that $S(X)$ is complete, we obtain the existence of common fixed point of A,B,S and T.

4. Corollaries and examples

In this section, we draw some corollaries from the main results of Section 3 and provide examples in support of our results.

By choosing $f = T$ and g, the identity map of X in Theorem 3.1, we have the following.

Corollary 4.1(Theorem 3, [19]). *Let (X,p) be a complete partial metric space and let $T : X \to X$ be a selfmapping. Suppose that there exist $\beta \in \mathfrak{F}$ and $L \geq 0$ such that*

$p(Tx,Ty) \leq \beta(M(x,y))M(x,y) + LN(x,y)$

holds for all $x,y \in X$, where $M(x,y) = \max\{p(x,y), p(x,Tx), p(y,Ty), \frac{1}{2}[p(x,Ty) + p(y,Tx)]\}$

and $N = \min\{p^w(x,Tx), p^w(y,Ty), p^w(x,Ty), p^w(y,Tx)\}$.

Then T *has a unique fixed point* $u \in X$. *Moreover,* $p(u,u) = 0$.

If $L = 0$ in (2.1), then we have the following corollary due to Theorem 3.1.

Corollary 4.2. *Let* (X,p) *be a partial metric space and let the pair* (f,g) *be generalized Geraghty type contraction maps. If* $f(X) \subseteq g(X)$, *the pair* (f,g) *is weakly compatible and* $g(X)$ *is a complete subspace of* X *then* f *and* g *have a unique common fixed point in* X.

The following is an example in support of Theorem 3.1, in which we show the importance of L.

Example 4.1. Let $X = [0,1]$. We define $p(x,y) = \begin{cases} 0 & \text{if } x = y \\ \max\{x,y\} & \text{if } x \neq y, \end{cases}$ for all $x,y \in X$.

Then (X,p) is a partial metric space. We define selfmaps $f,\ g$ on X by

$f(x) = \begin{cases} \frac{x}{20} & \text{if } 0 \leq x \leq \frac{1}{2} \\ \frac{1}{4} & \text{if } \frac{1}{2} < x \leq 1, \end{cases}$ $g(x) = \begin{cases} \frac{x}{2} & \text{if } 0 \leq x \leq \frac{1}{2} \\ \frac{1}{40} & \text{if } \frac{1}{2} < x \leq 1, \end{cases}$ and $\beta(t) = \begin{cases} 0 & \text{if } t = 0 \\ \frac{e^{\frac{-t}{10}}}{1+t} & \text{if } t > 0. \end{cases}$

Then clearly $f(X) \subseteq g(X)$. Without loss of generality, we assume that $x \geq y$.

We verify that the pair (f,g) is a Geraghty-Berinde type contraction maps. i.e., we show that $f,\ g$ satisfy the inequality (2.1).

Case (i): $x,y \in [0,\frac{1}{2}]$.

$p(fx,fy) = \frac{x}{20}; p(gx,gy) = \frac{x}{2}; p(gx,fx) = \frac{x}{2}$;

$p(gy,fy) = \frac{y}{2}; p(gx,fy) = \frac{x}{2}; p(gy,fx) = \max\{\frac{y}{2},\frac{x}{20}\} \leq \frac{x}{2}$.

Here $\frac{1}{2}[p(gx,fy)+p(gy,fx)] \leq \frac{x}{2}$. Therefore $M(x,y) = \frac{x}{2}$.

$p^w(gx,fx) = \frac{x}{2}; p^w(gx,fy) = \frac{x}{2}; p^w(gy,fx) = \max\{\frac{y}{2},\frac{x}{20}\}$.

We consider

$$\begin{aligned} p(fx,fy) = \frac{x}{20} &\leq \frac{e^{-\frac{x}{20}}}{1+\frac{x}{2}}\frac{x}{2} + 3\min\{\tfrac{x}{2},\tfrac{x}{2},\max\{\tfrac{y}{2},\tfrac{x}{20}\}\} \\ &= \beta(M(x,y))M(x,y) + LN(x,y) \text{ with } L = 3. \end{aligned}$$

Case (ii): $x,y \in (\frac{1}{2},1]$.

$p(fx,fy) = \frac{1}{4}; p(gx,gy) = \frac{1}{40}; p(gx,fx) = \frac{1}{4}; p(gy,fy) = \frac{1}{4}; \frac{1}{2}[p(gx,fy)+p(gy,fx)] = \frac{1}{4}$.

$p^w(gx,fx) = \frac{1}{4}; p^w(gx,fy) = \frac{1}{4}; p^w(gy,fx) = \frac{1}{4}$.

Therefore $M(x,y) = \frac{1}{4}$ and $N(x,y) = \frac{1}{4}$.

We consider

$$p(fx,fy) = \frac{1}{4} \leq \frac{e^{-\frac{1}{40}}}{1+\frac{1}{4}}\frac{1}{4} + 3(\tfrac{1}{4})$$

$$= \beta(\tfrac{1}{4})\tfrac{1}{4} + 3(\tfrac{1}{4})$$
$$= \beta(M(x,y))M(x,y) + 3\min\{p^w(gx,fx), p^w(gx,fy), p^w(gy,fx)\}$$
$$= \beta(M(x,y))M(x,y) + LN(x,y) \text{ with } L = 3.$$

Case (iii): $x \in (\frac{1}{2}, 1], y \in [0, \frac{1}{2}]$.

$p(fx,fy) = \frac{1}{4}; p(gx,gy) = \max\{\frac{1}{40}, \frac{y}{2}\} \leq \frac{1}{4}; p(gx,fx) = \frac{1}{4}, p(gy,fy) = \frac{y}{2}$,

$\frac{1}{2}[p(gx,fy) + p(gy,fx)] = \frac{1}{2}[\frac{1}{40} + \frac{1}{4}] = \frac{11}{80}$. $p^w(gx,fx) = \frac{1}{4}; p^w(gx,fy) = \frac{1}{40}; p^w(gy,fx) = \frac{1}{4}$.

Therefore $M(x,y) = \frac{1}{4}$ and $N(x,y) = \frac{1}{40}$.

We consider

$$p(fx,fy) = \tfrac{1}{4} \leq \tfrac{e^{-\frac{1}{40}}}{1+\frac{1}{4}}\tfrac{1}{4} + 3\min\{\tfrac{1}{4}, \tfrac{1}{40}, \tfrac{1}{4}\}$$
$$= \beta(M(x,y))M(x,y) + LN(x,y) \text{ with } L = 3.$$

From all the above cases, we have f and g satisfy the inequality (2.1) with $L = 3$.

Therefore f and g satisfy all the hypotheses of Theorem 3.1 and 0 is the unique common fixed point of f and g.

If $L = 0$ in the inequality (2.1) then the inequality (2.1) fails to hold, which shows the importance of L in the inequality (2.1).

For, by choosing $x = 1,\ y = \frac{2}{3}$. We have

$p(fx,fy) = \frac{1}{4} \nleq \beta(\frac{1}{4})\frac{1}{4} = \beta(M(1,\frac{2}{3}))M(1,\frac{2}{3})$ for any $\beta \in \mathfrak{F}$.

The following is an example in support of Corollary 4.2.

Example 4.2 Let $X = [0,1]$. We define $p(x,y) = \max\{x,y\}$ for all $x,y \in X$. Then (X,p) is a partial metric space. We define selfmaps $f,\ g$ on X by

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ \frac{x}{2} & \text{if } 0 < x < \frac{1}{2} \\ \frac{1}{3} & \text{if } \frac{1}{2} \leq x \leq 1, \end{cases} \quad g(x) = \frac{x(5-x)}{4} \text{ if } 0 \leq x \leq 1, \text{ and } \beta(t) = \begin{cases} 0 & \text{if } t = 0 \\ \frac{e^{\frac{-t}{2}}}{1+t} & \text{if } t > 0. \end{cases}$$

Then clearly $f(X) \subseteq g(X)$. Without loss of generality, we assume that $x \geq y$.

We verify that the pair (f,g) is a generalized Geraghty type contraction maps. i.e., we show that $f,\ g$ satisfy the inequality (2.1) with $L = 0$.

Case (i): $x = y = 0$.

In this case, trivially holds the inequality (2.1) with $L = 0$.

Case (ii): $x,y \in (0, \frac{1}{2})$.

$p(fx,fy) = \frac{x}{2}; p(gx,gy) = \frac{x(5-x)}{4}; p(gx,fx) = \frac{x(5-x)}{4}$;

$p(gy,fy)=\frac{y(5-y)}{4}; p(gx,fy)=\frac{x(5-x)}{4}; p(gy,fx)=\max\{\frac{y(5-y)}{4},\frac{x}{2}\}\leq\frac{x(5-x)}{4}$.
Here $\frac{1}{2}[p(gx,fy)+p(gy,fx)]\leq\frac{x(5-x)}{4}$. Therefore $M(x,y)=\frac{x(5-x)}{4}$.

We consider

$$p(fx,fy)=\frac{x}{2}\leq\frac{e^{-\frac{x(5-x)}{8}}}{1+\frac{x(5-x)}{4}}\frac{x(5-x)}{4}=\beta(\frac{x(5-x)}{4})\frac{x(5-x)}{4}=\beta(M(x,y))M(x,y).$$

Case (iii): $x,y\in[\frac{1}{2},1]$.

$p(fx,fy)=\frac{1}{3}; p(gx,gy)=\frac{x(5-x)}{4}; p(gx,fx)=\frac{x(5-x)}{4}$;
$p(gy,fy)=\frac{y(5-y)}{4}; p(gx,fy)=\frac{x(5-x)}{4}; p(gy,fx)=\frac{y(5-y)}{4}\leq\frac{x(5-x)}{4}$.
Here $\frac{1}{2}[p(gx,fy)+p(gy,fx)]\leq\frac{x(5-x)}{4}$. Therefore $M(x,y)=\frac{x(5-x)}{4}$.

We consider

$$p(fx,fy)=\frac{1}{3}\leq\frac{e^{-\frac{x(5-x)}{8}}}{1+\frac{x(5-x)}{4}}\frac{x(5-x)}{4}=\beta(\frac{x(5-x)}{4})\frac{x(5-x)}{4}=\beta(M(x,y))M(x,y).$$

Case (iv): $x\in(0,\frac{1}{2}), y=0$.

$p(fx,fy)=\frac{x}{2}; p(gx,gy)=\frac{x(5-x)}{4}; p(gx,fx)=\frac{x(5-x)}{4}; p(gy,fy)=0; p(gx,fy)=\frac{x(5-x)}{4}$;
$p(gy,fx)=\frac{x}{2}\leq\frac{x(5-x)}{4}$. Here $\frac{1}{2}[p(gx,fy)+p(gy,fx)]\leq\frac{x(5-x)}{4}$. Therefore $M(x,y)=\frac{x(5-x)}{4}$.

We consider

$$p(fx,fy)=\frac{x}{2}\leq\frac{e^{-\frac{x(5-x)}{8}}}{1+\frac{x(5-x)}{4}}\frac{x(5-x)}{4}=\beta(\frac{x(5-x)}{4})\frac{x(5-x)}{4}=\beta(M(x,y))M(x,y).$$

Case (v): $x\in[\frac{1}{2},1], y=0$.

$p(fx,fy)=\frac{1}{3}; p(gx,gy)=\frac{x(5-x)}{4}; p(gx,fx)=\frac{x(5-x)}{4}$;
$p(gy,fy)=0; p(gx,fy)=\frac{x(5-x)}{4}; p(gy,fx)=\frac{1}{3}\leq\frac{x(5-x)}{4}$.
Here $\frac{1}{2}[p(gx,fy)+p(gy,fx)]\leq\frac{x(5-x)}{4}$. Therefore $M(x,y)=\frac{x(5-x)}{4}$.

We consider

$$p(fx,fy)=\frac{1}{3}\leq\frac{e^{-\frac{x(5-x)}{8}}}{1+\frac{x(5-x)}{4}}\frac{x(5-x)}{4}=\beta(\frac{x(5-x)}{4})\frac{x(5-x)}{4}=\beta(M(x,y))M(x,y).$$

Case (vi): $x\in[\frac{1}{2},1], y\in(0,\frac{1}{2})$.

If $x\in[\frac{1}{2},\frac{2}{3}]$, then $p(fx,fy)=\frac{1}{3}; p(gx,gy)=\frac{x(5-x)}{4}; p(gx,fx)=\frac{x(5-x)}{4}$;
$p(gy,fy)=\frac{y(5-y)}{4}; p(gx,fy)=\frac{x(5-x)}{4}; p(gy,fx)=\max\{\frac{y(5-y)}{4},\frac{1}{3}\}\leq\frac{x(5-x)}{4}$.
Here $\frac{1}{2}[p(gx,fy)+p(gy,fx)]\leq\frac{x(5-x)}{4}$. Therefore $M(x,y)=\frac{x(5-x)}{4}$.

We consider

$$p(fx,fy)=\frac{1}{3}\leq\frac{e^{-\frac{x(5-x)}{8}}}{1+\frac{x(5-x)}{4}}\frac{x(5-x)}{4}=\beta(\frac{x(5-x)}{4})\frac{x(5-x)}{4}=\beta(M(x,y))M(x,y).$$

From all the above cases, we have f and g satisfy the inequality (2.1) with $L=0$.

Therefore f and g satisfy all the hypotheses of Corollary 4.2 and 0 is the unique common fixed point of f and g.

Here we observe that f and g fail to satisfy the generalized Geraghty type contraction condition with respect to the metric $d(x,y) = 2|x-y|$.

For, by choosing $x = \frac{1}{2}$, $y = 0$. We have
$\frac{2}{3} = 2|\frac{1}{3} - 0| = d(f(\frac{1}{2}), f(0)) \not\leq \beta(\frac{9}{16})\frac{9}{16} = \beta(M(\frac{1}{2},0))M(\frac{1}{2},0)$ for any $\beta \in \mathfrak{F}$.

The following is an example in support of Theorem 3.3.

Example 4.3. Let $X = [0,1]$. We define $p(x,y) = \max\{x,y\}$ for all $x, y \in X$. Then (X, p) is a partial metric space. We define selfmaps A, B, S and T on X by

$$A(x) = \frac{x}{6},\ B(x) = \begin{cases} 0 & \text{if } x \neq \frac{1}{2} \\ \frac{1}{8} & \text{if } x = \frac{1}{2}, \end{cases} \quad S(x) = \begin{cases} \frac{x}{2} & \text{if } 0 \leq x < \frac{1}{2} \\ \frac{1}{4} & \text{if } \frac{1}{2} \leq x \leq 1, \end{cases} \quad T(x) = \begin{cases} \frac{x}{3} & \text{if } 0 \leq x < \frac{1}{2} \\ \frac{1}{6} & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases}$$

and $\beta(t) = \begin{cases} 0 & \text{if } t = 0 \\ \frac{1}{1+t} & \text{if } t > 0 \end{cases}$ with $L \geq 0$.

Clearly A, B, S and T satisfy all the hypotheses of Theorem 3.3 and 0 is the unique common fixed point.

Conflict of Interests

The authors declare that there is no conflict of interests.

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