Available online at http://scik.org
J. Math. Comput. Sci. 9 (2019), No. 2, 206-224
https://doi.org/10.28919/jmcs/3965
ISSN: 1927-5307

FINDING THE APPROXIMATE ANALYTICAL SOLUTIONS OF 2n ($n \epsilon \mathbb{R}$) ORDER DIFFERENTIAL EQUATION WITH BOUNDARY VALUE PROBLEM USING VARIOUS TECHNIQUES

SHAILLY MAHAJAN[1,*], AND SUBASH KUMAR[2]

[1]Mewar University, Chittorgarh (Rajasthan), India

[2]Pathankot College of Management &Technology, Pathankot (Punjab), India

Abstract: This paper judges against the errors estimated by approximate analytical solutions obtained using homotopy perturbation method (HPM), and modified power series method. HPM is a combination of traditional perturbation method and the homotopy method. A numerical example has been considered to demonstrate the effectiveness, exactness and implementation of the method and the results of errors are compared. To attain sufficiently exact results with HPM, it is generally required to calculate at least two statements of the S -terms. However, it was exposed in the numerical examples that highly accurate results were obtained by calculating only one S-term of the series, revealing the effectiveness of the HPM solution. It is concluded that HPM is a powerful tool for solving high-order boundary value problem as it shows less error than MPSAM.

Keywords: homotopy perturbation method; modified power series method; higher order boundary value problems; approximate analytical solution; Error estimates.

2010 AMS Subject Classification: 34K10.

*Corresponding author
E-mail address: shaillymahajan@rediffmail.com

Received November 22, 2018

1. INTRODUCTION

The homotopy perturbation method applies to solve a system of differential equation. We consider a boundary value problem

$$y^{12}(t)+f(t)y(t)=g(t), \qquad t=[\ell,m]$$
$$y(\ell)=a, y^{1}(\ell)=b, y^{2}(\ell)=c, y^{3}(\ell)=d, y^{4}(\ell)=e, y^{5}(\ell)=f$$
$$y\ (m)=g\ , y^{1}(m)=h, y^{2}(m)=i, y^{3}(m)=j, y^{4}(m)=k, y^{5}(m)=l$$

where a, b ,c ,d ,e ,f , g , h ,I ,j ,k , l are finite real constant and the functions f (t) and g (t) are continuous on [ℓ ,m].Liao [3] proposed a powerful analytic method for nonlinear problem, namely the homotopy perturbation method. The solution of the boundary value problems has been obtained in term of convergent series with easily computable mechanism. The homotopy perturbation method (HPM) was introduced by He [1], [2] by J. H. In (HPM) method the solution is well thought-out as the summation of an infinite series which frequently converges swiftly to the exact solution. This trouble-free method has been applied to solve linear and non-linear equations. Since He applies this method for solving Blasius equation [4]. Obtained solutions in contrast with earlier HPM results provide the higher accuracy.

The modified PSAM developed in [6] takes a slight similarity of the Taylor series method (TSM). However, the methodology differs. In the case of TSM, we determine the successive derivative of the boundary value problem and estimate each derivative using the boundary conditions. in conclusion, to fulfill the boundary conditions, we assess the solution at such a point, and then the resultant arrangement of equations is solved to obtain the unknowns. While in the modified PSAM as in Power Series Approximation Method (PSAM) developed in [19] for the solution of generalized Nth order boundary value problems, the BVP and its boundary conditions are first transformed into systems of ODEs. The general solution is then given in power series in t where the constant $a_i s$equals $\frac{\alpha_i}{i!}$ and $\alpha_i s$ are the boundaries correspondingly.

To fulfill the boundary conditions, we estimate the general solution at such a point, say, t = 1; then the resultant classification of equations is solved to obtain the unknowns. Also, Z_P and Z_C are termed the particular and complementary solution are used for more compact generalized series solution. Numerical examples are well thought-out to show the rate of convergence of the modified PSAM as compared with the HPM and analytic solution obtainable in the text.

2. HOMOTOPY PERTURBATION METHOD.

We consider the following nonlinear differential equation

$$A(p) - \mathrm{f}\,(q) = 0, \quad \mathrm{q} \in \Omega \tag{1.1}$$

With the boundary condition

$$B\left(p, \frac{\partial p}{\partial u}\right) = 0, q \in \Gamma \tag{1.2}$$

Where A, B is a common differential operator and boundary operator respectively, u is known analytical function, and Γ is the boundary of the domain **Ω**. The operator A can be separated into two parts L and N, where L is linear, while N is nonlinear. So (1.1) can be rewritten as

$$L(p) + \mathrm{N}(p) - \mathrm{f}\,(q) = 0. \tag{1.3}$$

By Liao [3], we can construct a homotopy

$\nu(\mathrm{q}, \mathrm{S}) : \Omega \times [0, 1] \rightarrow \mathrm{R}$ which satisfies

$$H(\nu, \mathrm{S}) = (1 - \mathrm{S})[\mathrm{L}\,(\nu) - \mathrm{L}\,(p_0)] + \mathrm{S}[A(\nu) - \mathrm{f}\,(q)] = 0, \mathrm{S} \in [0, 1] \in \Omega \tag{1.4}$$

$$H(\nu, \mathrm{S}) = \mathrm{L}\,(\nu) - \mathrm{L}\,(p_0) + \mathrm{SL}\,(p_0) + \mathrm{S}[N(\nu) - \mathrm{f}\,(q)] = 0 \tag{1.5}$$

Where q ∈ Γ and S∈ [0, 1] is an embedding parameter, u_0 is an initial approximation of (1.1), which satisfies the boundary conditions. Perceptibly from Equations (1.4) and (1.5) we will have:

$$H(\nu, 0) = \mathrm{L}\,(\nu) - \mathrm{L}\,(p_0) = 0 \tag{1.6}$$

$$H(\nu, 1) = \mathrm{A}(\nu) - \mathrm{f}\,(q) = 0, \tag{1.7}$$

Altering process of S from zero to unity is just that of H(ν,S) from L (ν) –L(p_0) to A(ν) – f (q). In topology, this is called deformation, L (ν) – L (p_0) and A(ν) – f (q) is called homotopic. The embedding parameter S is introduced a great deal more logically, unaltered by mock factors. Additionally, it can be considered as a small parameter for 0 < S ≤ 1. So, it is very normal to suppose that the solution of (1.4), (1. 5) can be written as

$$\nu = \nu_0 + \mathrm{s}\nu_1 + \mathrm{s}^2\nu_2 + \ldots. \tag{1.8}$$

When s=1 then (1.8) become

$$\mathrm{p} = \lim_{\mathrm{s}\to 1} . \nu_0 + \nu_1 + \nu_2 + \ldots$$

The grouping of the perturbation method and the homotopy method is called the HPM, which eliminates the drawbacks of the traditional perturbation methods while observance of all its reward. The series (1.8) is convergent for nearly all cases. We take the common higher order boundary value problems of the type

$$y^{2u}(t) = f\,(t, y, y'(t), \ldots\ldots\ldots, y^{(2u-1)}(t)),\ 0 < t < 1 \tag{1.9}$$

Boundary condition

$$y^{(2k)}(0) = \ell_{2k} \qquad k = 0,1,2,3,\ldots,(u-1) \tag{1.10}$$

$$y^{(2k)}(1) = m_{2k} \qquad k = 0,1,2,3,\ldots,(u-1) \tag{1.11}$$

where $f\,(t, y, y^{'}(t), \ldots, y^{(2u-1)}(t))$ and y (t) are assumed real and for t ∈ [0, 1],

ℓ^{2k} and m^{2k}, , k = 0, 1, 2… (u – 1) are real finite constants,

The constants ℓ_{2k},k = 0, 1, 2, 3,…, (u – 1) illustrate the even order derivatives at the boundary t = 0.Using the transformation

$$y = y_1, \frac{dy}{dt} = y_2, \frac{d^2y}{dt^2} + y_3\ ,\ldots\ldots \frac{d^{2n-1}y}{dt^{2u-1}} + y_u, \frac{dy}{dx} = f\,(t, y, y_1, y_2, \ldots y_{2u-1}(x)) \tag{1.12}$$

$$y_1(\ell) = \ell_{0,}\ \ y_2(\ell) = \ell_1, y_3(\ell) = \ell_2, \ldots\ldots\ldots, y_{2u}(\ell) = \ell_{2u-1}, \tag{1.13}$$

$$y_1(m) = m_0, y_2(m) = m_1, y_3(m) = m_2, \ldots\ldots\ldots, y_{2u}(m) = m_{2u-1}, \tag{1.14}$$

We can write as

$$\begin{aligned} y_1 &= a + \int_0^t y_2(x)dx, \\ y_2 &= b + \int_0^t y_3(x)dx, \\ y_3 &= c + \int_0^t y_4(x)dx, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ y_{2u} &= a_{2u-1} + \int_0^t f\{x, y_1(x), y_2, (x), y_3(x), y_4(x) \ldots\ldots y_2(x)\}dx, \end{aligned} \tag{1.15}$$

Let’s consider (1.15)

$$\begin{aligned} L\,(y_{1,}\, y_2, y_3, \ldots, y_{2u}) = L_1(y_1, y_2, y_3, \ldots, y_{2u}), L_2(y_1, y_{2,}\, y_3, \ldots, y_{2u}), \ldots \\ \ldots\ldots\ldots\ldots L_u(y_{1,}\, y_2, y_3, \ldots, y_{2u}) = 0 \end{aligned} \tag{1.16}$$

This has solution $(f_1, f_2, \ldots\ldots, f_{2u})$

Where $L(y_{1,}\ y_2, y_3, \ldots, y_{2u}) = y_1 - \ell_0 - \int_0^t y_2(x)dx$ ………………………………

$$L_{2u}(y_{1,}\ y_2, y_3, \ldots, y_{2u}) = y_{2u} - \ell_{2u-1} - \int_0^t f\{x, y_1(x), y_2,(x), y_3(x), y_4(x)......y_{2n-1}(x)\}dx \quad (1.17)$$

By definition of Homotopy $H(y_{1,}\ y_2, y_3, \ldots, y_{2u}, S)$

$$H(y_{1,}\ y_2, y_3, \ldots, y_{2u}, 0) = F(y_1, y_2, y_3, \ldots, y_{2u}), H(y_1, y_{2,}\ y_3, \ldots, y_{2u}1) = L_n(y_{1,}\ y_2, y_3, \ldots, y_{2u}) \quad (1.18)$$

Where $F(y_1, y_2, y_3, \ldots, y_{2u}) = [F_1(y_1, y_2, y_3, \ldots, y_{2u}), \ldots\ldots, F_{2u}(y_1, y_2, y_3, \ldots, y_{2u})\ ]^T$ =

$$= \quad [y_1 - \ell_0, \ldots. y_{2u-1} - \ell_{2u-2}, y_{2u} - \ell_{2u-1}\ ]^T \quad (1.19)$$

$$H(y_{1,}\ y_2, y_3, \ldots, y_{2u}, S) = [H_1(y_1, y_2, y_3, \ldots, y_{2u}, S), H_n(y_1, y_{2,}\ y_3, \ldots, y_{2u}, S)]^T \quad (1.20)$$

Select a convex homotopy by:

$$H(y_{1,}\ y_2, y_3, \ldots, y_{2u}, S) = (1-S)\ F(y_1, y_2, y_3, \ldots, y_{2u}) + S\,L(y_1, y_{2,}\ y_3, \ldots, y_{2u}) = 0 \quad (1.21)$$

The convex homotopy (1.21) incessantly mark out an absolutely defined carve from a starting point $H(y_1 - \ell_0, \ldots. y_{2u-1} - \ell_{2u-2}, y_{2u} - \ell_{2u-1}, 0)$ be a solution $, H(f_1, f_{2,}\ f_3, \ldots, f_{2u}1)$ parameter **S** monotonically increasing from 0 to 1 as trivial problem $, F(y_1, y_{2,}\ y_3, \ldots, y_{2u}) = 0$.

And original problem

$$L(y_1, y_{2,}\ y_3, \ldots, y_{2u}) = 0$$

The uses the homotopy parameter **S** as an expanding parameter

$$y_1 = y_{10} + S\,y_{11} + S^2 y_{12} + S^3 y_{13} +, \ldots, y_{2u} = y_{2u_0} + S y_{2u_1} + S^2 y_{2u_2} + \ldots\ldots$$

Solution of (1.9) become

$$f_1 = \lim_{S\to 1} y_1 = y_{10} + y_{11} + y_{12} + y_{13} + \ldots\ldots\ldots\ldots$$

$$f_{2n} = \lim_{S\to 1} y_1 = y_{2n_0} + y_{2n_1} + y_{2n_2} + y_{2n_3} + \ldots\ldots\ldots\ldots \quad (1.22)$$

Convergence of the above series for the application of (HPM) to (1.15) rewrites {1.21) as follows

$$H_1(y_{1,}\ y_2, y_3, \ldots, y_{2u}, S) = (1-S)\ F_1(y_1, y_2, y_3, \ldots, y_{2u}), S\,L_1(y_1, y_{2,}\ y_3, \ldots, y_{2u}) = 0$$

………………………………………………………………………………

$$H_{2u}(y_{1,}\ y_2, y_3, \ldots, y_{2u}, S) = (1-S)\ F_{2u}(y_1, y_2, y_3, \ldots, y_{2u}), S\,L_{2u}(y_1, y_{2,}\ y_3, \ldots, y_{2u}) = 0 \quad (1.23)$$

Put in (1.17), (1.18), (1.20) into (1.23) we get

$$(1-S)(y_1-\ell_{0)})+(S)(y_2-\ell_1)\ -\int_0^t y_2(x)dx=0$$

……………………………………………………………

$$(1-S)(y_{2u}-\ell_{2u-1)})+(S)(y_{2u}-\ell_{2u-1})\ -\int_0^t f\{x,y_1(x),y_2,(x),y_3(x),y_4(x)......y_{2u-1}(x)\}dx=0 \quad (1.24)$$

By equating the terms with identical powers of **S**, like S^0. , S^1. , S^2. …., S^{2k}. Combining all the terms of Equations give the solution of the problem, by using the boundary conditions (1.10) and (1.11) we can obtain all parameters.

3. MODIFIED POWER SERIES APPROXIMATION METHOd (MPSAM)

We will investigate the PSAM and MPSAM as stated in [19] and [6],

$$y^n(t)+f(t)y(t)=g(t),\quad 0\le t\le 1 \tag{2.1}$$

With the boundary conditions

$$y^m(0)=\alpha_m,\quad y^m(1)=\beta_m,\qquad m=0,1,2,3,.........(n-1) \tag{2.2}$$

Where f (t), g (t), y (t) is assumed real and continuous on $0\le t\le 1$, α_m,β_m are finite real constants.

The given *n*th order BVP (2.1), (2.2) are transformed to systems of ODEs such that we have with the boundary conditions

$$y',(0)=\alpha_0,y''(0)=\alpha_1,y'''(0)=\alpha_2,..........y^n(0)=\alpha_{n-1}, \tag{2.4}$$

and $\frac{dy}{dt}=y',\frac{dy'}{dt}=y'',\frac{dy''}{dt}=y''',\frac{dy'''}{dt}=y'^v,..............................\frac{dy^n}{dt}=g(t)-f(t)y(t)$ (2.3)

$$y'(1)=\beta_0,y''(1)=\beta_1,y'''(1)=\beta_2,..........y^{2n}(1)=\beta_{n-1}, \tag{2.5}$$

By modified PSAM app. Solution is uniquely given as

$$y^n(t)=\sum_{i=0}^{N}a_it^i,\quad N<\infty \qquad ,a_n=\frac{\alpha_n}{n!} \tag{2.6}$$

Where a_i, i= 0(1) N are unknown constants to be determined and $t\in[0,1]$

If the definitions and propositions of equations (2.1) – (2.6) are sustained, then a more compact generalized series solution at the primary boundary, t = 0 can be written as

$y^n = Z_p + Z_C$ (2.7)

where $\mathbf{Z}_p$ and $\mathbf{Z}_C$ are termed the particular and complementary solutions, and are defined as

$$\mathrm{Zp} = \sum_{i=0}^{\frac{n-1}{2}} \frac{\alpha_i}{i!} t^i \ , \mathrm{Z_C} = \sum_{i=\frac{n}{2}}^{n-1} \frac{t_i}{i!} c_i$$

Where c_i, $i = \frac{n}{2}.....,(n-1)$ are the unknowns to be estimated, and n is the order of the boundary value problem. To prove the above result. We assume a series approximation of a Power series of the form

$$y^n = \sum_{i=0}^{\frac{n-1}{2}} a_i t^i + \sum_{i=\frac{n}{2}}^{n-1} a_i t^i c_i \ \ where \ \ \mathrm{Zp} = \sum_{i=0}^{\frac{n-1}{2}} a_i t^i \ , \mathrm{Z_C} = \sum_{i=\frac{n}{2}}^{n-1} a_i t^i c_i \ \ \text{both Zp,} \mathrm{Z_C} \text{ are individual} \quad (2.8)$$

Put the value of Zp in (2.3) at t=0 we get $\mathrm{y}' = a_1 + i\sum_{i=2}^{\frac{n-1}{2}} a_i t^{i-1}$ (2.9)

$$\mathrm{y}'(0) = \alpha_0, a_1 = \alpha_0 \Rightarrow \mathrm{y}' = \alpha_0 + i\sum_{i=2}^{\frac{n-1}{2}} a_i t^{i-1} \quad (2.10)$$

proceed in this way

$$\mathrm{y}''(0) = \alpha_1, a_2 = \frac{\alpha_1}{2!} \Rightarrow y'' = \alpha_1 + i(i-1)\sum_{i=3}^{\frac{n-1}{2}} a_i t^{i-2} \quad (2.11)$$

Continuing in this way we will get

Similarly we will solve for $\mathrm{Z_C}$ at t = 0, here m= 1, as $m = 0(1)(n-1)$. Substituting $\mathrm{Z_C}$ in (2.3) as we solved for of Zp, we get

$$a_{i+1} = \frac{\alpha_i}{(i+1)!} i \geq 0 \quad (2.14)$$

$$\text{thus } \mathrm{Z}_C = \sum_{i=\frac{n}{2}}^{\frac{n-1}{2}} \frac{t_i}{i!} c_i \quad (2.15)$$

put (2.13) and (2.15) in (2.7) we get $\mathrm{y_n} = \sum_{i=0}^{\frac{n-1}{2}} \frac{\alpha_i}{i!} t^i + \sum_{i=\frac{n}{2}}^{n-1} \frac{t_i}{i!} c_i$ (2.16)

This is required result

All the unknowns c_i . $i=\frac{n}{2}......(n-1)$ in (2.16) are estimated at secondary boundary x=1 and following relation

$$\sum_{i=\frac{n}{2}}^{n-1}\frac{c_i}{1!}=\beta_0 \ -\sum_{i=0}^{\frac{n-1}{2}}\frac{\alpha_i}{i!}\ , \quad \sum_{i=\frac{n}{2}}^{n-1}\frac{ic_i}{1!}=\beta_1 \ -\sum_{i=0}^{\frac{n-1}{2}}\frac{i\alpha_i}{i!} \qquad (\ 2.17)$$

$$\sum_{i=\frac{n}{2}}^{n-1}\frac{i^2c_i-ic_i}{1!}=\beta_1 \ -\sum_{i=0}^{\frac{n-1}{2}}\frac{i^2\alpha_i-i\alpha_i}{i!} \qquad (2.18)$$

$$\sum_{i=\frac{n}{2}}^{n-1}\frac{i^2c_i[pochhammer(i-n+1,n)]}{1!}=\beta_{n-1} \ -\sum_{i=0}^{\frac{n-1}{2}}\frac{i^2\alpha_i[pochhamer(i-n+1,n)]}{i!} \qquad (2.19)$$

where pochhammer is notation with negative index

4. Differential Transformation for Boundary Value Problem

Suppose differential transformation of deflection function Y(x) [16]

$$Y(K)=\frac{1}{k!}[y^k(x)]_{x=x_0} \qquad (2.20)$$

Where $x_0=0$,y(x) in the term of Differential transformation;

$$y(x)=\sum_{k=0}^{\infty}\left[\frac{(x-x_0)^k}{k!}\right]Y(k)\Rightarrow y(x)=\sum_{k=0}^{\infty}\left[\frac{(x)^k}{k!}\right]Y(k) \qquad (2.\ 21)$$

After simplification some recurrence equations are as m=0,1,2,3…..

$$Y(2m+1)=\sum_{k=0}^{\infty}\left[\frac{(2m)!}{(2m+k)!}\right]Y(.,.) \qquad (2.\ 22)$$

$$Y(2m+k+1)=\sum_{k=0}^{\infty}\left[\frac{(2m+1)!}{(2m+k+1)!}\right]Y(.,.) \qquad (2.23)$$

Y(.,.) is transformation function of linear or non linear function of f(x, y)

At x=0 boundary value problem in case of even –order or odd order. Respectively

$$Y(2l)=\frac{1}{2l}\alpha_{2l,} \qquad l=0,1,2,3...(2m-1) \qquad (2.24)$$

$$Y(2l+1)=\frac{1}{(2l+1)}\beta_{(2l+1),} \quad l=0,1,2,3...2m \tag{2.25}$$

Put (2.24),(2.25) in (2.22),(2.23) for $l=0,1,2,3...(m-1)$

$$y(x)=\sum_{k=0}^{\infty}[\frac{1}{(2l)!}\alpha_{2l,}\ ]Y(k)x^k \quad l=0,1,2,3...m \tag{2.26}$$

$$y(x)=\sum_{k=0}^{\infty}\left[\frac{1}{(2l+1)!}\right]\beta_{2l+1,}\ ]Y(k)x^k$$

where $Y^{(2r+1)}(0)=\delta_r, r=0,1,2....(m-1)$ *and* $y^{(2r)}(0)=\lambda_r\ , r=0,1,2....(m)$ are constant.

4. MAIN RESULTS

In this section linear and nonlinear BVPs will be tested by the Homotopy perturbation method and modified power series method of the twelfth and tenth order.

Example1: Consider the following linear twelfth- order problem solve first using HPM and by Modified PSAM ,and then by DTM

$$y^{(12)}(t)=2e^t y^2(t)+y^{(3)}(t)$$

With the following boundary conditions

$$y_0^0=1, y_0^2=1, y_0^4=1, y_0^6=1,\ y_0^8=1, y_0^{10}=1,$$
$$y_1^0=1, y_1^2=1, y_1^4=1, y_1^6=1, y_1^8=1, y_1^{10}=0.367879441$$

The exact solution is $y(t)=e^{-t}$ using the transformation (1.12)

$$y_1=1+\int_0^t y_2(x)dx\ \ y_2=a+\int_0^t y_3(x)\,dx\ \ y_3=1+\int_0^t y_4(x)\,dx\ \ y_4=b+\int_0^t y_5(x)\,dx\ \ y_5=1+\int_0^t y_6(x)\,dx$$

$$y_6=c+\int_0^t y_7(x)\,dx\ \ y_7=1+\int_0^t y_8(x)\,dx\ \ y_8=d+\int_0^t y_9(x)\,dx\ \ y_9=1+\int_0^t y_{10}(x)\,dx\ \ y_{10}=e+\int_0^t y_{11}(x)\,dx$$

$$y_{11}=1+\int_0^t y_{12}(x)\,dx\ \ y_{12}=f+\int_0^t 2e^t y_1^2(x)+y_1^3(x)dx$$

Comparing coefficient of S

Comparing coefficient of S^0

$y_{10}=1$	$y_{20}=a$	$y_{30}==1$	$y_{40}=b$	$y_{50}=1$	$y_{60}=c$
$y_{70}=1$	$y_{80}=d$	$y_{90}=1$	$y_{100}=e$	$y_{110}=1$	$y_{120}=f$

Coefficient of S^1

$y_{11}=at$	$y_{21}=t$	$y_{31}=bt$	$y_{41}=x$	$y_{51}=cx$	$y_{61}=x$
$y_{71}=dt$	$y_{81}=t$	$y_{91}=et$	$y_{101}=x$	$y_{111}=fx$	$y_{121}=2e^{x}-2$

Coefficient of S^2

$y_{12}=\frac{x^2}{2}$	$y_{22}=b\frac{x^2}{2}$	$y_{32}=\frac{x^2}{2}$	$y_{42}=c\frac{x^2}{2}$
$y_{52}=\frac{x^2}{2}$	$y_{62}=d\frac{x^2}{2}$	$y_{72}=\frac{x^2}{2}$	$y_{82}=e\frac{x^2}{2}$
$y_{92}=\frac{x^2}{2}$	$y_{102}=f\frac{x^2}{2}$	$y_{112}=2e^{x}-2x-2$	$y_{122}=4axe^{x}-4ae^{x}+4a$

Coefficient of S^3

$y_{13}=\frac{t^3}{6}$	$y_{23}=c\frac{t^3}{6}$
$y_{33}=\frac{t^3}{6}$	$y_{43}=d\frac{t^3}{6}$
$y_{53}=\frac{t^3}{6}$	$y_{63}=e\frac{t^3}{6}$
$y_{73}=\frac{t^3}{6}$	$y_{83}=f\frac{t^3}{6}$
$y_{93}=b\frac{t^3}{6}+2e^{t}-2t-2-t^{2}$	$y_{103}=\frac{t^3}{6}+4a(te^{t}-2e^{t}+2e^{t}+2+t)$
	$y_{103}=c\frac{t^3}{6}+2(a^{2}+1)(t^{2}e^{t}-2te^{t}+2e^{t}-2)$

Coefficient of S^4

$$y_{14} = c\frac{t^4}{24} \qquad y_{24} = \frac{t^4}{24}$$

$$y_{34} = d\,\frac{t^4}{24} \qquad y_{44} = \frac{t^4}{24}$$

$$y_{54} = e\frac{t^4}{24} \qquad y_{64} = \frac{t^4}{24}$$

$$y_{74} = f\,\frac{t^4}{24} \qquad y_{84} = b\frac{t^4}{24} - \frac{t^3}{3}2e^t - 2t - 2 - t^2$$

$$y_{94} = \frac{t^4}{24} + 4a(te^t - 3e^t + 3 + t + \frac{t^2}{2})$$

$$y_{104} = c\frac{t^3}{6} + 2(a^2+1)(t^2e^t - 4te^t + 6e^t - 2t - 6) + \frac{t^4}{24}$$

$$y_{114} = 2(a \;\; + \frac{b}{3}) + 2(a^2+1)(t^3e^t - 3t^2e^t + 6te^t - 6e^t + 6) + \frac{t^4}{24}$$

Coefficient of S^5

$$y_{15} = \frac{t^5}{120} \qquad y_{25} = d\,\frac{t^5}{120}$$

$$y_{35} = \frac{t^5}{120} \qquad y_{45} = e\frac{t^5}{120}$$

$$y_{55} = \frac{t^5}{120} \qquad y_{65} = f\,\frac{t^5}{120}$$

$$y_{75} = \frac{t^5}{120} \qquad y_{85} = 4a(te^t - 4e^t + 4 + 3t + \frac{t^3}{6} + \frac{t^2}{2}) \;\; + \frac{t^5}{120}$$

$$y_{95} = c\frac{t^5}{120} + 2(a^2 \;\; + 1)(t^2e^t - 6te^t + 12e^t - 6t - t^2 \;\; - 6)$$

$$y_{105} = 2(a + \frac{b}{3}) + (t^3e^t - 6t^2e^t + 18te^t - 24e^t + 6t - 24) + \frac{t^5}{120})$$

$$y_{114} = \frac{2}{3}(1 + ab) + (t^4e^t - 4t^3e^t + 12t^2e^t + 24e^t - 24te^t - 24) + d\,\frac{t^5}{120})$$

Proceeding in this way we get S^{12} then adding above coefficient from S^0 to S^{12}then

$$y^{12}(x) = 1 + \frac{1}{2}t^2 + \frac{1}{24}t^4 + \frac{1}{720}t^6 + \frac{1}{40320}t^8 + \frac{1}{3628800}t^{10} + \frac{1}{239500800}t^{12} + \frac{1}{39916800}ft^{11} + \frac{1}{362880}et^9 + at + \frac{1}{120}ct^5 + \frac{1}{6}bt^3 + \frac{1}{5040}dt^7$$

Using boundary condition (1.26) then we get

$. a = 0.9999940293,\ b = 1.000058885,\ c = 0.9994190942,$

$d = 1.005725028,\ e = 0.9434337955,\ f = 1.632120555$

$$y^{12}(x) = 1 - 0.9999940293t - 0.1666764809t^3 + \frac{1}{24}t^4 - 0.008328492451t^5 + \frac{1}{720}t^6 - 0.0001995486167\,t^7 + \frac{1}{40320}t^8 - 0.208710^{-9}t^9 + \frac{1}{3628800}t^{10} - 0.4088806105t^{11} + \frac{1}{239500800}t^{12}$$

Now we solve same problem using Modified PSAM

Using the boundary condition

$$y^{2k}(0) = 1,\quad y^{2k}(1) = e^{-1},\qquad k = 0,1,2,3,.........11 \tag{2.27}$$

The exact solution is $y(t) = e^{-t}$ by modified PSAM the general solution at boundary t=0

Now this expansion is given by

$$y^{12}(x) = 1 + tc_6 + \frac{1}{2}t^2 + \frac{1}{6}c_7t^3 + \frac{1}{24}t^4 + \frac{1}{120}c_8t^5 + \frac{1}{720}t^6 + \frac{1}{5040}c_9t^7 + \frac{1}{40320}t^8 + \frac{1}{362880}c_{10}t^9 + \frac{1}{3628800}t^{10} + \frac{1}{39916800}c_{11}t^{11} \tag{2.28}$$

Estimating c_i, $i = 6(7)11$ at the boundary t = 1 also, $c_6 = -1.000003$, $c_7 = -0.999973$, $c_8 = -1.000262$, $c_9 = -0.997392$, $c_{10} = -1.026767$ and $c_{11} = -0.632121$. Thus

$$y^{12}(x) = 1 - 1.000003t + \frac{1}{2}t^2 - 0.166662167t^3 + \frac{1}{24}t^4 - 0.00833551667t^5 + \frac{1}{720}t^6 - 0.000197895238t^7 + \frac{1}{40320}t^8 - 2.8294946E10^{-6}t^9 + \frac{1}{3628800}t^{10} - 1.58359638E10^{-8}t^{11}$$

Finally, we consider same problem of twelfth order subject to boundary condition

$$y^{2k}(0) = 1,\quad y^{2k}(1) = \frac{1}{e},\qquad k = 0,1,2,3,.........11 \tag{2.29}$$

Applying the operation of DTM, the following recurrence relation is obtained

$$y(K+12)=\frac{K!}{(K+12)!}\left[2\sum_{l=0}^{K}\sum_{S=0}^{l}\frac{1}{S!}(l-s)Y(K-1)+(k+1)(k+2)(K+3)Y(K+3)\right] \quad (2.30)$$

By using Eq, (220)]the following transformed boundary conditions at x=0 can be obtained

$$y(0)=1,\quad y(2)=\frac{1}{2!},y(4)=\frac{1}{4!}\ ,y(6)=\frac{1}{6!},y(8)=\frac{1}{8!},y(10)=\frac{1}{10!} \quad (2.31)$$

Where According to Eq (2.1)in [14] we have

$$y(1)=a_1=y'(0),\quad y(3)=a_2=\frac{y'''(0)}{3!}\quad ,y(5)=a_3=\frac{y^{v}(0)}{5!},\ y(7)=a_4=\frac{y^{vii}(0)}{7!}$$

$$y(9)=a_5=\frac{y^{ix}(0)}{9!}\ ,y(11)=a_6=\frac{y^{xi}(0)}{11!} \quad (2.32)$$

The constant $a_1,\ a_{2,}\ a_3,\ \ a_5\ \ a_{4,}\ a_6,$ are evaluated from the boundary condition. Given in equation (2.21) for x=1 by taking

Now let's solve same problem using differential transformation method

subject to boundary conditions [11]

$$Y^{(2l)}(0)=1,\ l=0,1,2..3,4.5\ \ and\ \ y^{(2l)}(1)=e^{-1}\ ,l=0,1,2.,4,5 \quad (2.33)$$

Applying DTM we get recurrence relation is The constant a_1,a_2,a_3,a_4,a_5,a_6 are evaluated from boundary condition by taking N=16

$$a_1,=-0.9999999967a_2=-0.1666666720,a_3=-0.0083333307,a_4=-0.0001984133,$$
$$a_5=-0.0000027557,a_6=-0.0000000251$$

then using inverse transformation rule in (2.21) we will get solution up to N=16

$$\begin{aligned}
y(x) = {} & 1-x+0.5x^2-0.166667x^3+0.0416667x^4-0.00833333x^5+0.00138889x^6\\
&-0.0000198113x^7+0.0000218016x^8-2.75566E-6x^9+2.75573E-7x^{10}\\
&-2.50566E-8x^{11}+2.08768E-9x^{12}-1.6059E-10x^{13}+1.11707E-11x^{14}-7.64716E-13x^{15}\\
&+4.77946E-14x^{16}+O(17)
\end{aligned} \quad (2.34)$$

Numerical result for non linear twelfth order BVP

Table (1. 1): Shows the comparison of results obtained by the HPM and modified PSAM using the error estimates and DTM

Table 1.1(Error Estimate)

X	Exact solution	Numerical solution of HPM	Numerical solution of MPSAM	Numerical solution of DTM(N=16)	Error of HPM	Error of MPSAM	Error of DTM
0.0	1.00000000	1.000000000	1.000000000	1.000000000	0.00000	0.000	0
0.1	0.904837418	0.904837579	0.9048371537	0.9048374184	-1.61×10^{-7}	2.6430×10^{-7}	-4×10^{-10}
0.2	0.818730753	0.818731060	0.8187302503	0.8187307537	-3.07×10^{-7}	5.0280×10^{-7}	-7×10^{-10}
0.3	0.740818221	0.740818643	0.7408175284	0.7108182215	-4.22×10^{-7}	6.9230×10^{-7}	0.03
0.4	0.670320046	0.670320543	0.6703192325	0.6703200170	-4.97×10^{-7}	8.1350×10^{-7}	2.9×10^{-8}
0.5	0.606530659	0.606531182	0.6065298043	0.6065306608	-5.21×10^{-7}	8.5540×10^{-7}	-1.8×10^{-9}
0.6	0.548811636	0.548812133	0.5488108225	0.5488116371	-4.98×10^{-7}	8.1360×10^{-7}	-1.1×10^{-9}
0.7	0.496585304	0.496585726	0.4965846118	0.4965853046	-4.22×10^{-7}	6.9200×10^{-7}	-6×10^{-10}
0.8	0.449328964	0.449329710	0.4493284612	0.449329746	-3.07×10^{-7}	5.0290×10^{-7}	-7.82×10^{-7}
0.9	0.406569659	0.406569821	0.4065693953	0.4065696601	-1.61×10^{-7}	2.6440×10^{-7}	-1.1×10^{-9}
1.0	0.3678794415	0.3678794412	0.3678794412	0.3678794412	$3.00 \times 10–10^{-7}$	0.0000	3×10^{-10}

Example: 2 Consider the following nonlinear boundary value problem of tenth order solve using HPM, Modified power series method and DTM

$$Y^{(10)}(x) = e^{(-x)} y^{(2)}(x), \quad 0 < x < 1,$$

With the boundary condition

$$y(0) = 1, \quad y''(0) = y^{(iv)}(0) = y^{(vi)}(0) = y^{(viii)}(0) \quad = 1$$
$$y(0) = e, \quad y''(1) = y^{(iv)}(1) = y^{(vi)}(1) = y^{(viii)}(1) \quad = e$$

The exact solution of the problem is $y(x) \quad = e^{(x)}$

Applying the convex homotopy method

$$y_0^{(x)}(x) + p\, y_1^{(x)}(x) + p^2\, y_2^{(x)}(x) = p(e^{-x}(y_0(x) + p\, y_1(x) + p^2\, y_2(x) + ...)^2)$$

Comparing the co-efficient of like powers of P

$$p^{(0)} : y_0(x) = 1$$

$$P^{(1)} : y_1(x) = Ax + \frac{1}{2!}x^2 + \frac{1}{3!}Bx^3 + \frac{1}{4!}x^4 + \frac{1}{5!}Cx^5 + \frac{1}{6!}x^6 + \frac{1}{7!}Dx^7 + \frac{1}{8!}x^8 + \frac{1}{9!}Ex^9$$
$$+ \frac{1}{10!}x^{10} + \frac{1}{11!}x^{11} + \frac{1}{12!}x^{12} + ...$$

$$P^{(2)} : y_2(x) = \frac{2}{11!}Ax^{11} + \left(-\frac{4}{12!}A + \frac{1}{239500800}\right)x^{12} + ...,$$

$$y(x)=1+Ax+\frac{1}{2!}x^2+\frac{1}{3!}Bx^3+\frac{1}{4!}x^4+\frac{1}{5!}Cx^5+\frac{1}{6!}x^6+\frac{1}{7!}Dx^7+\frac{1}{8!}x^8+\frac{1}{9!}Ex^9$$

$$+\frac{1}{10!}Dx^{10}+(-\frac{1}{19958400}A+\frac{1}{39916800}x^{11})+(-\frac{1}{119750400}A+\frac{1}{159667200})x^{12}+O(x^{13}).......,$$

$A=y^1(0),\quad B=y^3(0)\ ,\ C=y5(0), D=y^7(0)\quad E=y^9(0)$

Imposing the boundary condition at x=1 we obtained

$A=1.00001436\quad B=0.999858964\qquad C=1.001365775,$

$D=0.987457318\quad E=1.0932797434$

The series solution is given as:

$$y(x)=1+1.00001436x+\frac{1}{2!}x^2+0.1666431607x^3+\frac{1}{4!}x^4+0.008344714791x^5+\frac{1}{6!}x^6+$$

$$0.0001952471x^7+\frac{1}{8!}x^8+3.013\times10^{-6}x^9+\frac{1}{10!}x^{10}+2.51\times10^{-8}x^{11})-2.087\times10^{-9}x^{12}+...$$

Table (1.2) exhibits the solution and the series solution along with the error obtained by using the homotopy perturbation method. It is obvious that the error can be reduced further, and higher accuracy can be obtained by evaluating more components of y(x).

Now we will solve same problem using Modified PSAM with the boundary condition $y^{2k}(0)=1,\quad y^{2k}(1)=e,\qquad k=0,1,2,3,4..9$

The exact solution is $y(x)=e^x$

By Modified PSAM the general solution at boundary x=0 is given by

$$\mathrm{Z_p}=\sum_{i=0}^{\frac{\text{n-1}}{2}}\frac{\alpha_i}{i!}t^i\ ,\mathrm{Z_C}=\sum_{i=\frac{n}{2}}^{n-1}\frac{t_i}{i!}c_i\quad n=10$$

This expansion gives

$$\mathrm{y}^{10}(\mathrm{x})=1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{24}x^4+\frac{1}{120}c_5x^5+\frac{1}{720}c_6x^6+\frac{1}{5040}c_7x^7$$
$$+\frac{1}{40320}c_8x^8+\frac{1}{362880}c_9x^9$$

The all c_i ,i= 5(6)9 are estimated at the boundary x=1, hence using (2.17)-(2.19) we have

$$\frac{1}{120}c_5+\frac{1}{720}c_6+\frac{1}{5040}c_7\ +\frac{1}{40320}c_8+\frac{1}{362880}c_9=e-\frac{65}{24}$$

$$1+c_5+\frac{1}{2}c_6+\frac{1}{6}c_7 \ +\frac{1}{24}c_8+\frac{1}{120}c_9 = e-1$$

$$c_8+c_9 = e$$

After solving above equation, we obtain C_5=1.000496,C_6=0.990240,C_7=1.087700,C_8=0.561900 and

C_9 =2.156400 so put all these value in expansion

$$y_{10}(x)=1+x+\frac{1}{2}x^2 \ +\frac{1}{6}x^3+\frac{1}{24}x^4+0.00833747x^5+0.00137533x^6+0.00021582x^7+$$
$$0.000013935x^8+0.0000059425x^9$$

Now will solve same problem with DTM by considering boundary condition $Y^{(2l)}(0)=1,\ l=0,1,2..3,4.\ and\ y^{(2l)}(1)=e^{-1}\ ,l=0,1,2.3,4$ (2.35)

Applying the operation of DTM, the following recurrence relation is obtained

$$\mathrm{Y(K+10)}=\frac{\mathrm{K!}}{(\mathrm{K+10})!}\left[\sum_{l=0}^{K}\sum_{S=0}^{l}\frac{(-1)^s}{S!}Y(l-s)Y(K-1)\right] \quad (2.36)$$

By using Eq,(2.20)the following transformed boundary conditions at x=0 can be obtained

$$y(0)=1, \quad y(2)=\frac{1}{2!},y(4)=\frac{1}{4!}\ ,y(6)=\frac{1}{6!},y(8)=\frac{1}{8!}, \quad (2.37)$$

Where According to Eq (2.20) we have

$$y(1)=a_1=y'(0), \quad y(3)=a_2=\frac{y'''(0)}{3!} \quad ,y(5)=a_3=\frac{y^{v}(0)}{5!},$$
$$y(7)=a_4=\frac{y^{vii}(0)}{7!}\ y(9)=a_5=\frac{y^{ix}(0)}{9!}\ , \quad (2.38)$$

$$Y(10)=\frac{1}{3658800},Y(11)=\frac{1}{19958400}a_1-\frac{1}{39916800},Y(12)=\frac{1}{159667200}+\frac{1}{1239500800}a_1^{\ 2}-\frac{1}{119750400}a_1,$$

$$Y(13)=\frac{1}{518918400}a_2+\frac{1}{518918400}a_1-\frac{1}{889574400}-\frac{1}{1037839800}a_1^{\ 2}$$

$$Y(14)=\frac{1}{1037836800}+\frac{1}{1816214400}a_1a_2-\frac{1}{1816214400}a_2-\frac{1}{2724321600}a_1+\frac{1}{7264857600}a_1^{\ 2}$$

and soon

The constant $a_1,\ a_{2,}\ a_3,\ \ a_5\ \ a_{4,}$ are evaluated from the boundary condition. Given in equation (2.21) for x=1 by taking by taking N=12

$$a_1, = 1.000000124, a_2 = 0.9999819650, a_3 = 1.000157229, a_4 = 0.9985666714, a_5 = 1.009946626$$

For N=17 values are

$$a_1, = 09999698990, a_2 = 1.000278188, a_3 = 09973109664\, a_4 = 1.023991491, a_5 = .8383579606$$

Then by using invers transformation rule(2.22) we get following sesies solution is evaluated up to N=17

$$\begin{aligned}
&\text{y(x) = 1-0.9999698990x} - 0.5x^2 + 0.166730314x^3 + 0.041666667x^4 \\
&+0.008310924720x^5 + 0.001388888889x^6 + 0.0002031729149x^7 \\
&+0.00002480158730x^8 + .2310289794E - 5x^9 + 0.2755731922E - 6x^{10} \\
&+0.2505060019E - 7x^{11} + 0.2087675700E - 8x^{12} - 0.17670301912E - 8x^{13} \\
&+0.1145693000E - 10x^{14} + 0.182278295E - 9x^{15} + 0.2395822685E - 10x^{16} \\
&+0.1917087518 - 10x^{17}
\end{aligned}$$

Table (1.2) Shows the comparison of result obtained by modified PSAM and HPM for Example :2 using error estimates

X	Exact solution	Numerical solution of HPM	Numerical Solution of MPSAM	Numerical solution of DTM (N=17)	Error of HPM	Error of MPSAM	Error of DTM
0.0	1.00000000	1.000000000	1.000000000	1.00000000	0.00000	0.0000	0
0.1	1.105170918	1.10517233	1.1051709180	1.10516800	-1.41×10^{-6}	0.0000	2.918×10^{-6}
0.2	1.221402758	1.221405446	1.2214027590	1.22139700	-2.69×10^{-6}	1.0000×10^{-9}	5.758×10^{-6}
0.3	1.349858808	1.349862509	1.3498588110	1.34985100	-3.70×10^{-6}	3.0000E-09	7.808×10^{-6}
0.4	1.491824698	1.49182905	1.4918247080	1.49181500	-4.35×10^{-6}	1.0000×10^{-8}	9.698×10^{-6}
0.5	1.648721271	1.648725849	1.64872122880	1.648711000	-4.58×10^{-6}	1.7000E$\times 10^{-8}$	1.0271×10^{-5}
0.6	1.822118800	1.822123158	1.8221188250	1.82210900	-4.36×10^{-6}	2.5000×10^{-8}	9.8×10^{-6}
0.7	2.013752707	2.01375415	2.0137527370	2.01374400	-3.71×10^{-6}	2.9000×10^{-8}	8.707×10^{-6}
0.8	2.225540928	2.225543628	2.2255409540	2.22553400	-2.69×10^{-6}	2.6000×10^{-8}	6.928×10^{-6}
0.9	2.459603111	2.459604528	2.4596031270	2.45959900	-2.69×10^{-6}	1.6000×10^{-8}	4.111×10^{-6}
1.0	2.718281828	2.7182830	2.7182818320	2.71828000	-2.00×10^{-9}	4.0000×10^{-9}	1.828×10^{-6}

4. Conclusion

Table (1.1) and (1.2) shows the comparison of results obtained by the HPM and Modified PSAM. As shown in Table the HPM is far superior than Modified PSAM. Also, the numerical results obtained are presented in table. We presented the comparison of absolute errors obtained by the HPM and modified PSAM.The HPM shows the less error as compared to Modified PSAM error. Rate of convergent of HPM is more as compared with MPSAM. It can be concluded that Homotopy perturbation method is a highly efficient method for solving boundary value problems arising in various fields of engineering and science.

Conflict of Interests

The authors declare that there is no conflict of interests.

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