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COMMON FIXED POINTS OF GENERALIZED CONTRACTION MAPS IN METRIC SPACES

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Abstract. In this paper, we prove the existence of common fixed points of two pairs of selfmaps under the assumptions that these two pairs of maps are weakly compatible and satisfying a contractive condition. The same is extended to a sequence of selfmaps. Also, we prove the same with different hypotheses on two pairs of selfmaps in which one pair is compatible, reciprocally continuous and the other one is weakly compatible. Further, we prove the same with different hypotheses on two pairs of selfmaps in which either one of the pair satisfies the property (E.A) and restricting the completeness of X to its subspace. We provide examples in support of our results.

Keywords: common fixed points; complete metric space; weakly compatible maps; reciprocally continuous maps; property (E.A).

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1. Introduction

It is well known that fixed point theory has wide applications in applied sciences. The development of fixed point theory is based on the generalization of contraction conditions in one

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direction or/and generalization of ambiant spaces of the operator under consideration on the other. Banach contraction principle [5] which states that if (X,d) is complete metric space and $T : X \to X$ is a contraction map then T has a unique fixed point, is a fundamental result in this theory. Due to its importance and simplicity several authors have obtained many interesting extensions and generalizations of Banach contraction principle, some generalizations of contraction condition was obtained ([6]-[9], [12]). Recently, Hussain, Parvanch, Samet and Vetro [8] introduced a new contraction map, namely JS-contraction map and proved the existence and uniqueness of fixed points in complete metric spaces.

In 2002, Aamari and Moutawakil [1] introduced the notion of property (E.A). Different authors (G. V. R. Babu and G. N. Alemayehu [3], S. Mudgal [13], Talat Nazir and Mujahid Abbas [15]) applied this concept to prove the existence of common fixed points in metric spaces.

2. Preliminaries

We use the following definitions in our subsequent discussion.

Definition 2.1. [10] Let A and B be selfmaps of a metric space (X,d). The pair (A,B) is said to be a compatible pair on X, if $\lim_{n\to\infty} d(ABx_n, BAx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = t$, for some $t \in X$.

Definition 2.2. [11] Let A and B be selfmaps of a metric space (X,d). The pair (A,B) is said to be weakly compatible, if they commute at their coincidence points. i.e., $ABx = BAx$ whenever $Ax = Bx, x \in X$.

Every compatible pair of maps is weakly compatible, but its converse need not true [11].

Definition 2.3. [14] Let A and B be selfmaps of a metric space (X,d). Then A and B are said to be reciprocally continuous, if $\lim_{n\to\infty} ABx_n = At$ and $\lim_{n\to\infty} BAx_n = Bt$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = t$, for some $t \in X$.

Clearly, if A and B are continuous then they are reciprocal continuous but its converse need not be true [14].

Definition 2.4. [4] A pair of selfmaps on a metric space (X,d) is said to be noncompatible if there exists at least one sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = t$ for some $t \in X$ but

$\lim_{n\to\infty} d(ABx_n, BAx_n)$ is either non-zero or does not exist.

Definition 2.5. [1] Two selfmappings f and g of a metric space (X,d) are said to satisfy the property (E.A), if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = t$, for some $t \in X$.

Every pair of noncompatible selfmaps of a metric space (X,d) satisfies property (E.A), but its

converse need not be true (see example 1.3 [3]).

Jleli and Samet [9] introduced the class of functions Φ, where Φ is the set of function $\phi : [0,\infty) \to [1,\infty)$ satisfying the conditions:

(i) ϕ is non-decreasing

(ii) for each sequence $\{t_n\} \subseteq (0,\infty)$, $\lim_{n\to\infty} \phi(t_n) = 1 \Leftrightarrow \lim_{n\to\infty} t_n = 0$ and

(iii) there exist $r \in (0,1)$ and $l \in (0,\infty)$ such that $\lim_{t\to 0^+} \frac{\phi(t)-1}{t^r} = l$, and proved the existence of fixed points in generalized metric spaces.

Theorem 2.6. (Corollary 2.1, [9]) *Let (X,d) be a complete metric space and $T : X \to X$ be a given map. Suppose that there exist $\phi \in \Phi$ and $k \in (0,1)$ such that $x,y \in X$,*

$d(Tx,Ty) \neq 0 \Rightarrow \phi(d(Tx,Ty)) \leq [\phi(d(x,y))]^k$. Then T has a unique fixed point.

Theorem 2.6 is a generalization of Banach contraction principle.

In continuation to this study, Hussain, Parvaneh, Samet and Vetro [8] introduced a new class of functions Ψ and defined a new contraction condition, namely $JS-$contraction.

Ψ is the set of all functions $\psi : [0,\infty) \to [1,\infty)$ satisfying the following conditions:

(ψ_1) ψ is nondecreasing and $\psi(t) = 1$ if and only if $t = 0$;

(ψ_2) for each sequence $\{t_n\} \subseteq (0,\infty)$, $\lim_{n\to\infty} \psi(t_n) = 1 \Leftrightarrow \lim_{n\to\infty} t_n = 0$;

(ψ_3) there exist $r \in (0,1)$ and $l \in (0,\infty)$ such that $\lim_{t\to 0^+} \frac{\psi(t)-1}{t^r} = l$ and

(ψ_4) $\psi(a+b) \leq \psi(a)\psi(b)$ for all $a,b > 0$.

Definition 2.7. [8] Let (X,d) be a metric space. A selfmap $T : X \to X$ is said to be $JS-$contraction if there exist a function $\psi \in \Psi$ and a positive real numbers k_1,k_2,k_3,k_4 with $0 \leq k_1+k_2+k_3+2k_4 < 1$ such that

$$\psi(d(Tx,Ty)) \leq [\psi(d(x,y))]^{k_1}[\psi(d(x,Tx))]^{k_2}[\psi(d(y,Ty))]^{k_3}[\psi(d(x,Ty))+\psi(d(y,Tx))]^{k_4}$$

for all $x,y \in X$.

Theorem 2.8. [8] *Let (X,d) be a complete metric space and $T : X \to X$ be a continuous $JS-$contraction. Then T has a unique fixed point.*

In 2017, G. V. R. Babu and T. M. Dula [4] introduced new class of functions Ψ_1 which are different from the class of functions Ψ (see example 1 [4]) and defined $JS-\Psi_1-$contraction and proved the existence of fixed points in complete metric spaces and also proved the existence of common fixed points for a pair of selfmaps. Ψ_1 is the set of all functions $\psi : [0.\infty) \to [1,\infty)$ satisfying the following conditions:

(ψ_1) ψ is nondecreasing;

(ψ_2) ψ is continuous;

(ψ_3) $\psi(t) = 1$ if and only if $t = 0$ and

(ψ_4) $\psi(a+b) \leq \psi(a)\psi(b)$ for all $a,b > 0$.

Definition 2.9. [4] Let (X,d) be a metric space and $T : X \to X$ be selfmap. If there exist a function $\psi \in \Psi_1$ and a positive real numbers k_1,k_2,k_3,k_4 with $0 \leq k_1+k_2+k_3+2k_4 < 1$ such that

$$\psi(d(Tx,Ty)) \leq [\psi(d(x,y))]^{k_1}[\psi(d(x,Tx))]^{k_2}[\psi(d(y,Ty))]^{k_3}[\psi(d(x,Ty))+\psi(d(y,Tx))]^{k_4}$$

for all $x,y \in X$, then T is said to be a $JS-\Psi_1-$contraction.

Every contraction map with constant $k \in [0,1)$ is a $JS-\Psi_1-$contraction with $\psi(t) = e^t, t \geq 0$. But its converse is not true (see example 2 [4]).

Theorem 2.10. [4] *Let (X,d) be a complete metric space and $T : X \to X$ be a $JS-\Psi_1-$contraction. Then T has a unique fixed point.*

Definition 2.11. [4] Let (X,d) be a metric space. Let $T,S : X \to X$ be selfmaps. Then T is said to be $JS-\Psi_1$ with respect to S, if there exist a function $\psi \in \Psi_1$ and positive real numbers k_1,k_2,k_3,k_4 with $0 \leq k_1+k_2+k_3+2k_4 < 1$ such that

$$\psi(d(Tx,Ty)) \leq [\psi(d(Sx,Sy))]^{k_1}[\psi(d(Sx,Tx))]^{k_2}[\psi(d(Sy,Ty))]^{k_3}[\psi(d(Sx,Ty))+\psi(d(Sy,Tx))]^{k_4}$$

for all $x,y \in X$.

Theorem 2.12. [8] *Let (X,d) be a metric space and $f,g : X \to X$ be selfmaps of X, with $f(X) \subseteq g(X)$. If f is a $JS-\Psi_1-$contraction with respect to g, either $g(X)$ (or) $f(X)$ is complete and the pair (f,g) is weakly compatible, then f and g have a unique common fixed point.*

In Section 3, we extend the results of G. V. R. Babu and T. M. Dula [4] to two pairs of maps in which one of the pair is weakly compatible. The same is extend to a sequence of selfmaps. Also, we prove the existence of common fixed points with different hypotheses on two pairs of selfmaps in which one pair is compatible, reciprocally continuous and the other one is weakly compatible. Further, we prove the same with different hypotheses on two pairs of selfmaps in which either one of the pair satisfies the property (E.A) and restricting the completeness of X to its subspace. In Section 4, we draw some corollaries from our main results and provide examples in support of our results.

3. Main results

Let A, B, S and T be mappings from a metric space (X, d) into itself and satisfying

$A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$. (A)

Now, by (A), for any $x_0 \in X$, there exists $x_1 \in X$ such that $y_0 = Ax_0 = Tx_1$. In the same way for this x_1, we can choose a point $x_2 \in X$ such that $y_1 = Bx_1 = Sx_2$ and so on. In general, we can define a sequence $\{y_n\} \in X$ such that

$y_{2n} = Ax_{2n} = Tx_{2n+1}$ and $y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}$ for $n = 0, 1, 2, \ldots$. (B)

Lemma 3.1 Let (X, d) be a metric space. Assume that A, B, S and T are selfmaps of X which satisfy the following condition:

there exist $\psi \in \Psi_1$ and nonnegative real numbers k_1, k_2, k_3, k_4 with $0 \leq k_1 + k_2 + k_3 + 2k_4 < 1$ such that

$$\psi(d(Ax, By)) \leq [\psi(d(Sx, Ty))]^{k_1} [\psi(d(Sx, Ax))]^{k_2} [\psi(d(Ty, By))]^{k_3} \times [\psi(d(Sx, By)) + \psi(d(Ty, Ax))]^{k_4} \tag{3.1}$$

for all $x, y \in X$. Then we have the following:

(i) If $A(X) \subseteq T(X)$ and the pair (B, T) is weakly compatible, and if z is a common fixed point of A and S then z is a common fixed point of A, B, S and T and it is unique.

(ii) If $B(X) \subseteq S(X)$ and the pair (A, S) is weakly compatible, and if z is a common fixed point of B and T then z is a common fixed point of A, B, S and T and it is unique.

Proof. First, we assume that (i) holds. Let z be a common fixed point of A and S.

Then $Az = Sz = z.$ (3.2)

Since $A(X) \subseteq T(X)$, there exists $u \in X$ such that $Tu = z$. Now, from (3.2), we have

$Az = Sz = Tu = z.$ (3.3)

We now prove that $Az = Bu$. Suppose that $Az \neq Bu$.

We consider,

$$\begin{aligned}
\psi(d(Az,Bu)) &\leq [\psi(d(Sz,Tu))]^{k_1}[\psi(d(Sz,Az))]^{k_2}[\psi(d(Tu,Bu))]^{k_3} \\
&\quad \times [\psi(d(Sz,Bu)) + \psi(d(Tu,Az))]^{k_4} \\
&= [\psi(0)]^{k_1}[\psi(0)]^{k_2}[\psi(d(Az,Bu))]^{k_3}[\psi(d(Az,Bu)) + \psi(0)]^{k_4} \\
&\leq [\psi(0)]^{k_1+k_2}[\psi(d(Az,Bu))]^{k_3}[\psi(d(Az,Bu))]^{k_4}[\psi(0)]^{k_4} \\
&= [\psi(0)]^{k_1+k_2+k_4}[\psi(d(Az,Bu))]^{k_3+k_4} \\
&= [\psi(d(Az,Bu))]^{k_3+k_4} < [\psi(d(Az,Bu))],
\end{aligned}$$

a contradiction.

Therefore, $Az = Bu$ (3.4)

From (3.3) and (3.4), we get

$Az = Bu = Sz = Tu = z.$ (3.5)

Since the pair (B,T) is weakly compatible and $Tu = Bu$, we have

$BTu = TBu.$ i.e., $Bz = Tz.$ (3.6)

Now we prove that $Bz = z$. If $Bz \neq z$, then by the inequality (3.1), we get

$$\begin{aligned}
\psi(d(Bz,z)) = \psi(d(z,Bz)) = \psi(d(Az,Bz)) \\
&\leq [\psi(d(Sz,Tz))]^{k_1}[\psi(d(Sz,Az))]^{k_2}[\psi(d(Tz,Bz))]^{k_3} \\
&\quad \times [\psi(d(Sz,Bz)) + \psi(d(Tz,Az))]^{k_4} \\
&= [\psi(d(z,Bz))]^{k_1}[\psi(0)]^{k_2}[\psi(0)]^{k_3}[\psi(d(z,Bz)) + \psi(d(Bz,z))]^{k_4} \\
&\leq [\psi(d(z,Bz))]^{k_1+2k_4}[\psi(0)]^{k_2+k_3} \\
&= [\psi(d(z,Bz))]^{k_1+2k_4} < \psi(d(z,Bz)),
\end{aligned}$$

a contradiction.

Hence, $Bz = z$.

From (3.6), we have

$$Bz = Tz = z. \tag{3.7}$$

From (3.5) and (3.7), we get

$Az = Bz = Sz = Tz = z$.

Therefore, z is a common fixed point of A, B, S and T.

If z' is also a common fixed point of A, B, S and T with $z \neq z'$, then

$$\begin{aligned}
\psi(d(z,z')) &= \psi(d(Az,Bz')) \\
&\leq [\psi(d(Sz,Tz'))]^{k_1}[\psi(d(Sz,Az))]^{k_2}[\psi(d(Tz',Bz'))]^{k_3} \\
&\qquad \times [\psi(d(Sz,Bz')) + \psi(d(Tz',Az))]^{k_4} \\
&= [\psi(d(z,z'))]^{k_1}[\psi(0)]^{k_2}[\psi(0)]^{k_3}[\psi(d(z,z')) + \psi(d(z,z'))]^{k_4} \\
&\leq [\psi(d(z,z'))]^{k_1+2k_4}[\psi(0)]^{k_2+k_3} \\
&= [\psi(d(z,z'))]^{k_1+2k_4} \\
&< \psi(d(z,z')),
\end{aligned}$$

a contradiction.

Therefore, $z = z'$.

Hence, z is the unique common fixed point of A, B, S and T.

The proof of (ii) is similar to (i) and hence is omitted. □

Lemma 3.2. *Let A, B, S and T be selfmaps of a metric space (X,d) and satisfy (A) and the inequality* (2.1.1). *Then for any $x_0 \in X$, the sequence $\{y_n\}$ defined by (B) is Cauchy in X.*

Proof. Let $x_0 \in X$ and let $\{y_n\}$ be a sequence defined by (B).

Assume that $y_n = y_{n+1}$ for some n.

<u>*Case* (i)</u>: n even. We write $n = 2m, m \in \mathbb{N}$.

Now we consider

$$\begin{aligned}
\psi(d(y_{n+1},y_{n+2})) &= \psi(d(y_{2m+1},y_{2m+2})) \\
&= \psi(d(y_{2m+2},y_{2m+1})) \\
&= \psi(d(Ax_{2m+2},Bx_{2m+1})) \\
&\leq [\psi(d(Sx_{2m+2},Tx_{2m+1}))]^{k_1}[\psi(d(Sx_{2m+2},Ax_{2m+2}))]^{k_2}[\psi(d(Tx_{2m+1},Bx_{2m+1}))]^{k_3} \\
&\quad\times[\psi(d(Sx_{2m+2},Bx_{2m+1}))+\psi(d(Tx_{2m+1},Ax_{2m+2}))]^{k_4} \\
&\leq [\psi(d(y_{2m+1},y_{2m}))]^{k_1}[\psi(d(y_{2m+1},y_{2m+2}))]^{k_2}[\psi(d(y_{2m+1},y_{2m}))]^{k_3} \\
&\quad\times[\psi(d(y_{2m+1},y_{2m+1}))]_4^k[\psi(d(y_{2m},y_{2m+2}))]^{k_4} \\
&\leq [\psi(d(y_{2m+1},y_{2m}))]^{k_1+k_3}[\psi(d(y_{2m+1},y_{2m+2}))]^{k_2} \\
&\quad\times[\psi(d(y_{2m},y_{2m+1}))+\psi(d(y_{2m+1},y_{2m+2}))]^{k_4} \\
&\leq [\psi(d(y_{2m+1},y_{2m}))]^{k_1+k_3+k_4}[\psi(d(y_{2m+1},y_{2m+2}))]^{k_2+k_4} \\
&= [\psi(d(y_{2m+1},y_{2m+2}))]^{k_2+k_4} \\
&< \psi(d(y_{2m+1},y_{2m+2})) = \psi(d(y_{n+1},y_{n+2})),
\end{aligned}$$

a contradiction if $y_{2m+1} \neq y_{2m+2}$.

Therefore, $d(y_{2m+1},y_{2m+2}) = 0$ which implies that $y_{2m+2} = y_{2m+1} = y_{2m}$.

In general, we have $y_{2m+k} = y_{2m}$ for $k = 0,1,2,\dots$.

<u>*Case (ii)*</u>: n odd. We write $n = 2m+1$ for some $m \in \mathbb{N}$. We consider

$$\begin{aligned}
\psi(d(y_{n+1},y_{n+2})) &= \psi(d(y_{2m+2},y_{2m+3})) \\
&= \psi(d(Ax_{2m+2},Bx_{2m+3})) \\
&\leq [\psi(d(Sx_{2m+2},Tx_{2m+3}))]^{k_1}[\psi(d(Sx_{2m+2},Ax_{2m+2}))]^{k_2}[\psi(d(Tx_{2m+3},Bx_{2m+3}))]^{k_3} \\
&\quad\times[\psi(d(Sx_{2m+2},Bx_{2m+3}))+\psi(d(Tx_{2m+3},Ax_{2m+2}))]^{k_4} \\
&\leq [\psi(d(y_{2m+1},y_{2m+2}))]^{k_1}[\psi(d(y_{2m+1},y_{2m+2}))]^{k_2}[\psi(d(y_{2m+2},y_{2m+3}))]^{k_3} \\
&\quad\times[\psi(d(y_{2m+1},y_{2m+3}))]^{k_4}[\psi(d(y_{2m+2},y_{2m+2}))]^{k_4}
\end{aligned}$$

$$\leq [\psi(d(y_{2m+1},y_{2m+2}))]^{k_1+k_2}[\psi(d(y_{2m+2},y_{2m+3}))]^{k_3}$$
$$\times [\psi(d(y_{2m+1},y_{2m+2}))+\psi(d(y_{2m+2},y_{2m+3}))]^{k_4}[\psi(0)]^{k_4}$$
$$\leq [\psi(d(y_{2m+1},y_{2m+2}))]^{k_1+k_2+k_4}[\psi(d(y_{2m+2},y_{2m+3}))]^{k_3+k_4}$$
$$= [\psi(d(y_{2m+2},y_{2m+3}))]^{k_2+k_4}$$
$$< \psi(d(y_{2m+2},y_{2m+3})) = \psi(d(y_{n+1},y_{n+2})),$$

a contradiction if $y_{2m+2} \neq y_{2m+3}$.

Therefore, $d(y_{n+2},y_{n+3}) = 0$ implies that $y_{2m+3} = y_{2m+2} = y_{2m+1}$.

In general, we have $y_{2m+k} = y_{2m+1}$ for $k = 1,2,3,\dots$.

From Case (i) and Case (ii), we have $y_{n+k} = y_n$ for $k = 0,1,2,\dots$.

Hence, $\{y_{n+k}\}$ is a constant sequence and hence $\{y_n\}$ is Cauchy.

Now we assume that $y_n \neq y_{n+1}$, for all $n \in \mathbb{N}$.

If n is odd, then $n = 2m+1$ for some $m \in \mathbb{N}$.

We now consider

$$\psi(d(y_n,y_{n+1})) = \psi(d(y_{2m+1},y_{2m+2}))$$
$$= \psi(d(y_{2m+2},y_{2m+1}))$$
$$= \psi(d(Ax_{2m+2},Bx_{2m+1}))$$
$$\leq [\psi(d(Sx_{2m+2},Tx_{2m+1}))]^{k_1}[\psi(d(Sx_{2m+2},Ax_{2m+2}))]^{k_2}[\psi(d(Tx_{2m+1},Bx_{2m+1}))]^{k_3}$$
$$\times [\psi(d(Sx_{2m+2},Bx_{2m+1}))+\psi(d(Tx_{2m+1},Ax_{2m+2}))]^{k_4}$$
$$\leq [\psi(d(y_{2m+1},y_{2m}))]^{k_1}[\psi(d(y_{2m+1},y_{2m+2}))]^{k_2}[\psi(d(y_{2m+1},y_{2m}))]^{k_3}$$
$$\times [\psi(d(y_{2m+1},y_{2m+1}))]_4^k[\psi(d(y_{2m},y_{2m+2}))]^{k_4}$$
$$\leq [\psi(d(y_{2m+1},y_{2m}))]^{k_1+k_3}[\psi(d(y_{2m+1},y_{2m+2}))]^{k_2}$$
$$\times [\psi(0)]^{k_4}[\psi(d(y_{2m},y_{2m+1}))+\psi(d(y_{2m+1},y_{2m+2}))]^{k_4}$$
$$\leq [\psi(d(y_{2m+1},y_{2m}))]^{k_1+k_3+k_4}[\psi(d(y_{2m+1},y_{2m+2}))]^{k_2+k_4}$$
$$= [\psi(d(y_{n-1},y_n))]^{k_1+k_3+k_4}[\psi(d(y_n,y_{n+1}))]^{k_2+k_4}$$
$$\leq [\psi(d(y_{n-1},y_n))]^{\frac{k_1+k_3+k_4}{1-k_2-k_4}}$$

$$\leq [\psi(d(y_{n-2},y_{n-1}))]^{(\frac{k_1+k_3+k_4}{1-k_2-k_4})^2}$$

$$\vdots$$

$$\leq [\psi(d(y_0,y_1))]^{(\frac{k_1+k_3+k_4}{1-k_2-k_4})^n} \to 1 \text{ as } n\to\infty.$$

so that $\psi(d(y_n,y_{n+1})) \to 1$ as $n\to\infty$.

Hence by the property (ii) and (iii) of ψ, we have $d(y_n,y_{n+1}) \to 0$ as $n\to\infty$.

On the similar lines, if n is even, it follows that $d(y_n,y_{n+1}) \to 0$ as $n\to\infty$.

Therefore, $\lim\limits_{n\to\infty} d(y_n,y_{n+1}) = 0.$ (3.8)

We now prove that $\{y_n\}$ is Cauchy.

It is sufficient to show that $\{y_{2n}\}$ is Cauchy in X.

Otherwise, there is an $\varepsilon > 0$ and there exists sequences $\{2m_k\},\{2n_k\}$ with $2n_k > 2m_k > k$ such that

$d(y_{2m_k},y_{2n_k}) \geq \varepsilon$ and $d(y_{2m_k},y_{2n_k-2}) < \varepsilon.$ (3.9)

Now we prove that (i) $\lim\limits_{k\to\infty} d(y_{2m_k-2},y_{2n_k}) = \varepsilon.$

Since $\varepsilon \leq d(y_{2m_k},y_{2n_k})$ for all k, we have

$\varepsilon \leq \liminf\limits_{k\to\infty} d(y_{2m_k},y_{2n_k}).$ (3.10)

Now for each positive integer k, by the triangular inequality, we get

$$d(y_{2m_k},y_{2n_k}) \leq d(y_{2n_k},y_{2n_k-1}) + d(y_{2n_k-1},y_{2n_k-2}) + d(y_{2n_k-2},y_{2m_k}).$$

On taking limit superior as $k\to\infty$, from (3.8) and (3.9), we have

$\limsup\limits_{k\to\infty} d(y_{2m_k},y_{2n_k}) \leq \varepsilon.$ (3.11)

Hence, from (3.10) and (3.11), we get $\lim\limits_{k\to\infty} d(y_{2m_k},y_{2n_k})$ exists and

$\lim\limits_{k\to\infty} d(y_{2m_k},y_{2n_k}) = \varepsilon.$

In similar way, it is easy to see that

(ii) $\lim\limits_{k\to\infty} d(y_{2m_k+1},y_{2n_k}) = \varepsilon;$ (iii) $\lim\limits_{k\to\infty} d(y_{2n_k-1},y_{2m_k+1}) = \varepsilon.$

We now consider

$$\begin{aligned}
\psi(d(y_{2n_k},y_{2m_k+1})) &= \psi(d(Ax_{2n_k},Bx_{2m_k+1}))\\
&\leq [\psi(d(Sx_{2n_k},Tx_{2m_k+1}))]^{k_1}[\psi(d(Sx_{2n_k},Ax_{2n_k}))]^{k_2}[\psi(d(Tx_{2m_k+1},Bx_{2m_k+1}))]^{k_3}\\
&\quad \times [\psi(d(Sx_{2n_k},Bx_{2m_k+1})) + \psi(d(Tx_{2m_k+1},Ax_{2n_k}))]^{k_4}\\
&\leq [\psi(d(y_{2n_k-1},y_{2m_k+1}))]^{k_1}[\psi(d(y_{2n_k-1},y_{2n_k}))]^{k_2}[\psi(d(y_{2m_k},y_{2m_k+1}))]^{k_3}
\end{aligned}$$

$$\times [\psi(d(y_{2n_k-1}, y_{2m_k+1}))]^{k_4}[\psi(d(y_{2m_k}, y_{2n_k}))]^{k_4} \tag{3.12}$$

On letting $k \to \infty$ in (3.12), we get

$$\psi(\varepsilon) \leq [\psi(\varepsilon)]^{k_1}[\psi(0)]^{k_2}[\psi(0)]^{k_3}[\psi(\varepsilon)]^{k_4}[\psi(\varepsilon)]^{k_4}$$

$$= [\psi(\varepsilon)]^{k_1+2k_4} < \psi(\varepsilon),$$

a contradiction.

Therefore, $\{y_n\}$ is a Cauchy sequence in X. □

The following is the main result of this paper.

Theorem 3.3. *Let A, B, S and T be selfmaps on a complete metric space (X, d) and satisfy (A) and the inequality* (3.1). *If the pairs (A, S) and (B, T) are weakly compatible and one of the range sets $S(X), T(X), A(X)$ and $B(X)$ is closed, then for any $x_0 \in X$, the sequence $\{y_n\}$ defined by (B) is Cauchy in X and $\lim_{n\to\infty} y_n = z$(say), $z \in X$ and z is the unique common fixed point of A, B, S and T.*

Proof. By Lemma 3.2, the sequence $\{y_n\}$ is Cauchy in X.

Since X is complete, there exists $z \in X$ such that $\lim_{n\to\infty} y_n = z$. Thus,

$$\lim_{n\to\infty} y_{2n} = \lim_{n\to\infty} Ax_{2n} = \lim_{n\to\infty} Tx_{2n+1} = z \tag{3.13}$$

and

$$\lim_{n\to\infty} y_{2n+1} = \lim_{n\to\infty} Bx_{2n+1} = \lim_{n\to\infty} Sx_{2n+2} = z. \tag{3.14}$$

We now consider the following four cases.

Case (i). $S(X)$ is closed.

In this case $z \in S(X)$ and there exists $u \in X$ such that $z = Su$.

Now we claim that $Au = z$. Suppose that $Au \neq z$.

We now consider

$$\psi(d(Au, Bx_{2n+1})) \leq [\psi(d(Su, Tx_{2n+1}))]^{k_1}[\psi(d(Su, Au))]^{k_2}[\psi(d(Tx_{2n+1}, Bx_{2n+1}))]^{k_3}$$

$$\times [\psi(d(Su, Bx_{2n+1})) + \psi(d(Tx_{2n+1}, Au))]^{k_4}$$

$$\leq [\psi(d(z, Tx_{2n+1}))]^{k_1}[\psi(d(z, Au))]^{k_2}[\psi(d(Tx_{2n+1}, Bx_{2n+1}))]^{k_3}$$

$$\times [\psi(d(z, Bx_{2n+1}))]^{k_4}[\psi(d(Tx_{2n+1}, Au))]^{k_4} \tag{3.15}$$

On letting $n \to \infty$ in (3.15), using (3.13) and (3.14), we get

$$\psi(d(Au,z)) \leq [\psi(d(z,z))]^{k_1}[\psi(d(z,Au))]^{k_2}[\psi(d(z,z))]^{k_3}[\psi(d(z,z))]^{k_4}[\psi(d(z,Au))]^{k_4}$$
$$= [\psi(0)]^{k_1+k_3+k_4}[\psi(d(z,Au))]^{k_2+k_4} = [\psi(d(z,Au))]^{k_2+k_4} < \psi(d(z,Au)),$$

a contradiction.

Therefore, $Au = z = Su$. (3.16)

Since the pair (A,S) is weakly compatible and $Au = Su$, we have

$ASu = SAu$. i.e., $Az = Sz$. (3.17)

Now we prove that $Az = z$.

If $Az \neq z$, then

$$\psi(d(Az,Bx_{2n+1})) \leq [\psi(d(Sz,Tx_{2n+1}))]^{k_1}[\psi(d(Sz,Az))]^{k_2}[\psi(d(Tx_{2n+1},Bx_{2n+1}))]^{k_3}$$
$$\times [\psi(d(Sz,Bx_{2n+1})) + \psi(d(Tx_{2n+1},Az))]^{k_4}$$
$$\leq [\psi(d(Az,Tx_{2n+1}))]^{k_1}[\psi(d(Az,Az))]^{k_2}[\psi(d(Tx_{2n+1},Bx_{2n+1}))]^{k_3}$$
$$\times [\psi(d(Az,Bx_{2n+1}))]^{k_4}[\psi(d(Tx_{2n+1},Az))]^{k_4} \tag{3.18}$$

On letting $n \to \infty$ in (3.18), using (3.13) and (3.14), we get

$$\psi(d(Az,z)) \leq [\psi(d(Az,z))]^{k_1}[\psi(0)]^{k_2}[\psi(d(z,z))]^{k_3}[\psi(d(Az,z))]^{k_4}[\psi(d(z,Az))]^{k_4}$$
$$= [\psi(d(Az,z))]^{k_1+2k_4}[\psi(0)]^{k_2+k_3} = [\psi(d(Az,z))]^{k_1+2k_4} < \psi(d(Az,z)),$$

a contradiction. Hence, $Az = z$.

From (3.17), we get $Az = z = Sz$.

Hence, z is a common fixed point of A and S.

By Lemma 3.1, we get that z is a unique common fixed point of A, B, S and T.

Case (ii). $T(X)$ is closed.

In this case $z \in T(X)$ and there exists $u \in X$ such that $z = Tu$.

Now we claim that $Bu = z$. Suppose that $Bu \neq z$.

We now consider

$$\begin{aligned}\psi(d(Ax_{2n+2},Bu)) &\leq [\psi(d(Sx_{2n+2},Tu))]^{k_1}[\psi(d(Sx_{2n+2},Ax_{2n+2}))]^{k_2}[\psi(d(Tu,Bu))]^{k_3}\\ &\quad\times[\psi(d(Sx_{2n+2},Bu))+\psi(d(Tu,Ax_{2n+2}))]^{k_4}\\ &\leq [\psi(d(Sx_{2n+2},z))]^{k_1}[\psi(d(Sx_{2n+2},Ax_{2n+2}))]^{k_2}[\psi(d(z,Bu))]^{k_3}\\ &\quad\times[\psi(d(Sx_{2n+2},Bu))+\psi(d(z,Ax_{2n+2}))]^{k_4} \qquad (3.19)\end{aligned}$$

On letting $n\to\infty$ in (3.19), using (3.13) and (3.14), we get

$$\begin{aligned}\psi(d(z,Bu)) &\leq [\psi(d(z,z))]^{k_1}[\psi(d(z,z))]^{k_2}[\psi(d(z,Bu))]^{k_3}[\psi(d(z,Bu))^{k_4}+\psi(d(z,z))]^{k_4}\\ &\leq [\psi(0)]^{k_1+k_2+k_4}[\psi(d(z,Bu))]^{k_3+k_4}=[\psi(d(z,Bu))]^{k_3+k_4}<\psi(d(z,Bu)),\end{aligned}$$

a contradiction.

Therefore, $Bu=z=Tu$. (3.20)

Since the pair (B,T) is weakly compatible and $Bu=Tu$, we have

$BTu=TBu$. i.e., $Bz=Tz$. (3.21)

Now we prove that $Bz=z$. If $Bz\neq z$, then

$$\begin{aligned}\psi(d(Ax_{2n+2},Bz)) &\leq [\psi(d(Sx_{2n+2},Tz))]^{k_1}[\psi(d(Sx_{2n+2},Ax_{2n+2}))]^{k_2}[\psi(d(Tz,Bz))]^{k_3}\\ &\quad\times[\psi(d(Sx_{2n+2},Bz))+\psi(d(Tz,Ax_{2n+2}))]^{k_4}\\ &\leq [\psi(d(Sx_{2n+2},Bz))]^{k_1}[\psi(d(Sx_{2n+2},Ax_{2n+2}))]^{k_2}[\psi(d(Bz,Bz))]^{k_3}\\ &\quad\times[\psi(d(Sx_{2n+2},Bz))+\psi(d(Bz,Ax_{2n+2}))]^{k_4} \qquad (3.22)\end{aligned}$$

On letting $n\to\infty$ in (3.22), using (3.13) and (3.14), we get

$$\begin{aligned}\psi(d(z,Bz)) &\leq [\psi(d(z,Bz))]^{k_1}[\psi(d(z,z))]^{k_2}[\psi(0)]^{k_3}[\psi(d(z,Bz))^{k_4}+\psi(d(Bz,z))]^{k_4}\\ &\leq [\psi(d(z,Bz))]^{k_1+2k_4}[\psi(0)]^{k_2+k_3}=[\psi(d(z,Bz))]^{k_1+2k_4}<\psi(d(z,Bz)),\end{aligned}$$

a contradiction. Hence, $Bz=z$.

From (3.21), we get $Bz=Tz=z$.

Therefore, z is a common fixed point of B and T.

Hence, by Lemma 3.1, we get that z is the unique common fixed point of A,B,S and T.

Case (iii). $A(X)$ is closed.

Since $z \in A(X) \subseteq T(X)$, there exists $u \in X$ such that $z = Tu$.

Now we show that $Bu = z$. If $Bu \neq z$, then we consider

$$\psi(d(Ax_{2n+2}, Bu)) \leq [\psi(d(Sx_{2n+2}, Tu))]^{k_1}[\psi(d(Sx_{2n+2}, Ax_{2n+2}))]^{k_2}[\psi(d(Tu, Bu))]^{k_3} \times [\psi(d(Sx_{2n+2}, Bu)) + \psi(d(Tu, Ax_{2n+2}))]^{k_4}$$

$$\leq [\psi(d(Sx_{2n+2}, z))]^{k_1}[\psi(d(Sx_{2n+2}, Ax_{2n+2}))]^{k_2}[\psi(d(z, Bu))]^{k_3} \times [\psi(d(Sx_{2n+2}, Bu)) + \psi(d(z, Ax_{2n+2}))]^{k_4}$$

On letting $n \to \infty$, using (2.3.1) and (2.3.2), we get

$$\psi(d(z, Bu)) \leq [\psi(d(z,z))]^{k_1}[\psi(d(z,z))]^{k_2}[\psi(d(z, Bu))]^{k_3}[\psi(d(z, Bu))^{k_4} + \psi(d(z,z))]^{k_4}$$

$$\leq [\psi(0)]^{k_1+k_2+k_4}[\psi(d(z, Bu))]^{k_3+k_4} = [\psi(d(z, Bu))]^{k_3+k_4} < \psi(d(z, Bu)),$$

a contradiction.

Therefore $Bu = z = Tu$. Thus (3.20) holds. Now by Case (ii), the conclusion of the theorem follows.

Case (iv). $B(X)$ is closed.

Since $z \in B(X) \subseteq S(X)$, there exists $u \in X$ such that $z = Su$.

Now we show that $Au = z$.

If $Au \neq z$, then we consider

$$\psi(d(Au, Bx_{2n+1})) \leq [\psi(d(Su, Tx_{2n+1}))]^{k_1}[\psi(d(Su, Au))]^{k_2}[\psi(d(Tx_{2n+1}, Bx_{2n+1}))]^{k_3} \times [\psi(d(Su, Bx_{2n+1})) + \psi(d(Tx_{2n+1}, Au))]^{k_4}$$

$$\leq [\psi(d(z, Tx_{2n+1}))]^{k_1}[\psi(d(z, Au))]^{k_2}[\psi(d(Tx_{2n+1}, Bx_{2n+1}))]^{k_3} \times [\psi(d(z, Bx_{2n+1}))]^{k_4}[\psi(d(Tx_{2n+1}, Au))]^{k_4} \tag{3.23}$$

On letting $n \to \infty$ in (3.23), using (3.13) and (3.14), we get

$$\psi(d(Az, z)) \leq [\psi(d(Az, z))]^{k_1}[\psi(0)]^{k_2}[\psi(d(z,z))]^{k_3}[\psi(d(Az, z))]^{k_4}[\psi(d(z, Az))]^{k_4}$$

$$= [\psi(d(Az, z))]^{k_1+2k_4}[\psi(0)]^{k_2+k_3} = [\psi(d(Az, z))]^{k_1+2k_4} < \psi(d(Az, z)),$$

a contradiction.

Therefore $Au = z = Su$. Thus (3.16) holds.

Now by Case (i), the conclusion of the theorem follows. □

Theorem 3.4. *Let A,B,S and T be selfmaps on a metric space (X,d) and satisfy (A) and the inequality* (3.1). *If the pairs (A,S) and (B,T) are weakly compatible and either one of the set $(S(X),d),(T(X),d),(A(X),d)$ (or) $(B(X),d)$ is complete, then for any $x_0 \in X$, the sequence $\{y_n\}$ defined by (B) is Cauchy in X and $\lim_{n\to\infty} y_n = z$(say), $z \in X$ and z is the unique common fixed point of A,B,S and T.*

Proof. By Lemma 3.2, the sequence $\{y_n\}$ is Cauchy in X.

Suppose $S(X)$ is complete, then there exists $z \in S(X)$ such that $\lim_{n\to\infty} y_n = z$.

Thus,

$$\lim_{n\to\infty} y_{2n} = \lim_{n\to\infty} Ax_{2n} = \lim_{n\to\infty} Tx_{2n+1} = z \tag{3.24}$$

and

$$\lim_{n\to\infty} y_{2n+1} = \lim_{n\to\infty} Bx_{2n+1} = \lim_{n\to\infty} Sx_{2n+2} = z. \tag{3.25}$$

Since $z \in S(X)$, there exists $u \in X$ such that $z = Su$.

We now prove that $Au = z$. If $Au \neq z$, then

$$\begin{aligned}
\psi(d(Au,Bx_{2n+1})) &\leq [\psi(d(Su,Tx_{2n+1}))]^{k_1}[\psi(d(Su,Au))]^{k_2}[\psi(d(Tx_{2n+1},Bx_{2n+1}))]^{k_3} \\
&\quad \times [\psi(d(Su,Bx_{2n+1})) + \psi(d(Tx_{2n+1},Au))]^{k_4} \\
&\leq [\psi(d(z,Tx_{2n+1}))]^{k_1}[\psi(d(z,Au))]^{k_2}[\psi(d(Tx_{2n+1},Bx_{2n+1}))]^{k_3} \\
&\quad \times [\psi(d(z,Bx_{2n+1}))]^{k_4}[\psi(d(Tx_{2n+1},Au))]^{k_4}
\end{aligned} \tag{3.26}$$

On letting $n \to \infty$ in (3.26), using (3.24) and (3.25), we get

$$\begin{aligned}
\psi(d(Az,z)) &\leq [\psi(d(Az,z))]^{k_1}[\psi(0)]^{k_2}[\psi(d(z,z))]^{k_3}[\psi(d(Az,z))]^{k_4}[\psi(d(z,Az))]^{k_4} \\
&= [\psi(d(Az,z))]^{k_1+2k_4}[\psi(0)]^{k_2+k_3} = [\psi(d(Az,z))]^{k_1+2k_4} < \psi(d(Az,z)),
\end{aligned}$$

a contradiction.

Therefore, $Au = z = Su$.

Since the pair (A,S) is weakly compatible and $Au = Su$, we have

$$ASu = SAu. \text{ i.e., } Az = Sz. \tag{3.27}$$

Now we prove that $Az = z$. If suppose that $Az \neq z$, then

$$\begin{aligned}\psi(d(Az,Bx_{2n+1})) &\leq [\psi(d(Sz,Tx_{2n+1}))]^{k_1}[\psi(d(Sz,Az))]^{k_2}[\psi(d(Tx_{2n+1},Bx_{2n+1}))]^{k_3}\\ &\quad\times[\psi(d(Sz,Bx_{2n+1}))+\psi(d(Tx_{2n+1},Az))]^{k_4}\\ &\leq [\psi(d(Az,Tx_{2n+1}))]^{k_1}[\psi(d(Az,Az))]^{k_2}[\psi(d(Tx_{2n+1},Bx_{2n+1}))]^{k_3}\\ &\quad\times[\psi(d(Az,Bx_{2n+1}))]^{k_4}[\psi(d(Tx_{2n+1},Az))]^{k_4} \qquad (3.28)\end{aligned}$$

On letting $n \to \infty$ in (3.28), using (3.24) and (3.25), we get

$$\begin{aligned}\psi(d(Az,z)) &\leq [\psi(d(Az,z))]^{k_1}[\psi(0)]^{k_2}[\psi(d(z,z))]^{k_3}[\psi(d(Az,z))]^{k_4}[\psi(d(z,Az))]^{k_4}\\ &= [\psi(d(Az,z))]^{k_1+2k_4}[\psi(0)]^{k_2+k_3} = [\psi(d(Az,z))]^{k_1+2k_4} < \psi(d(Az,z)),\end{aligned}$$

a contradiction. Hence, $Az = z$.

From (3.27), we get $Az = Sz = z$.

Thus, z is a common fixed point of A and S.

By Lemma 3.1, we get z is the unique common fixed point of A, B, S and T.

In a similar way, it is easy to see that z is the unique common fixed point of A, B, S and T when either $T(X)$ or $A(X)$ or $B(X)$ is complete. □

Theorem 3.5. *Let A, B, S and T be selfmaps on a complete metric space (X, d) and satisfy (A) and the inequality* (3.1). *Further assume that either*

- (i) *(A, S) is reciprocal continuous and compatible pairs of maps, and (B, T) is a pair of weakly compatible maps (or)*
- (ii) *(B, T) is reciprocal continuous and compatible pairs of maps, and (A, S) is a pair of weakly compatible maps.*

Then A, B, S and T have a unique common fixed point.

Proof. By Lemma 3.2, for each $x_0 \in X$, the sequence $\{y_n\}$ defined by (B) is Cauchy in X.

Since X is complete, then there exists $z \in X$ such that $\lim_{n\to\infty} y_n = z$.

Consequently, the subsequences $\{y_{2n}\}$ and $\{y_{2n+1}\}$ are also converges to $z \in X$, we have

$$\lim_{n\to\infty} y_{2n} = \lim_{n\to\infty} Ax_{2n} = \lim_{n\to\infty} Tx_{2n+1} = z, \qquad (3.29)$$

and

$$\lim_{n\to\infty} y_{2n+1} = \lim_{n\to\infty} Bx_{2n+1} = \lim_{n\to\infty} Sx_{2n+2} = z. \qquad (3.30)$$

First, we assume that (i) holds.

Since (A,S) is reciprocal continuous, it follows that

$\lim_{n\to\infty} ASx_{2n+2} = Az$ and $\lim_{n\to\infty} SAx_{2n+2} = Sz$.

Since (A,S) is compatible, we have

$\lim_{n\to\infty} d(ASx_{2n+2}, SAx_{2n+2}) = 0$

which implies that $\lim_{n\to\infty} d(Az,Sz) = 0$ implies that $Az = Sz$. (3.31)

Since $A(X) \subseteq T(X)$, there exists $u \in X$ such that $Az = Tu$.

From (3.31), we have $Az = Sz = Tu$.

Now we prove that $Az = Bu$. Suppose that $Az \neq Bu$.

We now consider

$$\begin{aligned}
\psi(d(Az,Bu)) &\leq [\psi(d(Sz,Tu))]^{k_1}[\psi(d(Sz,Az))]^{k_2}[\psi(d(Tu,Bu))]^{k_3} \\
&\quad \times [\psi(d(Sz,Bu)) + \psi(d(Tu,Az))]^{k_4} \\
&\leq [\psi(d(Az,Az))]^{k_1}[\psi(d(Az,Az))]^{k_2}[\psi(d(Az,Bu))]^{k_3} \\
&\quad \times [\psi(d(Az,Bu))]^{k_4}[\psi(d(Az,Az))]^{k_4} \\
&= [\psi(0)]^{k_1+k_2+k_4}[\psi(d(Az,Bu))]^{k_3+k_4} \\
&= [\psi(d(Az,Bu))]^{k_3+k_4} < \psi(d(Az,Bu)),
\end{aligned}$$

a contradiction.

Therefore $Az = Bu = Sz = Tu$. (3.32)

Since every compatible pair is weakly compatible, we have (A,S) is weakly compatible and from (3.31), we have

$ASz = SAz$. i.e ,$AAz = SAz$.

Now we prove that $AAz = Az$. If possible, suppose that $AAz \neq Az$.

We now consider

$$\begin{aligned}
\psi(d(AAz,Az)) = \psi(d(AAz,Bu)) &\leq [\psi(d(SAz,Tu))]^{k_1}[\psi(d(SAz,AAz))]^{k_2}[\psi(d(Tu,Bu))]^{k_3} \\
&\quad \times [\psi(d(SAz,Bu)) + \psi(d(Tu,AAz))]^{k_4} \\
&\leq [\psi(d(AAz,Az))]^{k_1}[\psi(d(AAz,AAz))]^{k_2}[\psi(d(Az,Az))]^{k_3} \\
&\quad \times [\psi(d(AAz,Az))]^{k_4}[\psi(d(Az,AAz))]^{k_4}
\end{aligned}$$

$$= [\psi(d(AAz,Az))]^{k_1+2k_4}[\psi(0)]^{k_2+k_3}$$

$$= [\psi(d(AAz,Az))]^{k_1+2k_4}$$

$$< \psi(d(AAz,Az)),$$

a contradiction.

Therefore $AAz = Az$. Hence, $AAz = SAz = Az$, so that Az is a common fixed point of A and S.

Since (B,T) is weakly compatible and $Bu = Tu$, we have $BTu = TBu$.

From (3.32), we have $BAz = TAz$. (3.33)

We now prove that $BAz = Az$. Suppose that $BAz \neq Az$.

Now, we consider

$$\begin{aligned}\psi(d(BAz,Az)) = \psi(d(Az,BAz)) &\leq [\psi(d(Sz,TAz))]^{k_1}[\psi(d(Sz,Az))]^{k_2}[\psi(d(TAz,BAz))]^{k_3}\\ &\quad\times[\psi(d(Sz,BAz))+\psi(d(TAz,Az))]^{k_4}\\ &\leq [\psi(d(Az,BAz))]^{k_1}[\psi(d(Az,Az))]^{k_2}[\psi(d(BAz,BAz))]^{k_3}\\ &\quad\times[\psi(d(Az,BAz))]^{k_4}[\psi(d(BAz,Az))]^{k_4}\\ &= [\psi(d(BAz,Az))]^{k_1+2k_4}[\psi(0)]^{k_2+k_3}\\ &= [\psi(d(BAz,Az))]^{k_1+2k_4} < \psi(d(BAz,Az)),\end{aligned}$$

a contradiction.

Hence, $BAz = Az$. From (3.33), we get $BAz = TAz = Az$.

Hence, $AAz = BAz = SAz = TAz = Az$. (3.34)

Therefore Az is a common fixed point of A,B,S and T.

Now we show that $Az = z$. If $Az \neq z$, then

$$\begin{aligned}\psi(d(Az,Bx_{2n+1})) &\leq [\psi(d(Sz,Tx_{2n+1}))]^{k_1}[\psi(d(Sz,Az))]^{k_2}[\psi(d(Tx_{2n+1},Bx_{2n+1}))]^{k_3}\\ &\quad\times[\psi(d(Sz,Bx_{2n+1}))+\psi(d(Tx_{2n+1},Az))]^{k_4}\\ &\leq [\psi(d(Az,Tx_{2n+1}))]^{k_1}[\psi(d(Az,Az))]^{k_2}[\psi(d(Tx_{2n+1},Bx_{2n+1}))]^{k_3}\\ &\quad\times[\psi(d(Az,Bx_{2n+1}))]^{k_4}[\psi(d(Tx_{2n+1},Az))]^{k_4} \qquad (3.35)\end{aligned}$$

On letting $n \to \infty$ in (3.35), using (3.29) and (3.30), we get

$$\psi(d(Az,z)) \leq [\psi(d(Az,z))]^{k_1}[\psi(0)]^{k_2}[\psi(d(z,z))]^{k_3}[\psi(d(Az,z))]^{k_4}[\psi(d(z,Az))]^{k_4}$$
$$= [\psi(d(Az,z))]^{k_1+2k_4}[\psi(0)]^{k_2+k_3} = [\psi(d(Az,z))]^{k_1+2k_4} < \psi(d(Az,z)),$$

a contradiction.

Hence, $Az = z$. From (3.34), we get

$Az = Bz = Sz = Tz = z$.

Therefore z is a common fixed point of A, B, S and T.

In a similar way, under the assumption (ii), we obtain the existence of common fixed point of A, B, S and T. Uniqueness of common fixed point follows from the inequality (3.1). □

Theorem 3.6. *Let A, B, S and T be selfmaps on a complete metric space (X, d) and satisfy (A) and the inequality* (3.1). *If either*

(i) *S is continuous, (A, S) compatible and (B, T) is weakly compatible (or)*

(ii) *T is continuous, (B, T) compatible and (A, S) is a pair of weakly compatible maps,*

then A, B, S and T have a unique common fixed point.

Proof. By Lemma 3.2, for each $x_0 \in X$, the sequence $\{y_n\}$ defined by (B) is Cauchy in X.

Since X is complete, then there exists $z \in X$ such that $\lim_{n\to\infty} y_n = z$.

Consequently, the subsequences $\{y_{2n}\}$ and $\{y_{2n+1}\}$ are also converges to $z \in X$, we have

$$\lim_{n\to\infty} y_{2n} = \lim_{n\to\infty} Ax_{2n} = \lim_{n\to\infty} Tx_{2n+1} = z, \tag{3.36}$$

and

$$\lim_{n\to\infty} y_{2n+1} = \lim_{n\to\infty} Bx_{2n+1} = \lim_{n\to\infty} Sx_{2n+2} = z. \tag{3.37}$$

First, we assume that (i) holds.

Since (A, S) is compatible pair, we have

$\lim_{n\to\infty} d(SAx_{2n}, ASx_{2n}) = 0$, it follows that

$\lim_{n\to\infty} SAx_{2n} = \lim_{n\to\infty} ASx_{2n}$.

Since S is continuous, we have

$Sz = \lim_{n\to\infty} SAx_{2n} = \lim_{n\to\infty} ASx_{2n}$

Now we prove that $Sz = z$. If $Sz \neq z$, then consider

$$\begin{aligned}\psi(d(ASx_{2n+2}, Bx_{2n+1})) &\leq [\psi(d(SSx_{2n+2}, Tx_{2n+1}))]^{k_1}[\psi(d(SSx_{2n+2}, ASx_{2n+2}))]^{k_2} \\ &\quad \times [\psi(d(Tx_{2n+1}, Bx_{2n+1}))]^{k_3} \\ &\quad \times [\psi(d(SSx_{2n+2}, Bx_{2n+1})) + \psi(d(Tx_{2n+1}, ASx_{2n+2}))]^{k_4} \\ &\leq [\psi(d(SSx_{2n+2}, Tx_{2n+1}))]^{k_1}[\psi(d(SSx_{2n+2}, ASx_{2n+2}))]^{k_2} \\ &\quad \times [\psi(d(Tx_{2n+1}, Bx_{2n+1}))]^{k_3} \\ &\quad \times [\psi(d(SSx_{2n+2}, Bx_{2n+1}))]^{k_4}[\psi(d(Tx_{2n+1}, ASx_{2n+2}))]^{k_4} \quad (3.38)\end{aligned}$$

On letting $n \to \infty$ in (3.38), using (3.36) and (3.37), we get

$$\begin{aligned}\psi(d(Sz,z)) &\leq [\psi(d(Sz,z))]^{k_1}[\psi(d(Sz,Sz))]^{k_2}[\psi(d(z,z))]^{k_3}[\psi(d(Sz,z))]^{k_4}[\psi(d(z,Sz))]^{k_4} \\ &= [\psi(d(Sz,z))]^{k_1+2k_4}[\psi(0)]^{k_2+k_3} = [\psi(d(Sz,z))]^{k_1+2k_4} < \psi(d(Sz,z)),\end{aligned}$$

a contradiction.

Hence, $Sz = z$. (3.39)

We now prove that $Az = z$.

If possible, suppose that $Az \neq z$.

Now we consider

$$\begin{aligned}\psi(d(Az, Bx_{2n+1})) &\leq [\psi(d(Sz, Tx_{2n+1}))]^{k_1}[\psi(d(Sz, Az))]^{k_2}[\psi(d(Tx_{2n+1}, Bx_{2n+1}))]^{k_3} \\ &\quad \times [\psi(d(Sz, Bx_{2n+1})) + \psi(d(Tx_{2n+1}, Az))]^{k_4} \\ &\leq [\psi(d(Az, Tx_{2n+1}))]^{k_1}[\psi(d(Az, Az))]^{k_2}[\psi(d(Tx_{2n+1}, Bx_{2n+1}))]^{k_3} \\ &\quad \times [\psi(d(Az, Bx_{2n+1}))]^{k_4}[\psi(d(Tx_{2n+1}, Az))]^{k_4} \quad (3.40)\end{aligned}$$

On taking limits as $n \to \infty$ in (3.40), using (3.36) and (3.37), we get

$$\begin{aligned}\psi(d(Az,z)) &\leq [\psi(d(Az,z))]^{k_1}[\psi(0)]^{k_2}[\psi(d(z,z))]^{k_3}[\psi(d(Az,z))]^{k_4}[\psi(d(z,Az))]^{k_4} \\ &= [\psi(d(Az,z))]^{k_1+2k_4}[\psi(0)]^{k_2+k_3} = [\psi(d(Az,z))]^{k_1+2k_4} < \psi(d(Az,z)),\end{aligned}$$

a contradiction.

Therefore $d(Az,z) \leq 0$ which implies that $Az = z$. (3.41)

From (3.39) and (3.41), we get $Az = Sz = z$.

Therefore z is a common fixed point of A and S.

Hence, by Lemma 2.1, we get that z is a common fixed point of A,B,S and T.

In a similar way, under the assumption (ii), we can obtain the existence of common fixed point of A,B,S and T. Uniqueness of common fixed point follows from the inequality (3.1). □

Theorem 3.7. *Let A,B,S and T be selfmapings on a metric space (X,d) and satisfy (A) and the inequality (3.1). Assume that the pairs (A,S) and (B,T) are weakly compatible. If either (A,S) (or) (B,T) satisfies the property (E.A) and either $S(X)$ (or) $T(X)$ is a closed subspace of X, then A,B,S and T have a unique common fixed point.*

Proof. First suppose that the pair (B,T) satisfy the property (E.A) and $S(X)$ is closed.

Then there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = z$, for some $z \in X$.

Since $S(X)$ is closed, we have $z \in S(X)$. Then there exists $u \in X$ such that $Su = p$.

Now, we prove that $Au = Su$.

Suppose that $Au \neq Su$.

By the inequality (3.1), we get

$$\begin{aligned}\psi(d(Au,Bx_n)) &\leq [\psi(d(Su,Tx_n))]^{k_1}[\psi(d(Su,Au))]^{k_2}[\psi(d(Tx_n,Bx_n))]^{k_3}\\ &\quad\times[\psi(d(Su,Bx_n))+\psi(d(Tx_n,Au))]^{k_4}\\ &\leq [\psi(d(z,Tx_n))]^{k_1}[\psi(d(z,Au))]^{k_2}[\psi(d(Tx_n,Bx_n))]^{k_3}\\ &\quad\times[\psi(d(z,Bx_n))]^{k_4}[\psi(d(Tx_n,Au))]^{k_4}\end{aligned}$$

Letting $n \to \infty$, we obtain

$$\begin{aligned}\psi(d(Au,z)) &\leq [\psi(d(z,z))]^{k_1}[\psi(d(z,Au))]^{k_2}[\psi(d(z,z))]^{k_3}[\psi(d(z,z))]^{k_4}[\psi(d(z,Au))]^{k_4}\\ &= [\psi(0)]^{k_1+k_3+k_4}[\psi(d(z,Au))]^{k_2+k_4} = [\psi(d(Au,z))]^{k_2+k_4} < \psi(d(Au,z)),\end{aligned}$$

a contradiction.

Therefore $Au = z$ implies that $Au = Su = z$.

Further, since $A(X) \subseteq T(X)$ there exists $v \in X$ such that $Au = Tv = z$.

Therefore $Au = Su = Tv = z$.

Now, we prove that $Bv = Tv$.

On the contrary suppose that $Bv \neq Tv$.

Using the inequality (3.1), we obtain

$$\begin{aligned}\psi(d(Tv,Bv)) &= \psi(d(Au,Bv))\\ &\leq [\psi(d(Su,Tv))]^{k_1}[\psi(d(Su,Au))]^{k_2}[\psi(d(Tv,Bv))]^{k_3}\\ &\quad\times[\psi(d(Su,Bv))+\psi(d(Tv,Au))]^{k_4}\\ &\leq [\psi(d(Tv,Tv))]^{k_1}[\psi(d(Tv,Tv))]^{k_2}[\psi(d(Tv,Bv))]^{k_3}\\ &\quad\times[\psi(d(Tv,Bv))]^{k_4}[\psi(d(Tv,Tv))]^{k_4}\\ &= [\psi(0)]^{k_1+k_2+k_4}[\psi(d(Tv,Bv))]^{k_3+k_4}\\ &= [\psi(d(Tv,Bv))]^{k_3+k_4} < \psi(d(Tv,Bv)),\end{aligned}$$

a contradiction.

Therefore $Au = Bv = Su = Tv = z$.

Suppose that the pairs (A,S) and (B,T) are weakly compatible and $Au = Su = z$, we have $ASu = SAu$ which implies that $Az = Sz$.

We now show that $Az = z$.

Suppose that $d(Az,z) > 0$.

By the inequality (3.1), we obtain

$$\begin{aligned}\psi(d(Az,Bx_n)) &\leq [\psi(d(Sz,Tx_n))]^{k_1}[\psi(d(Sz,Az))]^{k_2}[\psi(d(Tx_n,Bx_n))]^{k_3}\\ &\quad\times[\psi(d(Sz,Bx_n))+\psi(d(Tx_n,Az))]^{k_4}\\ &\leq [\psi(d(Az,Tx_n))]^{k_1}[\psi(d(Az,Az))]^{k_2}[\psi(d(Tx_n,Bx_n))]^{k_3}\\ &\quad\times[\psi(d(Az,Bx_n))]^{k_4}[\psi(d(Tx_n,Az))]^{k_4}\end{aligned}$$

On taking as $n\to\infty$, we get

$$\begin{aligned}\psi(d(Au,z)) &\leq [\psi(d(Az,z))]^{k_1}[\psi(d(Az,Az))]^{k_2}[\psi(d(z,z))]^{k_3}\\ &\quad\times[\psi(d(Az,z))]^{k_4}[\psi(d(z,Az))]^{k_4}\\ &= [\psi(d(Az,z))]^{k_1+2k_4}[\psi(0)]^{k_2+k_3}\\ &= [\psi(d(Az,z))]^{k_1+2k_4} < \psi(d(Az,z)),\end{aligned}$$

a contradiction.

Therefore $Az = Sz = z$.

Now, weakly compatibility of B and T and $Bv = Tv = z$, we have

$BTv = TBv$ which implies that $Bz = Tz$.

We now show that $Bz = z$.

Suppose that $d(Bz, z) > 0$.

By the inequality (3.1), we obtain

$$\begin{aligned}
\psi(d(z,Bz)) &= \psi(d(Az,Bz)) \\
&\leq [\psi(d(Sz,Tz))]^{k_1}[\psi(d(Sz,Az))]^{k_2}[\psi(d(Tz,Bz))]^{k_3} \\
&\quad \times [\psi(d(Sz,Bz))+\psi(d(Tz,Az))]^{k_4} \\
&\leq [\psi(d(z,Bz))]^{k_1}[\psi(d(z,z))]^{k_2}[\psi(d(Bz,Bz))]^{k_3}[\psi(d(z,Bz))]^{k_4}[\psi(d(Bz,z))]^{k_4} \\
&= [\psi(d(z,Bz))]^{k_1+2k_4}[\psi(0)]^{k_2+k_3} = [\psi(d(z,Bz))]^{k_1+2k_4} < \psi(d(z,Bz)),
\end{aligned}$$

a contradiction.

Therefore $Az = Bz = Sz = Tz = z$.

Hence z is a common fixed point of A, B, S and T.

Now, we suppose that the pair (A,S) satisfy the property (E.A) and $T(X)$ is closed.

Then there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$, for some $z \in X$.

Since $T(X)$ is closed, we have $z \in T(X)$. Then there exists $u \in X$ such that $Tu = z$.

We now prove that $Bu = Tu$.

Suppose that $d(Bu, Tu) > 0$.

Using the inequality (3.1), we obtain

$$\begin{aligned}
\psi(d(Ax_n,Bu)) &\leq [\psi(d(Sx_n,Tu))]^{k_1}[\psi(d(Sx_n,Ax_n))]^{k_2}[\psi(d(Tu,Bu))]^{k_3} \\
&\quad \times [\psi(d(Sx_n,Bu))+\psi(d(Tu,Ax_n))]^{k_4} \\
&\leq [\psi(d(Sx_n,z))]^{k_1}[\psi(d(Sx_n,Ax_n))]^{k_2}[\psi(d(z,Bu))]^{k_3} \\
&\quad \times [\psi(d(Sx_n,Bu))]^{k_4}[\psi(d(z,Ax_n))]^{k_4}
\end{aligned}$$

On letting $n \to \infty$, we get

$$\begin{aligned}\psi(d(z,Bu)) &\leq [\psi(d(z,z))]^{k_1}[\psi(d(z,z))]^{k_2}[\psi(d(z,Bu))]^{k_3} \\ &\quad \times [\psi(d(z,Bu))]^{k_4}[\psi(d(z,z))]^{k_4} \\ &= [\psi(0)]^{k_1+k_2+k_4}[\psi(d(z,Bu))]^{k_3+2k_4} \\ &= [\psi(d(z,Bu))]^{k_3+k_4} < \psi(d(z,Bu)),\end{aligned}$$

a contradiction.

Therefore $d(z,Bu) = 0$ implies that $Bu = Tu = z$.

Further, since $B(X) \subseteq S(X)$ there exists $v \in X$ such that $Bu = Sv = z$.

Therefore $Bu = Sv = Tu = z$.

We now show that $Av = Sv$.

On the contrary suppose that $Av \neq Sv$.

From the inequality (3.1), we obtain

$$\begin{aligned}\psi(d(Av,Sv)) &= \psi(d(Av,Bu)) \\ &\leq [\psi(d(Sv,Tu))]^{k_1}[\psi(d(Sv,Av))]^{k_2}[\psi(d(Tu,Bu))]^{k_3} \\ &\quad \times [\psi(d(Sv,Bu)) + \psi(d(Tu,Av))]^{k_4} \\ &\leq [\psi(d(Sv,Sv))]^{k_1}[\psi(d(Sv,Av))]^{k_2}[\psi(d(Sv,Sv))]^{k_3} \\ &\quad \times [\psi(d(Sv,Sv))]^{k_4}[\psi(d(Sv,Av))]^{k_4} \\ &= [\psi(0)]^{k_1+k_3+k_4}[\psi(d(Sv,Av))]^{k_2+k_4} \\ &= [\psi(d(Av,Sv))]^{k_2+k_4} < \psi(d(Av,Sv)),\end{aligned}$$

a contradiction.

Therefore $Av = Bu = Sv = Tu = z$.

Suppose that the pairs (A,S) and (B,T) are weakly compatible and $Av = Sv = z$, we have $ASv = SAv$ which implies that $Az = Sz$.

We now show that $Az = z$.

Suppose that $d(Az,z) > 0$.

By the inequality (2.1.1), we obtain

$$\begin{aligned}\psi(d(Az,Bx_n)) &\leq [\psi(d(Sz,Tx_n))]^{k_1}[\psi(d(Sz,Az))]^{k_2}[\psi(d(Tx_n,Bx_n))]^{k_3}\\ &\quad\times[\psi(d(Sz,Bx_n))+\psi(d(Tx_n,Az))]^{k_4}\\ &\leq [\psi(d(Az,Tx_n))]^{k_1}[\psi(d(Az,Az))]^{k_2}[\psi(d(Tx_n,Bx_n))]^{k_3}\\ &\quad\times[\psi(d(Az,Bx_n))]^{k_4}[\psi(d(Tx_n,Az))]^{k_4}\end{aligned}$$

On taking as $n \to \infty$, we get

$$\begin{aligned}\psi(d(Au,z)) &\leq [\psi(d(Az,z))]^{k_1}[\psi(d(Az,Az))]^{k_2}[\psi(d(z,z))]^{k_3}\\ &\quad\times[\psi(d(Az,z))]^{k_4}[\psi(d(z,Az))]^{k_4}\\ &= [\psi(d(Az,z))]^{k_1+2k_4}[\psi(0)]^{k_2+k_3}\\ &= [\psi(d(Az,z))]^{k_1+2k_4} < \psi(d(Az,z)),\end{aligned}$$

a contradiction.

Therefore $Az = Sz = z$.

Now, weakly compatibility of B and T and $Bu = Tu = z$, we have

$BTu = TBu$ which implies that $Bz = Tz$.

We now show that $Bz = z$.

Suppose that $d(Bz,z) > 0$.

By the inequality (3.1), we obtain

$$\psi(d(Ax_n,Bz)) \leq [\psi(d(Sx_n,Tz))]^{k_1}[\psi(d(Sx_n,Ax_n))]^{k_2}[\psi(d(Tz,Bz))]^{k_3}$$

$$\begin{aligned}&\times[\psi(d(Sx_n,Bz))+\psi(d(Tz,Ax_n))]^{k_4}\\ &\leq [\psi(d(Sx_n,Bz))]^{k_1}[\psi(d(Sx_n,Ax_n))]^{k_2}[\psi(d(Bz,Bz))]^{k_3}\\ &\times[\psi(d(Sx_n,Bz))]^{k_4}[\psi(d(Bz,Ax_n))]^{k_4}\end{aligned}$$

On letting $n \to \infty$, we get

$$\begin{aligned}\psi(d(z,Bz)) &\leq [\psi(d(z,Bz))]^{k_1}[\psi(d(z,z))]^{k_2}[\psi(Bz,Bz)]^{k_3} \\ &\quad \times [\psi(d(z,Bz))]^{k_4}[\psi(d(Bz,z))]^{k_4} \\ &= [\psi(d(z,Bz))]^{k_1+2k_4}[\psi(0)]^{k_2+k_3} \\ &= [\psi(d(z,Bz))]^{k_1+2k_4} < \psi(d(z,Bz)),\end{aligned}$$

a contradiction.

Therefore $d(z,Bz) = 0$ implies that $Bz = Tz = z$. Hence $Az = Bz = Sz = Tz = z$.

Thus, z is a common fixed point of A, B, S and T.

Similarly, we can prove the result when the pair (B,T) satisfies the property (E.A) and $T(X)$ is closed. Also, it can be proved when the pair (A,S) satisfies the property (E.A) and $S(X)$ is closed.

□

4. Corollaries and examples

In this section, we draw some corollaries from the main results of Section 3 and provide examples in support of our results.

The following is an example in support of Theorem 3.3.

Example 4.1. Let $X = [0,1]$ with usual metric. We define selfmaps $A, B, S,$ T on X by

$$A(x) = \begin{cases} x^2 & \text{if } 0 \leq x < \frac{1}{2} \\ 0 & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases}, \quad B(x) = \begin{cases} \frac{x^2}{2} & \text{if } 0 \leq x < \frac{1}{2} \\ 0 & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases}$$

$$S(x) = \begin{cases} 2x^2 & \text{if } 0 \leq x \leq \frac{1}{2} \\ 1 & \text{if } \frac{1}{2} < x \leq 1 \end{cases} \quad \text{and } T(x) = \begin{cases} x^2 & \text{if } 0 \leq x < \frac{1}{2} \\ 0 & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}$$

Here $A(X) = [0, \frac{1}{4}), B(X) = [0, \frac{1}{8}), S(X) = [0, \frac{1}{2}] \cup \{1\}$ and $T(X) = [0, \frac{1}{4})$ so that $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$. We have $ASx = SAx$ whenever $Ax = Sx$ and $BTx = TBx$ whenever $Bx = Tx$, hence the pairs (A,S) and (B,T) are weakly compatible and the set $S(X)$ is closed. We define $\psi : [0,\infty) \to [1,\infty)$ by $\psi(t) = e^t$.

Then clearly $\psi \in \Psi_1$.

Now, we verify the inequality (3.1) with $k = \frac{1}{2}$ and k_2, k_3, k_4 are arbitrary non-negative real numbers such that $0 \leq k_1 + k_2 + k_3 + 2k_4 < 1$.

Since $\psi(t) = e^t$, we have

$$|Ax - By| \leq k_1|Sx - Ty| + k_2|Sx - Ax| + k_3|Ty - By| + k_4[|Sx - By| + |Ty - Ax|] \tag{4.1}$$

for all $x, y \in X$

We now verify the inequality (4.1).

Case (i): $x, y \in [0, \frac{1}{2})$.

$|Ax - By| = |x^2 - \frac{y^2}{2}|; |Sx - Ty| = |2x^2 - y^2|$.

We have

$$\begin{aligned} |Ax - By| &= |x^2 - \frac{y^2}{2}| \\ &= \frac{1}{2}|2x^2 - y^2| \\ &\leq \frac{1}{2}|Sx - Ty| + k_2|Sx - Ax| + k_3|Ty - By| + k_4[|Sx - By| + |Ty - Ax|]. \end{aligned}$$

Case (ii): $x = y = \frac{1}{2}$.

In this case $|Ax - By| = 0$ and the inequality (4.1) trivially holds.

Case (iii): $x, y \in (\frac{1}{2}, 1]$.

$|Ax - By| = 0$ and trivially holds the inequality (4.1) in this case.

Case (iv): $x \in [0, \frac{1}{2}), y = \frac{1}{2}$.

$|Ax - By| = x^2; |Sx - Ty| = 2x^2$.

We have

$$\begin{aligned} |Ax - By| &= x^2 \\ &= \frac{1}{2}(2x^2) \\ &\leq \frac{1}{2}|Sx - Ty| + k_2|Sx - Ax| + k_3|Ty - By| + k_4[|Sx - By| + |Ty - Ax|]. \end{aligned}$$

Case (v): $x = \frac{1}{2}, y \in [0, \frac{1}{2})$.

$|Ax - By| = \frac{y^2}{2}; |Sx - Ty| = |2x^2 - y^2|$. We have

$$\begin{aligned} |Ax - By| &= \frac{y^2}{2} \\ &= \frac{1}{2}|2x^2 - y^2| \\ &\leq \frac{1}{2}|Sx - Ty| + k_2|Sx - Ax| + k_3|Ty - By| + k_4[|Sx - By| + |Ty - Ax|]. \end{aligned}$$

Case (vi): $x = \frac{1}{2}, y \in (\frac{1}{2}, 1]$.

In this case $|Ax - By| = 0$ and the inequality (4.1) trivially holds.

Case (vii): $x \in (\frac{1}{2}, 1], y = \frac{1}{2}$.

$|Ax - By| = 0$ and trivially holds the inequality (4.1) in this case.

From the above all cases, A, B, S and T satisfy the inequality (4.1).

Therefore A, B, S and T satisfy all the hypotheses of Theorem 3.3 and 0 is the unique common fixed point of A, B, S and T.

Corollary 4.2. Let $\{A_n\}_{n=1}^{\infty}, S$ and T be selfmaps on a complete metric space (X, d) satisfying $A_1 \subseteq S(X)$ and $A_1 \subseteq T(X)$. Assume that there exists $\psi \in \Psi_1$ such that

$$\begin{aligned}\psi(d(A_1x, A_jy)) \leq{} & [\psi(d(Sx, Ty))]^{k_1}[\psi(d(Sx, A_1x))]^{k_2}[\psi(d(Ty, A_jy))]^{k_3} \\ & \times [\psi(d(Sx, A_jy)) + \psi(d(Ty, A_1x))]^{k_4} \qquad (4.2)\end{aligned}$$

for all $x, y \in X$ and $j = 1, 2, 3, \ldots$. If the pairs (A_1, S) and (A_1, T) are weakly compatible and one of the range sets $A_1(X), S(X)$ and $T(X)$ is closed, then $\{A_n\}_{n=1}^{\infty}, S$ and T have a unique common fixed point in X.

Proof. Under the assumptions on A_1, S and T, the existence of common fixed point z of A_1, S and T follows by choosing $A = B = A_1$ in Theorem 2.3.

Therefore $A_1z = Sz = Tz = z$.

Now, let $j \in \mathbb{N}$ with $j \neq 1$.

We now consider

$$\begin{aligned}\psi(d(z, A_jz)) &= d(A_1z, A_jz) \\ &\leq [\psi(d(Sz, Tz))]^{k_1}[\psi(d(Sz, A_1z))]^{k_2}[\psi(d(Tz, A_jz))]^{k_3} \\ &\qquad \times [\psi(d(Sz, A_jz)) + \psi(d(Tz, A_1z))]^{k_4} \\ &\leq [\psi(d(z, z))]^{k_1}[\psi(d(z, z))]^{k_2}[\psi(d(z, A_jz))]^{k_3} \\ &\qquad \times [\psi(d(z, A_jz))]^{k_4}[\psi(d(z, z))]^{k_4} \\ &= [\psi(0)]^{k_1+k_2+k_4}[\psi(d(z, A_jz))]^{k_3+k_4} \\ &= [\psi(d(z, A_jz))]^{k_3+k_4} < \psi(d(z, A_jz)),\end{aligned}$$

a contradiction if $A_j z \neq z$. Therefore $A_j z = z$ for $j = 1, 2, 3, \ldots$.

Uniqueness of common fixed point follows from the inequality (4.2).

Hence, $\{A_n\}_{n=1}^{\infty}, S$ and T have a unique common fixed point in X. □

The following is an example in support of Theorem 3.5.

Example 4.3. Let $X = [0,2]$ with usual metric. We define selfmaps $A, B, S,$ T on X by

$$A(x) = \begin{cases} \frac{2}{3} & \text{if } 0 \le x < 2 \\ \frac{1}{2} & \text{if } x = 2, \end{cases} \quad B(x) = \begin{cases} \frac{2}{3} & \text{if } 0 \le x < 2 \\ \frac{1}{3} & \text{if } x = 2 \end{cases}$$

$$S(x) = \begin{cases} \frac{2}{3} & \text{if } 0 \le x < \frac{2}{3} \\ \frac{4}{3} - x & \text{if } \frac{2}{3} \le x \le 1 \\ 0 & \text{if } 1 < x \le 2 \end{cases} \text{ and } T(x) = \begin{cases} \frac{1}{3} & \text{if } x = 0 \\ 1 - \frac{x}{2} & \text{if } 0 < x \le 1 \\ \frac{4}{3} & \text{if } 1 < x \le 2 \end{cases}$$

Here $A(X) = \{\frac{1}{2}, \frac{2}{3}\}, B(X) = \{\frac{1}{3}, \frac{2}{3}\}, S(X) = [\frac{1}{3}, \frac{2}{3}] \cup \{0\}$ and $T(X) = [\frac{1}{2}, 1) \cup \{\frac{1}{3}, \frac{4}{3}\}$ so that $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$. Clearly the pair (A, S) is reciprocally continuous and compatible and the pair (B, T) is weakly compatible. We define $\psi : [0, \infty) \to [1, \infty)$ by $\psi(t) = e^t$. Then clearly $\psi \in \Psi_1$.

Now, we verify the inequality (3.1) with $k = \frac{1}{2}$ and k_2, k_3, k_4 are arbitrary non-negative real numbers such that $0 \le k_1 + k_2 + k_3 + 2k_4 < 1$. It is enough to verify the inequality (4.1).

Case (1): $x = y = 0$.

$|Ax - By| = 0$ and trivially holds the inequality (4.1).

Case (2): $x = y = 2$.

$|Ax - By| = \frac{1}{6}; d(Sx, Ty) = \frac{4}{3}$. We have

$$\begin{aligned} |Ax - By| = \frac{1}{6} &\le \frac{1}{2}(\frac{4}{3}) \\ &\le \frac{1}{2}|Sx - Ty| + k_2|Sx - Ax| + k_3|Ty - By| + k_4[|Sx - By| + |Ty - Ax|]. \end{aligned}$$

Case (3): $x, y \in (0, \frac{2}{3})$.

$|Ax - By| = 0$ and trivially holds the inequality (4.1).

Case (4): $x, y \in [\frac{2}{3}, 1]$.

$|Ax - By| = 0$ and trivially holds the inequality (4.1).

Case (5): $x, y \in (1, 2)$.

$|Ax - By| = 0$ and trivially holds the inequality (4.1).

Case (6): $x=0, y=2$.

$|Ax-By|=\frac{1}{3}; |Sx-Ty|=\frac{2}{3}$. We have

$$\begin{aligned}|Ax-By| &= \frac{1}{3}=\frac{1}{2}(\frac{2}{3})\\ &\leq \frac{1}{2}|Sx-Ty|+k_2|Sx-Ax|+k_3|Ty-By|+k_4[|Sx-By|+|Ty-Ax|].\end{aligned}$$

Case (7): $x=0, y\in(0,\frac{2}{3})$.

$|Ax-By|=0$ and trivially holds the inequality (4.1).

Case (8): $x=0, y\in[\frac{2}{3},1]$.

$|Ax-By|=0$ and trivially holds the inequality (4.1).

Case (9): $x=0, y\in(1,2)$.

$|Ax-By|=0$ and trivially holds the inequality (4.1).

Case (10): $x=2, y=0$.

$|Ax-By|=\frac{1}{6}; |Sx-Ty|=\frac{1}{3}$. We have

$$\begin{aligned}|Ax-By| &= \frac{1}{6}=\frac{1}{2}(\frac{1}{3})\\ &\leq \frac{1}{2}|Sx-Ty|+k_2|Sx-Ax|+k_3|Ty-By|+k_4[|Sx-By|+|Ty-Ax|].\end{aligned}$$

Case (11): $x=2, y\in(0,\frac{2}{3})$.

$|Ax-By|=\frac{1}{6}; |Sx-Ty|=1-\frac{y}{2}$. We have

$$|Ax-By|=\frac{1}{6}\leq\frac{1}{2}(1-\frac{y}{2})$$

$$\leq \frac{1}{2}|Sx-Ty|+k_2|Sx-Ax|+k_3|Ty-By|+k_4[|Sx-By|+|Ty-Ax|].$$

Case (12): $x=2, y\in[\frac{2}{3},1]$.

$|Ax-By|=\frac{1}{6}; |Sx-Ty|=1-\frac{y}{2}$. We have

$$\begin{aligned}|Ax-By| &= \frac{1}{6}\leq\frac{1}{2}(1-\frac{y}{2})\\ &\leq \frac{1}{2}|Sx-Ty|+k_2|Sx-Ax|+k_3|Ty-By|+k_4[|Sx-By|+|Ty-Ax|].\end{aligned}$$

Case (13): $x = 2, y \in (1,2)$.

$|Ax - By| = \frac{1}{6}; d(Sx,Ty) = \frac{4}{3}$. We have

$$\begin{aligned}|Ax - By| = \frac{1}{6} &\leq \frac{1}{2}(\frac{4}{3}) \\ &\leq \frac{1}{2}|Sx - Ty| + k_2|Sx - Ax| + k_3|Ty - By| + k_4[|Sx - By| + |Ty - Ax|].\end{aligned}$$

Case (14): $x \in (0, \frac{2}{3}), y = 0$.

$|Ax - By| = 0$ and trivially holds the inequality (4.1) in this case.

Case (15): $x \in (0, \frac{2}{3}), y = 2$.

$|Ax - By| = \frac{1}{3}; d(Sx,Ty) = \frac{2}{3}$. We have

$$\begin{aligned}|Ax - By| = \frac{1}{3} &= \frac{1}{2}(\frac{2}{3}) \\ &\leq \frac{1}{2}|Sx - Ty| + k_2|Sx - Ax| + k_3|Ty - By| + k_4[|Sx - By| + |Ty - Ax|].\end{aligned}$$

Case (16): $x \in (0, \frac{2}{3}), y \in [\frac{2}{3}, 1]$.

$|Ax - By| = 0$ and trivially holds the inequality (4.1) in this case.

Case (17): $x \in (0, \frac{2}{3}), y \in (1,2)$.

$|Ax - By| = 0$ and trivially holds the inequality (4.1) in this case.

Case (18): $x \in [\frac{2}{3}, 1], y = 0$.

$|Ax - By| = 0$ and trivially holds the inequality (4.1) in this case.

Case (19): $x \in [\frac{2}{3}, 1], y = 2$.

$|Ax - By| = \frac{1}{3}; d(Sx,Ty) = x$. We have

$$\begin{aligned}|Ax - By| = \frac{1}{3} &\leq \frac{1}{2}(x) \\ &\leq \frac{1}{2}|Sx - Ty| + k_2|Sx - Ax| + k_3|Ty - By| + k_4[|Sx - By| + |Ty - Ax|].\end{aligned}$$

Case (20): $x \in [\frac{2}{3}, 1], y \in (0, \frac{2}{3})$.

$|Ax - By| = 0$ and trivially holds the inequality (4.1) in this case.

Case (21): $x \in [\frac{2}{3}, 1], y \in (1,2)$.

$|Ax - By| = 0$ and trivially holds the inequality (4.1) in this case.

Case (22): $x \in (1,2), y = 0$.

$|Ax - By| = 0$ and trivially holds the inequality (4.1) in this case.

Case (23): $x \in (1,2), y = 2$.

$|Ax - By| = \frac{1}{3}; d(Sx,Ty) = \frac{4}{3}$. We have

$$|Ax - By| = \frac{1}{3} \leq \frac{1}{2}(\frac{4}{3}) \leq \frac{1}{2}|Sx - Ty| + k_2|Sx - Ax| + k_3|Ty - By| + k_4[|Sx - By| + |Ty - Ax|].$$

Case (24): $x \in (1,2), y \in (0, \frac{2}{3})$.

$|Ax - By| = 0$ and trivially holds the inequality (4.1) in this case.

Case (25): $x \in (1,2), y \in [\frac{2}{3}, 1]$.

$|Ax - By| = 0$ and trivially holds the inequality (4.1) in this case.

From the above all cases, A, B, S and T satisfy the inequality (4.1).

Therefore A, B, S and T satisfy all the hypotheses of Theorem 3.5 and $\frac{2}{3}$ is the unique common fixed point of A, B, S and T.

Example 4.4. Let $X = (0,1]$ with usual metric. We define selfmaps $A, B, S,$ T on X by

$$A(x) = \begin{cases} \frac{1}{3} & \text{if } 0 < x < \frac{2}{5} \\ \frac{2}{5} & \text{if } \frac{2}{5} \leq x \leq 1, \end{cases} \quad B(x) = \begin{cases} \frac{1}{4} & \text{if } 0 < x < \frac{2}{5} \\ \frac{2}{5} & \text{if } \frac{2}{5} \leq x \leq 1, \end{cases}$$

$$S(x) = \begin{cases} \frac{2}{3} & \text{if } 0 < x < \frac{2}{5} \\ \frac{1}{2} - \frac{x}{4} & \text{if } \frac{2}{5} \leq x \leq 1, \end{cases} \text{ and } T(x) = \begin{cases} 1 & \text{if } 0 < x < \frac{2}{5} \\ \frac{3}{5} - \frac{x}{2} & \text{if } \frac{2}{5} \leq x \leq 1, \end{cases}$$

Here $A(X) = \{\frac{1}{3}, \frac{2}{5}\}, B(X) = \{\frac{1}{4}, \frac{2}{5}\}, S(X) = [\frac{1}{4}, \frac{2}{5}] \cup \{\frac{2}{3}\}$ and $T(X) = [\frac{1}{10}, \frac{2}{5}] \cup \{1\}$.

Clearly $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$.

Since there is a sequence $\{x_n\} = \frac{2}{5} + \frac{1}{n}, n \geq 2$ with $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \frac{2}{5}$ and $\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = \frac{2}{5}$, the pairs (A,S) and (B,T) satisfy the property (E.A) and $S(X)$ is closed. We define $\psi : [0,\infty) \to [1,\infty)$ by $\psi(t) = e^t$.

Then clearly $\psi \in \Psi_1$.

Now, we verify the inequality (3.1) with $k = \frac{1}{2}$ and k_2, k_3, k_4 are arbitrary non-negative real numbers such that $0 \leq k_1 + k_2 + k_3 + 2k_4 < 1$. It is enough to verify the inequality (4.1).

Case (i): $x, y \in (0, \frac{2}{5})$.

$|Ax - By| = \frac{1}{12}; d(Sx,Ty) = \frac{1}{2}$. We have

$$|Ax - By| = \frac{1}{12} \leq \frac{1}{2}(\frac{1}{2}) \leq \frac{1}{2}|Sx - Ty| + k_2|Sx - Ax| + k_3|Ty - By| + k_4[|Sx - By| + |Ty - Ax|].$$

<u>*Case* (*ii*)</u>: $x, y \in [\frac{2}{5}, 1]$.

$|Ax - By| = 0$ and trivially holds the inequality (4.1) in this case.

<u>*Case* (*iii*)</u>: $x \in (0, \frac{2}{5}), y \in [\frac{2}{5}, 1]$.

$|Ax - By| = \frac{1}{15}; d(Sx, Ty) = (\frac{y}{2} + \frac{1}{15})$. We have

$$|Ax - By| = \frac{1}{15} \leq \frac{1}{2}(\frac{y}{2} + \frac{1}{15})$$
$$\leq \frac{1}{2}|Sx - Ty| + k_2|Sx - Ax| + k_3|Ty - By| + k_4[|Sx - By| + |Ty - Ax|].$$

<u>*Case* (*iv*)</u>: $x \in [\frac{1}{2}, 1], y \in [0, \frac{1}{2})$.

$|Ax - By| = \frac{3}{20}; d(Sx, Ty) = (\frac{x}{4} + \frac{1}{2})$. We have

$$|Ax - By| = \frac{3}{20} \leq \frac{1}{2}(\frac{x}{4} + \frac{1}{2})$$
$$\leq \frac{1}{2}|Sx - Ty| + k_2|Sx - Ax| + k_3|Ty - By| + k_4[|Sx - By| + |Ty - Ax|].$$

From the above four cases, A, B, S and T satisfy the inequality (4.1).

Therefore A, B, S and T satisfy all the hypotheses of Theorem 2.7 and $\frac{2}{5}$ is the unique common fixed point of A, B, S and T.

Remark 4.5. Theorem 2.12, follows as a corollary to Theorem 3.3 by choosing $A = B = f$ and $T = S = g$.

Conflict of Interests

The authors declare that there is no conflict of interests.

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