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PRODUCTS OF DIFFERENTIATION AND WEIGHTED COMPOSITION OPERATORS FROM HARDY SPACES TO WEIGHTED-TYPE SPACES

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Abstract. Let $\mathscr{H}(\mathbb{D})$ be the space of all analytic functions on the open unit disk $\mathbb{D}$. Let ψ_1 and ψ_2 be analytic functions on $\mathbb{D}$, and ϕ be an analytic self-map of $\mathbb{D}$. We consider the operator $T_{\psi_1,\psi_2,\phi}$ that is defined on $\mathscr{H}(\mathbb{D})$ by

$$\left(T_{\psi_1,\psi_2,\phi}f\right)(z) = \psi_1(z)f(\phi(z)) + \psi_2(z)f'(\phi(z)).$$

In this paper, we characterize the boundedness and compactness of the operator $T_{\psi_1,\psi_2,\phi}$ that act from the Hardy spaces H^p into the weighted-type space H^∞_μ and the little weighted-type space $H^\infty_{\mu,0}$.

Keywords: weighted composition operators; multiplication operators; differentiation operators; Hardy spaces; weighted-type spaces; compact operators; bounded operators; radial weights.

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1. Introduction

Let $\mathbb{D}$ be the open unit disk $\{z \in \mathbb{C} : |z| < 1\}$ in the complex plane $\mathbb{C}$, $\mathscr{H}(\mathbb{D})$ be the space of all analytic functions on $\mathbb{D}$, and $H^\infty = H^\infty(\mathbb{D})$ be the space of all bounded analytic functions on

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$\mathbb{D}$ such that $\|f\|_\infty = \sup_{z\in\mathbb{D}} |f(z)|$ is finite. For $0 < p < \infty$, the Hardy space $H^p(\mathbb{D})$ consists of all analytic functions f on $\mathbb{D}$ that satisfy

$$\|f\|_{H^p}^p = \sup_{0<r<1} \int_{\partial\mathbb{D}} |f(r\zeta)|^p d\sigma(\zeta) < \infty,$$

where σ is the normalized Lebesgue measure on the boundary of the unit disk. For f belongs to $H^p(\mathbb{D})$, it follows from Fatou's theorem that the radial limit

$$f^*(\zeta) = \lim_{r\to 1^-} f(r\zeta)$$

exists for almost all ζ on $\partial\mathbb{D}$. Moreover,

$$\|f\|_{H^p}^p = \int_{\partial\mathbb{D}} |f^*(\zeta)|^p d\sigma(\zeta),$$

for all finite values of p. In this paper we consider only radial weights, these arise from the positive and continuous functions $\mu : [0,1) \to (0,\infty)$. We define μ on $\mathbb{D}$ as $\mu(z) = \mu(|z|)$ for each $z \in \mathbb{D}$. Given a radial weight $\mu : \mathbb{D} \to \mathbb{C}$, the weighted-type space $H_\mu^\infty = H_\mu^\infty(\mathbb{D})$ consists of all analytic functions f on $\mathbb{D}$ such that

$$\|f\|_{H_\mu^\infty} = \sup_{z\in\mathbb{D}} \mu(z)|f(z)| < \infty.$$

The little weighted-type space $H_{\mu,0}^\infty$ is a closed subspace of H_μ^∞ that contains all those functions $f \in H_\mu^\infty$ such that

$$\lim_{|z|\to 1} \mu(z)|f(z)| = 0.$$

If $\mu \equiv 1$, we get the space $H^\infty(\mathbb{D})$ of all bounded analytic functions on $\mathbb{D}$. if $\mu(z) = (1-|z|^2)^\beta$, for $-1 < \beta < \infty$, we get the Bergman-type spaces H_β^∞ and $H_{\beta,0}^\infty$. These Bergman-type spaces are sometimes called growth or Bers type spaces, see, for example [3] and [20].

Suppose ϕ is an analytic function mapping $\mathbb{D}$ into itself and ψ is an analytic function on $\mathbb{D}$, the weighted composition operator $W_{\psi,\phi}$ is defined on the space $\mathscr{H}(\mathbb{D})$ of all analytic functions on $\mathbb{D}$ by

$$(W_{\psi,\phi} f)(z) = \psi(z) C_\phi f(z) = \psi(z) f(\phi(z)),$$

for all $f \in \mathscr{H}(\mathbb{D})$ and $z \in \mathbb{D}$. It is well known that the weighted composition operator $W_{\phi,\psi} f = \psi(f \circ \phi)$ defines a linear operator $W_{\psi,\phi}$ which acts boundedly on various spaces of analytic or harmonic functions on $\mathbb{D}$. In recent years, considerable interest has emerged in the study of the weighted composition operators, on spaces of analytic functions, with the goal of explaining the operator-theoretic properties of $W_{\psi,\varphi}$ in terms of the function-theoretic properties of the induced maps φ and ψ. For the overview of the field, we refer to the monographs [1], [2], [3], [10], [18] and [19].

The differentiation operator D is defined on $\mathscr{H}(\mathbb{D})$ as $Df = f'$ and $D^0 f = f$ for $f \in \mathscr{H}(\mathbb{D})$. For $\psi \in \mathscr{H}(\mathbb{D})$ the multiplication operator M_ψ is defined by $M_\psi f(z) = \psi(z)f(z)$, for $f \in \mathscr{H}(\mathbb{D})$ and $z \in \mathbb{D}$. Therefore, the weighted composition operator can be defined as the product of the multiplication and composition operators, that is $W_{\psi,\phi} = M_\psi C_\phi$.

Suppose ψ_1 and ψ_2 are analytic functions on $\mathbb{D}$, and ϕ is an analytic self-map of $\mathbb{D}$. In this paper we consider the operator $T_{\psi_1,\psi_2,\phi}$ that defined on $\mathscr{H}(\mathbb{D})$ by $\left(T_{\psi_1,\psi_2,\phi} f\right)(z) = \psi_1(z)f(\phi(z)) + \psi_2(z)f'(\phi(z))$. Moreover, this operator can be written as $\left(T_{\psi_1,\psi_2,\phi} f\right)(z) = M_{\psi_1} C_\phi D^0 f(z) + M_{\psi_2} C_\phi D f(z) = W_{\psi_1,\phi} D^0 f(z) + W_{\psi_2,\phi} D f(z)$. This operator has been introduced and studied on weighted Bergman spaces by the authors of [11] and [12].

In this paper, we characterize the boundedness and compactness of the operator $T_{\psi_1,\psi_2,\phi}$ from the Hardy spaces H^p to the weighted-type spaces H^∞_μ and $H^\infty_{\mu,0}$. As a consequence of the main results, we get the characterization of the products operators $M_\psi C_\phi D$, $M_\psi D C_\phi$, $C_\phi M_\psi D$, $D M_\psi C_\phi$, $C_\phi D M_\psi$, and $D C_\phi M_\psi$ that act from the Hardy spaces to the weighted-type spaces. If we set $\mu(z) = (1-|z|^2)^\beta$ for $\beta > -1$, then by simple modifying all the results of this paper one could also obtain similar results for all the previous operators act from Hardy spaces to the bloch-type spaces $\mathscr{B}^\infty_\beta$ and the little bloch-type spaces $\mathscr{B}^\infty_{\beta,0}$.

2. Preliminaries

The following are some auxiliary propositions which will be used in the proofs of this paper main results. The next proposition is a standard result on the Hardy spaces and it is curial for our work in this paper. For the proof, see for example [17].

Proposition 1.1. *Let n be a nonnegative integer, let $0 < p < \infty$. If $f \in H^p$, then*

$$|f^{(n)}(z)| \leq C \frac{\|f\|_{H^p}}{(1-|z|^2)^{n+1/p}},$$

for some positive constant C independent of f.

The following proposition is a basic fact about the compactness of the operator $T_{\psi_1,\psi_2,\phi}$. The proposition's proof follows by similar arguments to those outlined in (Proposition 3.11, [1]). So we omit the proof's details.

Proposition 1.2. *Let $0 < p < \infty$. Let $\psi_1, \psi_2 \in \mathscr{H}(\mathbb{D})$, and ϕ be an analytic self-map of $\mathbb{D}$ such that $T_{\psi_1,\psi_2,\phi} : H^p \to H^\infty_\mu$ is bounded. Then $T_{\psi_1,\psi_2,\phi}$ is compact if and only if whenever $\{f_k\}$ is bounded sequence in H^p and $f_k \to 0$ uniformly on compact subsets of $\mathbb{D}$, then $\lim_{k\to\infty} \|T_{\psi_1,\psi_2,\phi} f_k\|_{H^\infty_\mu} = 0$.*

The following proposition can be proved using arguments similar to those outlined in (Lemma 1, [8]). Thus, we omit the proof's details.

Proposition 1.3. *A closed set K in $H^\infty_{\mu,0}$ is compact if and only if it is bounded and satisfies $\lim_{|z|\to 1} \sup_{f\in K} \mu(z)|f(z)| = 0$.*

3. Main results

In this section we characterize the boundedness and compactness of the operator $T_{\psi_1,\psi_2,\phi}$ that acts from the Hardy spaces H^p into the weighted-type spaces H^∞_μ and $H^\infty_{\mu,0}$. The following theorem gives necessary and sufficient conditions for the boundedness of the operator $T_{\psi_1,\psi_2,\phi}$ that acts from Hardy spaces into the weighted-type spaces H^∞_μ.

Theorem 3.1. *Let ϕ be an analytic self-map of $\mathbb{D}$ and $\psi_1, \psi_2 \in \mathscr{H}(\mathbb{D})$. Then $T_{\psi_1,\psi_2,\phi} : H^p \to H^\infty_\mu$ is bounded if and only if the following conditions hold*

$$\sup_{z\in\mathbb{D}} \frac{\mu(z)|\psi_1(z)|}{(1-|\phi(z)|^2)^{1/p}} < \infty; \tag{1}$$

$$\sup_{z\in\mathbb{D}} \frac{\mu(z)|\psi_2(z)|}{(1-|\phi(z)|^2)^{1+1/p}} < \infty. \tag{2}$$

Proof. First, suppose that $T_{\psi_1,\psi_2,\phi} : H^p \to H^\infty_\mu$ is bounded. For $f(z) = 1$, we have that

$$\sup_{z\in\mathbb{D}} \mu(z)|\psi_1(z)| = \|T_{\psi_1,\psi_2,\phi} f\|_{H^\infty_\mu} < \infty. \tag{3}$$

For $f(z) = z$, we have that

$$\sup_{z\in\mathbb{D}} \mu(z)\,|\psi_1(z)\phi(z) + \psi_2(z)| = \|T_{\psi_1,\psi_2,\phi} f\|_{H^\infty_\mu} < \infty. \tag{4}$$

Since ϕ is a self-map of $\mathbb{D}$, from (3) and (4) we get that

$$\sup_{z\in\mathbb{D}} \mu(z)|\psi_2(z)| < \infty. \tag{5}$$

For a fixed $w \in \mathbb{D}$, set

$$f_w(z) = (2+1/p)\,\frac{1-|\phi(w)|^2}{\left(1-z\overline{\phi(w)}\right)^{1+1/p}} - (1+1/p)\,\frac{\left(1-|\phi(w)|^2\right)^2}{\left(1-z\overline{\phi(w)}\right)^{2+1/p}}.$$

It is easy to check that $f_w \in H^p(\mathbb{D})$ and $\sup_{z\in\mathbb{D}} \|f_w\|_{H^p} \leq C$, for some positive constant C. Moreover, $f'_w(\phi(w)) = 0$ and

$$f_w(\phi(w)) = \frac{1}{\left(1-|\phi(w)|^2\right)^{1/p}}.$$

Hence, for every $w \in \mathbb{D}$ we have

$$\begin{aligned} C &\geq \|T_{\psi_1,\psi_2,\phi} f_w\|_{H^\infty_\mu} \\ &= \sup_{w\in\mathbb{D}} \mu(w)\left|\psi_1(w) f_w(\phi(w)) + \psi_2(w) f'_w(\phi(w))\right| \\ &\geq \frac{\mu(w)|\psi_1(w)|}{\left(1-|\phi(w)|^2\right)^{1/p}}, \end{aligned}$$

which gives condition (1). To show condition (2) holds, for a fixed $w \in \mathbb{D}$, set

$$g_w(z) = \frac{\left(1-|\phi(w)|^2\right)^2}{\left(1-z\overline{\phi(w)}\right)^{2+1/p}} - \frac{1-|\phi(w)|^2}{\left(1-z\overline{\phi(w)}\right)^{1+1/p}}.$$

Then, $g_w(\phi(w)) = 0$ and

$$g'_w(\phi(w)) = \frac{\overline{\phi(w)}}{\left(1-|\phi(w)|^2\right)^{1+1/p}}.$$

Since $g_w \in H^p$, by the boundedness of the operator $T_{\psi_1,\psi_2,\phi}$ we get for every $w \in \mathbb{D}$

$$\begin{aligned} C &\geq \|T_{\psi_1,\psi_2,\phi} g_w\|_{H^\infty_\mu} \\ &\geq \frac{\mu(w)|\psi_2(w)||\phi(w)|}{\left(1-|\phi(w)|^2\right)^{1+1/p}}. \end{aligned} \tag{6}$$

On the one hand, from(5) we get

$$\sup_{|\phi(w)|\leq\frac{1}{2}} \frac{\mu(w)|\psi_2(w)|}{(1-|\phi(w)|^2)^{1+1/p}} \leq \left(\frac{4}{3}\right)^{1+1/p} \sup_{|\phi(w)|\leq\frac{1}{2}} \mu(w)|\psi_2(w)|$$

$$< \infty. \tag{7}$$

On the other hand, from (6) and

$$\sup_{\frac{1}{2}<|\phi(w)|<1} \frac{\mu(w)|\psi_2(w)|}{(1-|\phi(w)|^2)^{1+1/p}} \leq 2 \sup_{\frac{1}{2}<|\phi(w)|<1} \frac{\mu(w)|\psi_2(w)||\phi(w)|}{(1-|\phi(w)|^2)^{1+1/p}}$$

$$\leq 2C. \tag{8}$$

Therefore, (7) and (8) give condition (2).

For the converse, suppose that conditions (1) and (2) hold. Then for any $z \in \mathbb{D}$ and $f \in H^P$, by using Proposition 1.1, we get

$$\begin{aligned}
&\mu(z)\left|\left(T_{\psi_1,\psi_2,\phi}f\right)(z)\right| \\
&= \mu(z)\left|\psi_1(z)f(\phi(z)) + \psi_2(z)f'(\phi(z))\right| \\
&\leq \frac{C\mu(z)|\psi_1(z)|\|f\|_{H^p}}{(1-|\phi(z)|^2)^{1/p}} + \frac{C\mu(z)|\psi_2(z)|\|f\|_{H^p}}{(1-|\phi(z)|^2)^{1+1/p}}.
\end{aligned}$$

Taking supremum of both sides over all $z \in \mathbb{D}$, we get

$$\|T_{\psi_1,\psi_2,\phi}f\|_{H^\infty_\mu} = \sup_{z\in\mathbb{D}} \mu(z)\left|\left(T_{\psi_1,\psi_2,\phi}f\right)(z)\right| < \infty,$$

which gives the boundedness of $T_{\psi_1,\psi_2,\phi}$. This completes the proof.

The following theorem characterizes the compactness of the operator $T_{\psi_1,\psi_2,\phi}$ that acts from Hardy spaces into the weighted-type spaces H^∞_μ.

Theorem 3.2. *Let ϕ be an analytic self-map of $\mathbb{D}$ and $\psi_1, \psi_2 \in \mathscr{H}(\mathbb{D})$. Then $T_{\psi_1,\psi_2,\phi} : H^p \to H^\infty_\mu$ is compact if and only if ψ_1, ψ_2 are in H^∞_μ and the following conditions hold*

$$\lim_{|\phi(z)|\to 1} \frac{\mu(z)|\psi_1(z)|}{(1-|\phi(z)|^2)^{1/p}} = 0; \tag{9}$$

$$\lim_{|\phi(z)|\to 1} \frac{\mu(z)|\psi_2(z)|}{(1-|\phi(z)|^2)^{1+1/p}} = 0. \tag{10}$$

Proof. First, assume that ψ_1, ψ_2 are in H^∞_μ and that the growth conditions (9) and (10) hold. Using $\sup_{z\in\mathbb{D}} \mu(z)|\psi_1(z)|$ is finite and condition (9) we get condition (1) in Theorem 3.1. Similarly, using $\sup_{z\in\mathbb{D}} \mu(z)|\psi_2(z)|$ is finite and condition (10) we get condition (2) in Theorem 3.1. Therefore, Theorem 3.1 gives $T_{\psi_1,\psi_2,\phi}$ is bounded.

Now let $\{f_k\}$ be a bounded sequence in H^p, say bounded by a positive constant M, such that $f_k \to 0$ uniformly on compact subsets of $\mathbb{D}$. To show that $T_{\psi_1,\psi_2,\phi}$ is compact, by Proposition 1.2, it suffices to show $\|T_{\psi_1,\psi_2,\phi} f_k\|_{H^\infty_\mu} \to 0$ as $k \to \infty$. Using the growth conditions (9) and (10), for any $\varepsilon > 0$ there exists $\delta \in (0,1)$ such that when $\delta < |\phi(z)| < 1$

$$\frac{\mu(z)|\psi_1(z)|}{(1-|\phi(z)|^2)^{1/p}} < \varepsilon. \tag{11}$$

$$\frac{\mu(z)|\psi_2(z)|}{(1-|\phi(z)|^2)^{1+1/p}} < \varepsilon. \tag{12}$$

Therefore, since $T_{\psi_1,\psi_2,\phi}$ is bounded, for $\delta < |\phi(z)| < 1$ we have

$$\begin{aligned}
&\mu(z)\left|\left(T_{\psi_1,\psi_2,\phi} f_k\right)(z)\right| \\
&= \mu(z)\left|\psi_1(z) f_k(\phi(z)) + \psi_2(z) f_k'(\phi(z))\right| \\
&\le C \sup_{z\in\mathbb{D}} \|f_k\|_{H^p} \left[\frac{\mu(z)|\psi_1(z)|}{(1-|\phi(z)|^2)^{1/p}} + \frac{\mu(z)|\psi_2(z)|}{(1-|\phi(z)|^2)^{1+1/p}}\right] \\
&\le 2CM\varepsilon. \qquad (13)
\end{aligned}$$

On the other hand, if $|\phi(z)| \le \delta$ then we have

$$\begin{aligned}
&\mu(z)\left|\left(T_{\psi_1,\psi_2,\phi} f_k\right)(z)\right| \\
&= \mu(z)\left|\psi_1(z) f_k(\phi(z)) + \psi_2(z) f_k'(\phi(z))\right| \\
&\le \|\psi_1\|_{H_\mu^\infty} \sup_{|w|\le\delta} |f_k(w)| + \|\psi_2\|_{H_\mu^\infty} \sup_{|w|\le\delta} |f_k'(w)|
\end{aligned}$$

In this case, since ψ_1, ψ_2 are in H_μ^∞, we get by using Cauchy's estimate

$$\|T_{\psi_1,\psi_2,\phi} f_k\|_{H_\mu^\infty} \to 0 \quad \text{as } k \to \infty. \tag{14}$$

Hence, (13) and (14) give the compactness of $T_{\psi_1,\psi_2,\phi}$.

For the converse, assume that $T_{\psi_1,\psi_2,\phi} : H^p \to H_\mu^\infty$ is compact. Then it is obvious that $T_{\psi_1,\psi_2,\phi}$ is bounded. Hence, from the proof of Theorem 3.1, we have ψ_1 and ψ_2 are in H_μ^∞. To complete the proof, let $\{z_k\}$ be a sequence in $\mathbb{D}$ such that $|\phi(z_k)| \to 1$ as $k \to \infty$.

On the one hand, consider the test function

$$f_k(z) = (2+1/p)\,\frac{1-|\phi(z_k)|^2}{\left(1-z\overline{\phi(z_k)}\right)^{1+1/p}} - (1+1/p)\,\frac{\left(1-|\phi(z_k)|^2\right)^2}{\left(1-z\overline{\phi(z_k)}\right)^{2+1/p}}.$$

It is clear that f_k converges to zero on compact subsets of $\mathbb{D}$. Moreover, $f_k'(\phi(z_k)) = 0$ and

$$f_k(\phi(z_k)) = \frac{1}{\left(1-|\phi(z_k)|^2\right)^{1/p}}.$$

Now,

$$\begin{aligned}
\|T_{\psi_1,\psi_2,\phi} f_k\|_{H_\mu^\infty} &\ge \mu(z_k)\left|\psi_1(z_k) f_k(\phi(z_k)) + \psi_2(z_k) f_k'(\phi(z_k))\right| \\
&= \frac{\mu(z_k)|\psi_1(z_k)|}{\left(1-|\phi(z_k)|^2\right)^{1/p}}.
\end{aligned}$$

Since $T_{\psi_1,\psi_2,\phi}$ is compact and $f_k \to 0$ on compact subsets of $\mathbb{D}$, by using Proposition 1.2, we get

$$\lim_{k\to\infty} \|T_{\psi_1,\psi_2,\phi} f_k\|_{H_\mu^\infty} = 0.$$

Therefore, since $|\phi(z_k)| \to 1$ as $k \to \infty$, we get

$$\lim_{k\to\infty} \frac{\mu(z_k)|\psi_1(z_k)|}{(1-|\phi(z_k)|^2)^{1/p}} = 0,$$

which gives condition (9).

On the other hand, consider the test function

$$g_k(z) = \frac{\left(1-|\phi(z_k)|^2\right)^2}{\left(1-z\overline{\phi(z_k)}\right)^{2+1/p}} - \frac{1-|\phi(z_k)|^2}{\left(1-z\overline{\phi(z_k)}\right)^{1+1/p}}.$$

It is obvious that $g_k \to 0$ on compact subsets of $\mathbb{D}$. Moreover, $g_k(\phi(z_k)) = 0$ and

$$g_k'(\phi(z_k)) = \frac{\overline{\phi(z_k)}}{(1-|\phi(z_k)|^2)^{1+1/p}}.$$

Using the compactness of $T_{\psi_1,\psi_2,\phi}$ we get

$$\begin{aligned} 0 &= \lim_{k\to\infty} \|T_{\psi_1,\psi_2,\phi} g_k\|_{H_\mu^\infty} \\ &\geq \frac{\mu(z_k)|\psi_2(z_k)||\phi(z_k)|}{(1-|\phi(z_k)|^2)^{1+1/p}}. \end{aligned}$$

Since $|\phi(z_k)| \to 1$ as $k \to \infty$, we get

$$\lim_{k\to\infty} \frac{\mu(z_k)|\psi_2(z_k)|}{(1-|\phi(z_k)|^2)^{1+1/p}} = 0,$$

which gives condition (10). This completes the proof.

The next theorem gives necessary and sufficient conditions for the boundedness of the operator $T_{\psi_1,\psi_2,\phi}$ that acts from Hardy spaces into the weighted-type spaces $H_{\mu,0}^\infty$.

Theorem 3.3. *Let ϕ be an analytic self-map of $\mathbb{D}$ and $\psi_1, \psi_2 \in \mathscr{H}(\mathbb{D})$. Then $T_{\psi_1,\psi_2,\phi} : H^p \to H_{\mu,0}^\infty$ is bounded if and only if ψ_1, ψ_2 are in $H_{\mu,0}^\infty$ and $T_{\psi_1,\psi_2,\phi} : H^p \to H_\mu^\infty$ is bounded.*

Proof. First, suppose that $T_{\psi_1,\psi_2,\phi} : H^p \to H_\mu^\infty$ is bounded and that ψ_1, ψ_2 are in $H_{\mu,0}^\infty$. Then for each polynomial p, we have

$$\begin{aligned} \mu(z)\left|\left(T_{\psi_1,\psi_2,\phi} p\right)(z)\right| &= \mu(z)\left|\psi_1(z)p(\phi(z)) + \psi_2(z)p'(\phi(z))\right| \\ &\leq \mu(z)|\psi_1(z)|\|p\|_\infty + \mu(z)|\psi_2(z)|\|p'\|_\infty \end{aligned}$$

Since the polynomials p and p' are in H^∞, taking limit as $|z| \to 1$ we get that $T_{\psi_1,\psi_2,\phi} p$ is in H_μ^∞. Now let $f \in H^p$. Since the set of all polynomials is dense in H^p, there is a sequence of polynomials $\{p_n\}$ such that $\|f - p_n\|_{H^p} \to 0$ as $n \to \infty$. By the boundedness of $T_{\psi_1,\psi_2,\phi} : H^p \to H_\mu^\infty$, we have

$$\|T_{\psi_1,\psi_2,\phi} f - T_{\psi_1,\psi_2,\phi} p_n\|_{H_\mu^\infty} \leq \|T_{\psi_1,\psi_2,\phi}\|\|f - p_n\|_{H^p}.$$

Since $T_{\psi_1,\psi_2,\phi} : H^p \to H_\mu^\infty$ is bounded, we get that

$$\lim_{n\to\infty} \|T_{\psi_1,\psi_2,\phi}(f-p_n)\|_{H_\mu^\infty} = 0.$$

Since $H_{\mu,0}^\infty$ is closed subset of H_μ^∞, we get $T_{\psi_1,\psi_2,\phi} : H^p \to H_{\mu,0}^\infty$ is bounded.

For the converse, suppose that $T_{\psi_1,\psi_2,\phi} : H^p \to H_{\mu,0}^\infty$ is bounded. Then it is clear that $T_{\psi_1,\psi_2,\phi} : H^p \to H_\mu^\infty$ is bounded. Taking $f(z) = 1$, we get

$$0 = \lim_{|z|\to 1} \left| \left(T_{\psi_1,\psi_2,\phi} f\right)(z)\right|$$

$$= \lim_{|z|\to 1} \mu(z)|\psi_1(z)|. \tag{15}$$

Thus, ψ_1 is in $H_{\mu,0}^\infty$. Similarly, taking $f(z) = z$, we get

$$0 = \lim_{|z|\to 1} \left| \left(T_{\psi_1,\psi_2,\phi} f\right)(z)\right|$$

$$= \lim_{|z|\to 1} \mu(z)\left|\psi_1(z)\phi(z) + \psi_2(z)\right|. \tag{16}$$

By using (15), (16) and the boundedness of ϕ, we get

$$\lim_{|z|\to 1} \mu(z)|\psi_2(z)| = 0.$$

Therefore, ψ_2 is in $H_{\mu,0}^\infty$. This completes the proof.

The next theorem characterizes the compactness of the operator $T_{\psi_1,\psi_2,\phi}$ that acts from Hardy spaces into the weighted-type spaces $H_{\mu,0}^\infty$.

Theorem 3.4. *Let ϕ be an analytic self-map of $\mathbb{D}$ and $\psi_1, \psi_2 \in \mathscr{H}(\mathbb{D})$. Then $T_{\psi_1,\psi_2,\phi} : H^p \to H_{\mu,0}^\infty$ is compact if and only if the following conditions hold*

$$\lim_{|z|\to 1} \frac{\mu(z)|\psi_1(z)|}{(1-|\phi(z)|^2)^{1/p}} = 0; \tag{17}$$

$$\lim_{|z|\to 1} \frac{\mu(z)|\psi_2(z)|}{(1-|\phi(z)|^2)^{1+1/p}} = 0. \tag{18}$$

Proof. First, assume that $T_{\psi_1,\psi_2,\phi} : H^p \to H^\infty_{\mu,0}$ is compact. Then it is clear that $T_{\psi_1,\psi_2,\phi} : H^p \to H^\infty_{\mu,0}$ is bounded. Then, from Theorem 3.3, we get ψ_1 and ψ_2 are in $H^\infty_{\mu,0}$, that is

$$\lim_{|z|\to 1} \mu(z)|\psi_1(z)| = 0. \tag{19}$$

$$\lim_{|z|\to 1} \mu(z)|\psi_2(z)| = 0. \tag{20}$$

If $\|\phi\|_\infty < 1$, then from (19) and (20) we get

$$\lim_{|z|\to 1} \frac{\mu(z)|\psi_1(z)|}{(1-|\phi(z)|^2)^{1/p}} \le \left(1-\|\phi\|_\infty^2\right)^{-1/p} \lim_{|z|\to 1} \mu(z)|\psi_1(z)| = 0;$$

and

$$\lim_{|z|\to 1} \frac{\mu(z)|\psi_2(z)|}{(1-|\phi(z)|^2)^{1+1/p}} \le \left(1-\|\phi\|_\infty^2\right)^{-(1+1/p)} \lim_{|z|\to 1} \mu(z)|\psi_2(z)| = 0.$$

Hence, we get conditions (17) and (18).

Now, assume that $\|\phi\|_\infty = 1$. The compactness of $T_{\psi_1,\psi_2,\phi} : H^p \to H^\infty_{\mu,0}$ implies that $T_{\psi_1,\psi_2,\phi} : H^p \to H^\infty_\mu$ is compact. Therefore, by Theorem 3.2, we have

$$\lim_{|\phi(z)|\to 1} \frac{\mu(z)|\psi_1(z)|}{(1-|\phi(z)|^2)^{1/p}} = 0. \tag{21}$$

$$\lim_{|\phi(z)|\to 1} \frac{\mu(z)|\psi_2(z)|}{(1-|\phi(z)|^2)^{1+1/p}} = 0. \tag{22}$$

By using (21), for every $\varepsilon > 0$ there is $r \in (0,1)$ such that

$$\frac{\mu(z)|\psi_1(z)|}{(1-|\phi(z)|^2)^{1/p}} < \varepsilon,$$

whenever $r < |\phi(z)| < 1$. By using (19), there exists $\delta \in (0,1)$ such that

$$\mu(z)|\psi_1(z)| < \varepsilon(1-r^2)^{1/p},$$

whenever $\delta < |z| < 1$. Therefore, if $\delta < |z| < 1$ and $r < |\phi(z)| < 1$ then

$$\frac{\mu(z)|\psi_1(z)|}{(1-|\phi(z)|^2)^{1/p}} < \varepsilon. \tag{23}$$

On the other hand, if $\delta < |z| < 1$ and $|\phi(z)| \le r$ we obtain

$$\frac{\mu(z)|\psi_1(z)|}{(1-|\phi(z)|^2)^{1/p}} \le \frac{\mu(z)|\psi_1(z)|}{(1-r^2)^{1/p}} < \varepsilon. \tag{24}$$

Now combining (23) and (24) we obtain (17). Similarly, using (20) and (22) we obtain (18).

Second, assume that conditions (17) and (18) hold. It is clear that conditions (1) and (2) hold. Hence, by Theorem 3.1, we have $T_{\psi_1,\psi_2,\phi} : H^p \to H^\infty_\mu$ is bounded. On the other hand for any $f \in H^p$, by Proposition 1.1, there exists $C > 0$ such that

$$\mu(z)\left|\left(T_{\psi_1,\psi_2,\phi}f\right)(z)\right| = \mu(z)\left|\psi_1(z)f(\phi(z)) + \psi_2(z)f'(\phi(z))\right|$$

$$\leq C\|f\|_{H^p}\left[\frac{\mu(z)|\psi_1(z)|}{(1-|\phi(z)|^2)^{1/p}} + \frac{\mu(z)|\psi_2(z)|}{(1-|\phi(z)|^2)^{(1+1/p)}}\right]. \tag{25}$$

So using (17) and (18) we have

$$\lim_{|z|\to 1}\mu(z)\left|\left(T_{\psi_1,\psi_2,\phi}f\right)(z)\right| = 0,$$

which gives that $T_{\psi_1,\psi_2,\phi}f$ is in $H^\infty_{\mu,0}$. Thus, $T_{\psi_1,\psi_2,\phi} : H^p \to H^\infty_{\mu,0}$ is bounded. Now, let $K = \{f \in H^p : \|f\|_{H^p} \leq 1\}$. Taking supremum in (25) over all $f \in K$ and then letting $|z| \to 1$, we get

$$\lim_{|z|\to 1}\sup_{f\in K}\mu\left|\left(T_{\psi_1,\psi_2,\phi}f\right)(z)\right| = 0.$$

Hence, by Proposition 1.3, we get that the operator $T_{\psi_1,\psi_2,\phi} : H^p \to H^\infty_{\mu,0}$ is compact. This completes the proof.

Note that all products of composition, multiplication, and differentiation operators can be obtained from the operator $T_{\psi_1,\psi_2,\phi}$ as follow: $T_{0,\psi,\phi} = M_\psi C_\phi D$, $T_{0,\psi\phi',\phi} = M_\psi DC_\phi$, $T_{0,\psi\circ\phi,\phi} = C_\phi M_\psi D$, $T_{\psi',\psi\phi',\phi} = DM_\psi C_\phi$, $T_{\psi'\circ\phi,\psi\circ\phi,\phi} = C_\phi DM_\psi$, and $T_{(\psi'\circ\phi)\phi',(\psi\circ\phi)\phi',\phi} = DC_\phi M_\psi$. These operator have been investigated on spaces of analytic functions by many authors, see for example [4], [5], [6], [7], [9], [13], [14], [15], [16], [20], and the references therein.

Moreover, for $f \in \mathscr{H}(\mathbb{D})$ and $z \in \mathbb{D}$, those operators can be written as

$$\left(M_\psi C_\phi Df\right)(z) = \psi(z)f'(\phi(z));$$

$$\left(M_\psi DC_\phi f\right)(z) = \psi(z)\phi'(z)f'(\phi(z));$$

$$\left(C_\phi M_\psi Df\right)(z) = \psi(\phi(z))f'(\phi(z));$$

$$\left(DM_\psi C_\phi f\right)(z) = \psi'(z)f(\phi(z)) + \psi(z)\phi'(z)f'(\phi(z));$$

$$\left(C_\phi DM_\psi f\right)(z) = \psi'(\phi(z))f(\phi(z)) + \psi(\phi(z))f'(\phi(z));$$

$$\left(DC_\phi M_\psi f\right)(z) = \psi'(\phi(z))\phi'(z)f(\phi(z)) + \psi(\phi(z))\phi'(z)f'(\phi(z)).$$

The following are immediate corollaries of the main results of this paper. These corollaries characterize the boundedness and compactness of the products operators $M_\psi C_\phi D$, $M_\psi DC_\phi$, and $C_\phi M_\psi D$. similarly, one could characterize the other product operators.

Corollary 3.5. *Let ϕ be an analytic self-map of $\mathbb{D}$ and $\psi \in \mathscr{H}(\mathbb{D})$.*

(1) $M_\psi C_\phi D : H^p \to H^\infty_\mu$ is bounded if and only if

$$\sup_{z\in\mathbb{D}} \frac{\mu(z)|\psi(z)|}{(1-|\phi(z)|^2)^{1+1/p}} < \infty.$$

(2) $M_\psi C_\phi D : H^p \to H^\infty_\mu$ is compact if and only if ψ is in H^∞_μ and

$$\lim_{|\phi(z)|\to 1} \frac{\mu(z)|\psi(z)|}{(1-|\phi(z)|^2)^{1+1/p}} = 0.$$

(3) $M_\psi C_\phi D : H^p \to H^\infty_{\mu,0}$ is compact if and only if

$$\lim_{|z|\to 1} \frac{\mu(z)|\psi(z)|}{(1-|\phi(z)|^2)^{1+1/p}} = 0.$$

Corollary 3.6. *Let ϕ be an analytic self-map of $\mathbb{D}$ and $\psi \in \mathscr{H}(\mathbb{D})$.*

(1) $M_\psi DC_\phi : H^p \to H^\infty_\mu$ is bounded if and only if

$$\sup_{z\in\mathbb{D}} \frac{\mu(z)|\psi(z)\phi'(z)|}{(1-|\phi(z)|^2)^{1+1/p}} < \infty.$$

(2) $M_\psi DC_\phi : H^p \to H^\infty_\mu$ is compact if and only if $\psi\phi'$ is in H^∞_μ and

$$\lim_{|\phi(z)|\to 1} \frac{\mu(z)|\psi(z)\phi'(z)|}{(1-|\phi(z)|^2)^{1+1/p}} = 0.$$

(3) $M_\psi DC_\phi : H^p \to H^\infty_{\mu,0}$ is compact if and only if

$$\lim_{|z|\to 1} \frac{\mu(z)|\psi(z)\phi'(z)|}{(1-|\phi(z)|^2)^{1+1/p}} = 0.$$

Corollary 3.7. *Let ϕ be an analytic self-map of $\mathbb{D}$ and $\psi \in \mathscr{H}(\mathbb{D})$.*

(1) $C_\phi M_\psi D : H^p \to H^\infty_\mu$ is bounded if and only if

$$\sup_{z\in\mathbb{D}} \frac{\mu(z)|\psi(\phi(z))|}{(1-|\phi(z)|^2)^{1+1/p}} < \infty.$$

(2) $C_\phi M_\psi D : H^p \to H^\infty_\mu$ is compact if and only if ψ is in H^∞_μ and

$$\lim_{|\phi(z)|\to 1} \frac{\mu(z)|\psi(\phi(z))|}{(1-|\phi(z)|^2)^{1+1/p}} = 0.$$

(3) $C_\phi M_\psi D : H^p \to H^\infty_{\mu,0}$ *is compact if and only if*

$$\lim_{|z|\to 1} \frac{\mu(z)|\psi(\phi(z))|}{(1-|\phi(z)|^2)^{1+1/p}} = 0.$$

Conflict of Interests

The authors declare that there is no conflict of interests.

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