Available online at http://scik.org
J. Math. Comput. Sci. 9 (2019), No. 2, 182-193
https://doi.org/10.28919/jmcs/3932
ISSN: 1927-5307

SAMPLING THEOREM ASSOCIATED WITH Q-DIRAC SYSTEM

FATMA HIRA*

Department of Mathematics, Faculty of Science and Arts, Hitit University, Çorum, 19040, Turkey

Abstract. This paper deals with $q-$analogue of sampling theory associated with $q-$Dirac system. We derive sampling representation for transform whose kernel is a solution of this $q-$Dirac system. As a special case, three examples are given.

Keywords: sampling theory; $q-$Dirac system.

2010 AMS Subject Classification: 94A20, 33D15, 35Q41.

1. Introduction

Consider the following $q-$Dirac system

$$\begin{cases} -\dfrac{1}{q}D_{q^{-1}}y_2 + p(x)y_1 = \lambda y_1, \\ D_q y_1 + r(x)y_2 = \lambda y_2, \end{cases} \tag{1.1}$$

$$k_{11}y_1(0) + k_{12}y_2(0) = 0, \tag{1.2}$$

*Corresponding author
E-mail address: fatmahira@yahoo.com.tr
Received October 12, 2018

$$k_{21}y_1(a)+k_{22}y_2\left(aq^{-1}\right)=0, \tag{1.3}$$

where k_{ij} $(i,j=1,2)$ are real numbers, λ is a complex eigenvalue parameter, $y(x)=\begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}$, $p(x)$ and $r(x)$ are real-valued functions defined on $[0,a]$ and continuous at zero and $p(x)$, $r(x)\in L_q^1(0,a)$ (see $[1,2]$).

The papers in $q-$Dirac system are few, see $[1-3]$. However, sampling theories associated with $q-$Dirac system do not exist as far as we know. So that we will construct a $q-$analogue of sampling theorem for $q-$Dirac system (1.1)-(1.3), building on recent results in $[1,2]$. To achieve our aim we will briefly give the spectral analysis of the problem (1.1)-(1.3). Then we derive sampling theorem using solution. In the last section we give three examples illustrating the obtained results.

2. Notations and Preliminaries

We state the $q-$notations and results which will be needed for the derivation of the sampling theorem. Throughout this paper q is a positive number with $0<q<1$.

A set $A\subseteq\mathbb{R}$ is called q-geometric if, for every $x\in A$, $qx\in A$. Let f be a real or complex-valued function defined on a q-geometric set A. The q-difference operator is defined by

$$D_qf(x):=\frac{f(x)-f(qx)}{x(1-q)},\ x\neq 0. \tag{2.1}$$

If $0\in A$, the q-derivative at zero is defined to be

$$D_qf(0):=\lim_{n\to\infty}\frac{f(xq^n)-f(0)}{xq^n},\ x\in A, \tag{2.2}$$

if the limit exists and does not depend on x. Also, for $x\in A$, $D_{q^{-1}}$ is defined to be

$$D_{q^{-1}}f(x):=\begin{cases}\dfrac{f(x)-f\left(q^{-1}x\right)}{x(1-q^{-1})}, & x\in A\setminus\{0\},\\ D_qf(0), & x=0,\end{cases} \tag{2.3}$$

provided that $D_qf(0)$ exists. The following relation can be verified directly from the definition

$$D_{q^{-1}}f(x)=\left(D_qf\right)\left(xq^{-1}\right). \tag{2.4}$$

A right inverse, q-integration, of the q-difference operator D_q is defined by Jackson [4] as

$$\int_0^x f(t)\,d_qt := x(1-q)\sum_{n=0}^{\infty} q^n f(xq^n),\ x\in A, \tag{2.5}$$

provided that the series converges. A q-analog of the fundamental theorem of calculus is given by

$$D_q\int_0^x f(t)\,d_qt = f(x),\ \int_0^x D_qf(t)\,d_qt = f(x) - \lim_{n\to\infty} f(xq^n), \tag{2.6}$$

where $\lim_{n\to\infty} f(xq^n)$ can be replaced by $f(0)$ if f is q-regular at zero, that is, if $\lim_{n\to\infty} f(xq^n) = f(0)$, for all $x\in A$. Throughout this paper, we deal only with functions q-regular at zero.

The q-type product formula is given by

$$D_q(fg)(x) = g(x)D_qf(x) + f(qx)D_qg(x), \tag{2.7}$$

and hence the q-integration by parts is given by

$$\int_0^a g(x)D_qf(x)\,d_qx = (fg)(a) - (fg)(0) - \int_0^a D_qg(x)f(qx)\,d_qx, \tag{2.8}$$

where f and g are q-regular at zero.

For more results and properties in q-calculus, readers are referred to the recent works $[5-8]$.

The basic trigonometric functions $\cos(z;q)$ and $\sin(z;q)$ are defined on $\mathbb{C}$ by

$$\cos(z;q) := \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}(z(1-q))^{2n}}{(q;q)_{2n}}, \tag{2.9}$$

$$\sin(z;q) := \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)}(z(1-q))^{2n+1}}{(q;q)_{2n+1}}, \tag{2.10}$$

and they are q-analogs of the cosine and sine functions. $\cos(.;q)$ and $\sin(.;q)$ have only real and simple zeros $\{\pm x_m\}_{m=1}^{\infty}$ and $\{0,\pm y_m\}_{m=1}^{\infty}$, respectively, where $x_m,\ y_m>0, m\geqslant 1$ and

$$x_m = (1-q)^{-1}q^{-m+1/2+\varepsilon_m(1/2)} \text{ if } q^3<\left(1-q^2\right)^2, \tag{2.11}$$

$$y_m = (1-q)^{-1}q^{-m+\varepsilon_m(-1/2)} \text{ if } q<\left(1-q^2\right)^2. \tag{2.12}$$

Moreover, for any $q\in(0,1)$, (2.11) and (2.12) hold for sufficiently large m, cf. $[5,9-11]$.

Let $L_q^2(0,a)$ be the space of all complex valued functions defined on $[0,a]$ such that

$$\|f\| := \left(\int_0^a |f(x)|^2 d_q x \right)^{1\backslash 2} < \infty. \tag{2.13}$$

The space $L_q^2(0,a)$ is a separable Hilbert space with the inner product (see [12])

$$\langle f,g\rangle := \int_0^a f(x)\overline{g(x)} d_q x,\ f,g \in L_q^2(0,a). \tag{2.14}$$

Let H_q be the Hilbert space

$$H_q := \left\{ y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix},\ y_1(x), y_2(x) \in L_q^2(0,a) \right\}.$$

The inner product of H_q is defined by

$$\langle y(.), z(.)\rangle_{H_q} := \int_0^a y^{\top}(x) z(x) d_q x, \tag{2.15}$$

where $\top$ denotes the matrix transpose, $y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}, z(x) = \begin{pmatrix} z_1(x) \\ z_2(x) \end{pmatrix} \in H_q,\ y_i(.), z_i(.) \in L_q^2(0,a)\ (i = 1,2)$.

It is known [1,2] that the problem (1.1)-(1.3) has a countable number of eigenvalues $\{\lambda_n\}_{n=-\infty}^{\infty}$ which are real and simple, and to every eigenvalue λ_n, there corresponds a vector-valued eigenfunction $y_n^{\top}(x,\lambda_n) = (y_{n,1}(x,\lambda_n), y_{n,2}(x,\lambda_n))$. Moreover, vector-valued eigenfunctions belonging to different eigenvalues are orthogonal, i.e.,

$$\begin{aligned} &\int_0^a y_n^{\top}(x,\lambda_n) y_m(x,\lambda_m) d_q x \\ &= \int_0^a \{y_{n,1}(x,\lambda_n) y_{m,1}(x,\lambda_m) + y_{n,2}(x,\lambda_n) y_{m,2}(x,\lambda_m)\} d_q x = 0, \quad \text{for } \lambda_n \neq \lambda_m. \end{aligned}$$

Let $y_1(x,\lambda_1) = \begin{pmatrix} y_{11}(x,\lambda_1) \\ y_{12}(x,\lambda_1) \end{pmatrix}$ and $y_2(x,\lambda_2) = \begin{pmatrix} y_{21}(x,\lambda_2) \\ y_{22}(x,\lambda_2) \end{pmatrix}$ be two solutions of (1.1): hence

$$\begin{cases} -\frac{1}{q} D_{q^{-1}} y_{12} + \{p(x) - \lambda_1\} y_{11} = 0, \\ D_q y_{11} + \{r(x) - \lambda_1\} y_{12} = 0, \end{cases} \tag{2.16}$$

and

$$\begin{cases} -\frac{1}{q}D_{q^{-1}}y_{22}+\{p(x)-\lambda_2\}y_{21}=0, \\ D_q y_{21}+\{r(x)-\lambda_2\}y_{22}=0. \end{cases} \tag{2.17}$$

Multiplying (2.16) by y_{21} and y_{22} and (2.17) by $-y_{11}$ and $-y_{22}$ respectively, and adding them together also using the formula (2.4) we obtain

$$\begin{aligned} &D_q\left\{y_{11}(x,\lambda_1)\,y_{22}\left(xq^{-1},\lambda_2\right)-y_{12}\left(xq^{-1},\lambda_1\right)y_{21}(x,\lambda_2)\right\} \\ &=(\lambda_1-\lambda_2)\left\{y_{11}(x,\lambda_1)\,y_{21}(x,\lambda_2)+y_{12}(x,\lambda_1)\,y_{22}(x,\lambda_2)\right\}. \end{aligned} \tag{2.18}$$

Let $y(x)=\begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}$, $z(x)=\begin{pmatrix} z_1(x) \\ z_2(x) \end{pmatrix}\in H_q$. Then the q-Wronskian of $y(x)$ and $z(x)$ is defined by

$$W(y,z)(x):=y_1(x)\,z_2\left(xq^{-1}\right)-z_1(x)\,y_2\left(xq^{-1}\right). \tag{2.19}$$

Let us consider the next initial value problem

$$\begin{cases} -\frac{1}{q}D_{q^{-1}}y_2+p(x)\,y_1=\lambda y_1, \\ D_q y_1+r(x)\,y_2=\lambda y_2, \end{cases} \tag{2.20}$$

$$y_1(0)=k_{12},\quad y_2(0)=-k_{11}. \tag{2.21}$$

By virtue of Theorem 1 in $[1]$, this problem has a unique solution $\phi(x,\lambda)=\begin{pmatrix} \phi_1(x,\lambda) \\ \phi_2(x,\lambda) \end{pmatrix}$. It is obvious that $\phi(x,\lambda)$ satisfies the boundary condition (1.2) and this function is uniformly bounded on the subsets of the form $[0,a]\times\Omega$ where $\Omega\subset\mathbb{C}$ is compact. The proof is similar to the one in the proof of Lemma 3.1 in $[13]$. To find the eigenvalues of the $q-$Dirac system (1.1)-(1.3) we have to insert this function into the boundary condition (1.3) and find the roots of the obtained equation. So, putting the function $\phi(x,\lambda)$ into the boundary condition (1.3) we get the following equation whose zeros are the eigenvalues of the $q-$Dirac system (1.1)-(1.3)

$$\omega(\lambda)=-\left\{k_{21}\phi_1(a,\lambda)+k_{22}\phi_2\left(aq^{-1},\lambda\right)\right\}. \tag{2.22}$$

It is also known that if $\{\phi_n(.)\}_{n=-\infty}^{\infty}$ denotes a set of vector-valued eigenfunctions corresponding $\{\lambda_n\}_{n=-\infty}^{\infty}$, then $\{\phi_n(.)\}_{n=-\infty}^{\infty}$ is a complete orthogonal set of H_q. For more details about how to obtain the solutions and the eigenvalues for $q-$Dirac system see $[1,2]$, similar to the classical case of Dirac system [14] and $q-$Sturm-Liouville problems $[15,16]$.

3. The Sampling Theory

The WKS (Whittaker-Kotel'nikov-Shannon) $[17-19]$ sampling theorem has been generalized in many different ways. The connection between the WKS sampling theorem and boundary value problems was first observed by Weiss [20] and followed by Kramer $[21]$. In [22], sampling theorem is introduced where sampling representations are derived for integral transforms whose kernels are solutions of one-dimensional regular Dirac systems. In recent years, the connection between sampling theorems and $q-$boundary value problems has been the focus of many research papers. In $[12,23]$, $q-$versions of the classical sampling theorem of WKS as well as Kramer's analytic theorem were introduced. These results were extended to $q-$Sturm-Liouville problems in [13,24], singular q-Sturm-Liouville problem in [25] and the $q,\omega-$Hahn-Sturm-Liouville problem in $[26]$.

In this section, we state and prove $q-$analogue of sampling theorem associated with $q-$Dirac system (1.1)-(1.3), inspired by the classical case $[22]$.

Theorem 3.1. *Let* $f(x)=\begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix} \in H_q$ *and* $F(\lambda)$ *be the* $q-$*type transform*

$$F(\lambda)=\int_0^a f^\top(x)\,\phi(x,\lambda)\,d_qx,\ \lambda\in\mathbb{C}, \tag{3.1}$$

where $\phi(x,\lambda)$ *is the solution defined above. Then* $F(\lambda)$ *is an entire function that can be reconstructed using its values at the points* $\{\lambda_n\}_{n=-\infty}^{\infty}$ *by means of the sampling form*

$$F(\lambda)=\sum_{n=-\infty}^{\infty} F(\lambda_n)\frac{\omega(\lambda)}{(\lambda-\lambda_n)\,\omega'(\lambda_n)}, \tag{3.2}$$

where $\omega(\lambda)$ *is defined in (2.22). The series (3.2) converges absolutely on* $\mathbb{C}$ *and uniformly on compact subsets of* $\mathbb{C}$.

Proof. Since $\phi(x,\lambda)$ is in H_q for any λ, we have

$$\phi(x,\lambda)=\sum_{n=-\infty}^{\infty}\widehat{\phi}_n\frac{\phi_n(x)}{\|\phi_n\|_{H_q}^2}, \tag{3.3}$$

where

$$\widehat{\phi}_n = \int_0^a \phi^\top(x,\lambda)\,\phi_n(x)\,d_qx = \int_0^a \left\{\phi_1(x,\lambda)\,\phi_{n,1}(x) + \phi_2(x,\lambda)\,\phi_{n,2}(x)\right\} d_qx, \tag{3.4}$$

$\phi^\top(x,\lambda) = (\phi_1(x,\lambda),\ \phi_2(x,\lambda))$ and $\phi_n^\top(x) = (\phi_{n,1}(x),\ \phi_{n,2}(x))$ is the vector-valued eigenfunction corresponding to the eigenvalue λ_n.

Since f is in H_q, it has the Fourier expansion

$$f(x) = \sum_{n=-\infty}^{\infty} \widehat{f}_n \frac{\phi_n(x)}{\|\phi_n\|_{H_q}^2}, \tag{3.5}$$

where

$$\widehat{f}_n = \int_0^a f^\top(x)\,\phi_n(x)\,d_qx = \int_0^a \left\{f_1(x)\,\phi_{n,1}(x) + f_2(x)\,\phi_{n,2}(x)\right\} d_qx. \tag{3.6}$$

In view of Parseval's relation and definition (3.1), we obtain

$$F(\lambda) = \sum_{n=-\infty}^{\infty} F(\lambda_n)\,\frac{\widehat{\phi}_n}{\|\phi_n\|_{H_q}^2}. \tag{3.7}$$

Let $\lambda \in \mathbb{C}$, $\lambda \neq \lambda_n$ and $n \in \mathbb{N}$ be fixed. From relation (2.18), with $y_{11}(x) = \phi_1(x,\lambda)$, $y_{12}(x) = \phi_2(x,\lambda)$ and $y_{21}(x) = \phi_{n,1}(x)$, $y_{22}(x) = \phi_{n,2}(x)$, we obtain

$$(\lambda - \lambda_n) \int_0^a \left\{\phi_1(x,\lambda)\,\phi_{n,1}(x) + \phi_2(x,\lambda)\,\phi_{n,2}(x)\right\} d_qx = W(\phi(.,\lambda),\phi_n(.))|_{x=a} - W(\phi(.,\lambda),\phi_n(.))|_{x=0}. \tag{3.8}$$

From (2.19) and the definition of $\phi(.,\lambda)$, we have

$$(\lambda - \lambda_n) \int_0^a \left\{\phi_1(x,\lambda)\,\phi_{n,1}(x) + \phi_2(x,\lambda)\,\phi_{n,2}(x)\right\} d_qx = \phi_1(a,\lambda)\,\phi_{n,2}\left(aq^{-1}\right) - \phi_{n,1}(a)\,\phi_2\left(aq^{-1},\lambda\right). \tag{3.9}$$

Assume that $k_{22} \neq 0$. Since $\phi_n(.)$ is an eigenfunction, then it satisfies (1.3). Hence

$$\phi_{n,2}\left(aq^{-1}\right) = -\frac{k_{21}}{k_{22}}\phi_{n,1}(a). \tag{3.10}$$

Substituting from (3.10) in (3.9), we obtain

$$\begin{aligned}(3.11)\qquad &(\lambda-\lambda_n)\int_0^a\left\{\phi_1(x,\lambda)\,\phi_{n,1}(x)+\phi_2(x,\lambda)\,\phi_{n,2}(x)\right\}d_qx\\ &=-\phi_{n,1}(a)\left\{\frac{k_{21}}{k_{22}}\phi_1(a,\lambda)+\phi_2\left(aq^{-1},\lambda\right)\right\}\\ &=\frac{\omega(\lambda)\,\phi_{n,1}(a)}{k_{22}}\end{aligned}$$

provided that $k_{22}\neq 0$. Similarly, we can show that

$$\begin{aligned}(3.12)\qquad &(\lambda-\lambda_n)\int_0^a\left\{\phi_1(x,\lambda)\,\phi_{n,1}(x)+\phi_2(x,\lambda)\,\phi_{n,2}(x)\right\}d_qx\\ &=\frac{\omega(\lambda)\,\phi_{n,2}\left(aq^{-1}\right)}{k_{21}}\end{aligned}$$

provided that $k_{21}\neq 0$. Differentiating with respect to λ and taking the limit as $\lambda\to\lambda_n$, we obtain

$$\begin{aligned}(3.13)\qquad \|\phi_n\|_{H_q}^2&=\int_0^a\phi_n^\top(x)\,\phi_n(x)\,d_qx\\ &=\frac{\omega'(\lambda_n)\,\phi_{n,1}(a)}{k_{22}},\end{aligned}$$

and

$$\begin{aligned}(3.14)\qquad \|\phi_n\|_{H_q}^2&=\int_0^a\phi_n^\top(x)\,\phi_n(x)\,d_qx\\ &=\frac{\omega'(\lambda_n)\,\phi_{n,2}\left(aq^{-1}\right)}{k_{21}}.\end{aligned}$$

From (3.4), (3.11) and (3.13), we have for $k_{22}\neq 0$,

$$(3.15)\qquad \frac{\widehat{\phi}_n}{\|\phi_n\|_{H_q}^2}=\frac{\omega(\lambda)}{(\lambda-\lambda_n)\,\omega'(\lambda_n)},$$

and if $k_{21}\neq 0$, we use (3.4), (3.12) and (3.14) to obtain the same result. Therefore from (3.7) and (3.15) we get (3.2) when λ is not an eigenvalue. Now we investigate the convergence of (3.2). Using Cauchy-Schwarz inequality for $\lambda\in\mathbb{C}$.

$$\begin{aligned}(3.16)\qquad &\sum_{k=-\infty}^{\infty}\left|F(\lambda_k)\frac{\omega(\lambda)}{(\lambda-\lambda_k)\,\omega'(\lambda_k)}\right|=\sum_{k=-\infty}^{\infty}\left|\widehat{f}_k\frac{\widehat{\phi}_k}{\|\phi_k\|_{H_q}^2}\right|\\ &\leq\left(\sum_{k=-\infty}^{\infty}\left|\frac{\widehat{f}_k}{\|\phi_k\|_{H_q}}\right|^2\right)^{1\backslash 2}\left(\sum_{k=-\infty}^{\infty}\left|\frac{\widehat{\phi}_k}{\|\phi_k\|_{H_q}}\right|^2\right)^{1\backslash 2}<\infty,\end{aligned}$$

since $f(.),\ \phi(.,\lambda)\in H_q$, then the two series in the right-hand side of (3.16) converge. Thus series (3.2) converge absolutely on $\mathbb{C}$. As for uniform convergence on compact subsets of $\mathbb{C}$, let

$\Omega_M := \{\lambda \in \mathbb{C},\ |\lambda| \leq M\}$ M is a fixed positive number. Let $\lambda \in \Omega_M$ and $N > 0$. Define $\Gamma_N(\lambda)$ to be

$$\Gamma_N(\lambda) = \left| F(\lambda) - \sum_{k=-N}^{N} F(\lambda_k) \frac{\omega(\lambda)}{(\lambda - \lambda_k)\,\omega'(\lambda_k)} \right|. \tag{3.17}$$

By Cauchy-Schwarz inequality

$$\Gamma_N(\lambda) \leq \|\phi(.,\lambda)\|_{H_q} \left(\sum_{k=-N}^{N} \frac{\left|\widehat{f}_k\right|^2}{\|\phi_k\|_{H_q}^2} \right)^{1\backslash 2}.$$

Since the function $\phi(.,\lambda)$ is uniformly bounded on the subsets of $\mathbb{C}$, we can find a positive constant C_Ω which is independent of λ such that $\|\phi(.,\lambda)\|_{H_q} \leq C_\Omega$, $\lambda \in \Omega_M$. Thus

$$\Gamma_N(\lambda) \leq C_\Omega \left(\sum_{k=-N}^{N} \frac{\left|\widehat{f}_k\right|^2}{\|\phi_k\|_{H_q}^2} \right)^{1\backslash 2} \to 0 \text{ as } N \to \infty.$$

Hence (3.2) converges uniformly on compact subsets of $\mathbb{C}$. Thus $F(\lambda)$ is an entire function and the proof is complete.

4. Examples

In this section we give three examples illustrating the sampling theorem of the previous section.

Example 4.1. Consider $q-$Dirac system (1.1)-(1.3) in which $p(x) = 0 = r(x)$:

$$\begin{cases} -\frac{1}{q} D_{q^{-1}} y_2 = \lambda y_1, \\ D_q y_1 = \lambda y_2, \end{cases} \tag{4.1}$$

$$y_1(0) = 0, \tag{4.2}$$

$$y_2\left(\pi q^{-1}\right) = 0. \tag{4.3}$$

It is easy to see that a solution (4.1) and (4.2) is given by

$$\phi^{\top}(x,\lambda) = \left(\sin(\lambda x; q),\ \cos(\lambda \sqrt{q} x; q)\right).$$

By substituting this solution in (4.3), we obtain $\omega(\lambda) = \cos\left(\lambda q^{-1\backslash 2}\pi; q\right)$, hence, the eigenvalues are $\lambda_n = \dfrac{q^{1-n+\varepsilon_n(1\backslash 2)}}{(1-q)\pi}$. Applying Theorem 3.1, the $q-$transforms

$$F(\lambda) = \int_0^\pi f^\top(x)\,\phi(x,\lambda)\,d_qx = \int_0^\pi \left\{f_1(x)\sin(\lambda x; q) + f_2(x)\cos\left(\lambda\sqrt{q}x; q\right)\right\} d_qx, \tag{4.4}$$

for some f_1 and $f_2 \in L_q^2(0,\pi)$, then it has the sampling formula

$$F(\lambda) = \sum_{n=-\infty}^{\infty} F(\lambda_n)\frac{\cos\left(\lambda q^{-1\backslash 2}\pi; q\right)}{(\lambda-\lambda_n)\,\omega'(\lambda_n)}. \tag{4.5}$$

Example 4.2. Consider $q-$Dirac equation (4.1) together with the following boundary conditions

$$y_2(0) = 0, \tag{4.6}$$

$$y_1(\pi) = 0. \tag{4.7}$$

In this case $\phi^\top(x,\lambda) = \left(\cos(\lambda x; q),\ -\sqrt{q}\sin\left(\lambda\sqrt{q}x; q\right)\right)$. Since $\omega(\lambda) = \cos(\lambda\pi; q)$, then the eigenvalues are given by $\lambda_n = \dfrac{q^{-n+1\backslash 2+\varepsilon_n(1\backslash 2)}}{(1-q)\pi}$. Applying Theorem 3.1 above to the $q-$transform

$$F(\lambda) = \int_0^\pi \{f_1(x)\cos(\lambda x; q) - f_2(x)\sqrt{q}\sin(\lambda\sqrt{q}x; q)\}\, d_qx, \tag{4.8}$$

for some f_1 and $f_2 \in L_q^2(0,\pi)$, then we obtain

$$F(\lambda) = \sum_{n=-\infty}^{\infty} F(\lambda_n)\frac{\cos(\lambda\pi; q)}{(\lambda-\lambda_n)\,\omega'(\lambda_n)}. \tag{4.9}$$

Example 4.3. Consider $q-$Dirac equation (4.1) together with the following boundary conditions

$$y_1(0) + y_2(0) = 0, \tag{4.10}$$

$$y_2\left(\pi q^{-1}\right) = 0. \tag{4.11}$$

In this case

$$\phi^{\top}(x,\lambda)=\left(\cos(\lambda x;q)-\sin(\lambda x;q),\ -\sqrt{q}\sin(\lambda\sqrt{q}x;q)-\cos(\lambda\sqrt{q}x;q)\right).$$

Since $\omega(\lambda)=-\sqrt{q}\sin\left(\lambda q^{-1\backslash 2}\pi;q\right)-\cos\left(\lambda q^{-1\backslash 2}\pi;q\right)$, then the eigenvalues of this problem are the solutions of equation

$$\sqrt{q}\sin\left(\lambda q^{-1\backslash 2}\pi;q\right)=-\cos\left(\lambda q^{-1\backslash 2}\pi;q\right). \tag{4.12}$$

Applying Theorem 3.1 above to the $q-$transform

$$\begin{aligned}F(\lambda)=&\int_0^{\pi}\{f_1(x)(\cos(\lambda x;q)-\sin(\lambda x;q))\\&-f_2(x)\left(\sqrt{q}\sin\left(\lambda\sqrt{q}x;q\right)+\cos\left(\lambda\sqrt{q}x;q\right)\right)\}d_qx,\end{aligned} \tag{4.13}$$

for some f_1 and $f_2\in L_q^2(0,\pi)$, then we obtain

$$F(\lambda)=\sum_{n=-\infty}^{\infty}F(\lambda_n)\frac{-\sqrt{q}\sin\left(\lambda q^{-1\backslash 2}\pi;q\right)-\cos\left(\lambda q^{-1\backslash 2}\pi;q\right)}{(\lambda-\lambda_n)\,\omega'(\lambda_n)}. \tag{4.14}$$

Conflict of Interests

The authors declare that there is no conflict of interests.

REFERENCES

[1] B.P. Allahverdiev, H. Tuna, One-dimensional q-Dirac equation, Math. Meth. Appl. Sci., 40 (2017), 7287–7306.

[2] F. Hıra, Eigenvalues and eigenfunctions of q-Dirac system, J. Sci. Arts 4 (45) (2018), 963-972.

[3] B.P. Allahverdiev, H. Tuna, Dissipative q-Dirac operator with general boundary conditions, Quaest. Math., 41 (2) (2018), 239-255.

[4] F.H. Jackson, On q-definite integrals, Q. J. Pure Appl. Math., 41 (1910), 193-203.

[5] G. Gasper, M. Rahman, Basic Hypergeometric Series, Cambridge Univ. Press, New York, 1990.

[6] V. Kac, P. Cheung, Quantum Calculus, Springer, New York, 2002.

[7] G. Bangerezako, An Introduction to q-Difference Equations, preprint, 2008.

[8] M.H. Annaby, Z.S. Mansour, q-Fractional Calculus and Equations, Springer, 2056, 2012.

[9] G.E. Andrews, R. Askey, R. Roy, Special Functions, Cambridge Univ. Press, Cambridge, 1999.

[10] M.H. Annaby, Z.S. Mansour, On the zeros of basic finite Hankel transforms, J. Math. Anal. Appl., 323 (2006), 1091-1103.

[11] M.H. Annaby, Z.S. Mansour, A basic analog of a theorem of Pólya, Math. Z., 258 (2008), 363-379.

[12] M.H. Annaby, q-type sampling theorems, Result. Math., 44 (2003), 214-225.

[13] M.H. Annaby, J. Bustoz, M.E.H. Ismail, On sampling theory and basic Sturm-Liouville systems, J. Comput. Appl. Math., 206 (2007). 73-85.

[14] B.M. Levitan, I.S. Sargsjan, Sturm-Liouville and Dirac Operators, Kluwer, Dordrecht, 1991.

[15] M.H. Annaby, Z.S. Mansour, Basic Sturm-Liouville problems, J. Phys. A:Math. Gen., 38 (2005), 3775-3797.

[16] M.H. Annaby, Z.S. Mansour, Asymptotic formulae for eigenvalues and eigenfunctions of q-Sturm-Liouville problems, Math. Nachr., 284 (2011), 443-470.

[17] E. Whittaker, On the functions which are represented by the expansion of the interpolation theory, Proc. Roy. Soc. Edinburgh Sect., A 35 (1915), 181–194.

[18] V. Kotel'nikov, On the carrying capacity of the "ether" and wire in telecommunications, in: Material for the All-Union Conference on Questions, Izd. Red. Upr. Svyazi RKKA, Moscow, 1933.

[19] C.E. Shannon, Communications in the presence of noise, Proc. IRE 37 (1949), 10–21.

[20] P. Weiss, Sampling theorems associated with Sturm–Liouville systems, Bull. Amer. Math. Soc., 163 (1957), 242.

[21] H.P. Kramer, A generalized sampling theorem, J. Math. Phys., 38 (1959), 68–72.

[22] A.I. Zayed, G.A. García, Sampling theorem associated with a Dirac operator and the Hartley transform, J. Math. Anal. Appl., 214 (1997), 587-598.

[23] M.E.H. Ismail, A.I. Zayed, A q-analogue of the Whittaker-Shannon-Kotel'nikov sampling theorem, Proc. Amer. Math. Soc., 131 (2003), 3711-3719.

[24] L.D. Abreu, Sampling theory associated with q-difference equations of the Sturm-Liouville type, J. Phys. A:Math. Gen., 38 (2005), 10311-10319.

[25] M.A. Annaby, H.A. Hassan, Z.S. Mansour, Sampling theorems associated with singular q-Sturm-Liouville problems, Results. Math., 62 (2012), 121-136.

[26] M.A. Annaby, H.A. Hassan, Sampling theorems for Jackson-Nörlund transforms associated with Hahn-difference operators, J. Math. Anal. Appl. 464 (2018), 493-506.