Available online at http://scik.org
J. Math. Comput. Sci. 9 (2019), No. 3, 239-253
https://doi.org/10.28919/jmcs/3983
ISSN: 1927-5307

SOLUTION OF AN EQUATION IN POISSON PARTIAL DERIVATIVES WITH CONDITIONS OF DIRICHLET USING TECHNIQUES OF THE INVERSE MOMENTS PROBLEM

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Abstract. In this paper, it will be shown that finding solutions from the Helmholtz equation and the non-linear Poisson equation under Dirichlet conditions is equivalent to solving an integral equation, which can be treated as a generalized two-dimensional moment problem over a domain that is considered rectangular in principle. We will see that an approximate solution of the equation in partial derivatives can be found using the techniques of generalized inverse moments problem and bounds for the error of the estimated solution. The method consists of two steps.In each one an integral equation is solved numerically using the two-dimensional inverse moments problem techniques. We illustrate the different cases with examples.

Keywords: equation in Poisson partial derivatives; integral equations; generalized moment problem.

2010 AMS Subject Classification: 45Q05, 45D05, 44A60.

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Received December 11, 2018

1. Introduction

You want to find $w(x,t)$ such that

$$w_{xx} + w_{tt} = f(x,t)$$

about a domain $E = (a_1, b_1) \times (a_2, b_2)$ or $E = (a_1, b_1) \times (a_2, \infty)$.
using the problem generalized moments techniques.

The problem has been solved using other techniques such as the Galerkin method [1], or the method of finite differences in regions of irregular shape [2].

The objective of this work is to show that we can solve the problem using the techniques of inverse moments problem. We focus the study on the numerical approximation.

The generalized moments problem [3, 4]is to find a function $f(x)$ about a domain $\Omega \subset R^d$ that satisfies the sequence of equations

$$\mu_i = \int_{\Omega} g_i(x) f(x) dx \quad i\varepsilon N \tag{1}$$

where N is the set of the natural numbers,$(g_i(x))$ is a given sequence of functions in $L^2(\Omega)$ linearly independent known and the succession of real numbers $\{\mu_i\}_{i\varepsilon N}$ are known data. The problem of Hausdorff moments [3, 4], is to find a function $f(x)$ en (a,b) such that

$$\mu_i = \int_a^b x^i f(x) dx \quad i\varepsilon N \tag{2}$$

In this case $g_i(x) = x^i$ with i belonging to the set N.
If the integration interval is $(0,\infty)$ we have the problem of Stieltjes moments; if the integration interval is $(-\infty,\infty)$ we have the problem of Hamburger moments [3, 4].
The moments problem is an ill-conditioned problem in the sense that there may be no solution and if there is no continuous dependence on the given data [3, 4]. There are several methods to build regularized solutions. One of them is the truncated expansion method [3]. This method is to approximate(1) with the finite moments problem

$$\mu_i = \int_{\Omega} g_i(x) f(x) dx \quad i = 1, 2, ..., n \tag{3}$$

where it is considered as approximate solution of $f(x)$ to $p_n(x) = \sum_{i=0}^{n} \lambda_i \phi_i(x)$, and the functions $\{\phi_i(x)\}_{i=1,\ldots,n}$ result of orthonormalize $\langle g_1, g_2, \ldots, g_n \rangle$ being λ_i the coefficients based on the data μ_i . In the subspace generated by $\langle g_1, g_2, \ldots, g_n \rangle$ the solution is stable. If $n \varepsilon N$ is chosen in an appropriate way then the solution of (3) it approaches the solution of the original problem (1). In the case where the data μ_i are inaccurate the convergence theorems should be applied and error estimates for the regularized solution (pg. 19 a 30 de [3]).

2. Resolution of the Poisson equation

You want to find $w(x,t)$ such that

(4)
$$w_{xx} + w_{tt} = f(x,t)$$

about a domain $E = (a_1, b_1) \times (a_2, b_2)$ $E = (a_1, b_1) \times (a_2, \infty)$.
We consider

(5)
$$w_{xx} - kw_{tt} = -(k+1)w_{tt} + f(x,t) = G(x,t)$$

If $w_x \neq w_t$ we can take $k = 1$.

We consider as auxiliary function

$$u(m,r,x,t) = e^{-m(x+1)} e^{-r(t+1)}$$

If the region E is bounded the conditions are:

(6)
$$\begin{array}{ll} w(a_1,t) = k_1(t) & w(b_1,t) = k_2(t) \\ w(x,a_2) = h_1(x) & w(x,b_2) = h_2(x) \end{array}$$

If the region E it is not bounded the conditions are:

(7)
$$w(a_1,t) = k_1(t) \qquad w(b_1,t) = k_2(t) \qquad w(x,a_2) = h_1(x)$$

We define the vector field

$$F^* = (F_1(w), F_2(w)) = (w_x, -kw_t)$$

As $udiv(F^*) = uG(x,t)$ we have to:

$$\iint_E udiv(F^*)dA = \iint_E uG(x,t)dA$$

Moreover, as $udiv(F^*) = div(uF^*) - F^*.\nabla u$, then

$$\iint_E udiv(F^*)dA = \iint_E div(uF^*)dA - \iint_E F^*.\nabla udA \tag{8}$$

where $\nabla u = (u_x, u_t)$.

Besides that

$$\begin{aligned}\iint_E div(uF^*)dA = \iint_E (uw_x)_x - (ukw_t)_t\, dA = \\ \iint_E udiv(F^*)dA + \iint_E (u_x w_x - u_t kw_t)\, dA\end{aligned} \tag{9}$$

Then from (8) y (9):

$$\iint_E (u_x w_x - u_t kw_t)\, dA = \iint_E F^*.\nabla udA \tag{10}$$

On the other hand, it can be proven, after several calculations that, integrating by parts:

$$\iint_E F^*.\nabla udA = A(m,r) + B(m,r) - \iint_E uw(m^2 - kr^2)dA = \varphi(m,r) \tag{11}$$

with

$$A(m,r) = \int_{a_2}^{b_2} (-m)u(m,r,b_1,t)w(b_1,t) - (-m)u(m,r,a_1,t)w(a_1,t)dt$$

$$B(m,r) = \int_{a_1}^{b_1} (-r)u(m,r,x,b_2)(-k)w(x,b_2) - (-r)u(m,r,x,a_2)(-k)w(x,a_2)dx$$

If $m = \sqrt{k}r$, instead of (10) y (11) :

$$\iint_E (-\sqrt{k}r)uw_x - (-r)kw_t udA = \varphi(\sqrt{k}r, r)$$

$$\therefore \iint_E u(-\sqrt{k}w_x + kw_t)dA = \frac{\varphi(\sqrt{k}r, r)}{r}$$

with

$$\begin{aligned}\frac{\varphi(\sqrt{k}r, r)}{r} = \int_{a_2}^{b_2} -\sqrt{k}u(\sqrt{k}r, r, b_1, t)w(b_1, t) + \sqrt{k}u(\sqrt{k}r, r, a_1, t)w(a_1, t)dt + \\ + \int_{a_1}^{b_1} -u(\sqrt{k}r, r, x, b_2)(-k)w(x, b_2) + u(\sqrt{k}r, r, x, a_2)(-k)w(x, a_2)dx\end{aligned}$$

We note $\varphi_1(r) = \dfrac{\varphi(\sqrt{k}r, r)}{r}$, then

$$\iint_E u(-\sqrt{k}w_x + kw_t)dA = \varphi_1(r) \tag{12}$$

To solve this integral equation we take a base $\psi_i(r) = r^i e^{-r} \quad i = 0, 1, 2, ..., n$ of $L^2(E)$.
Then we multiply both members of (12) by $\psi_i(r) = r^i e^{-r}$ and we integrate with respect to r, we obtain

$$\iint_E H_i(x,t)(-\sqrt{k}w_x + kw_t)dA = \int_{a_2}^{b_2} \varphi_1(r)\psi_i(r)dr = \mu_i \quad i = 0, 1, 2, ..., n \tag{13}$$

where $H_i(x,t) = \int_{a_2}^{b_2} u(-\sqrt{k}r, r, x, t)\psi_i(r)dr$.
We can interpret (13) as a generalized two-dimensional moment problem. We solve it numerically with the truncated expansion method and we found an approximation $p_n(x,t)$ for $-\sqrt{k}w_x + kw_t$.

3. Solution of the generalized moments problem

We can apply the detailed truncated expansion method in [4] and generalized in [1] and [5] to find an approximation $p_n(x,t)$ of $-\sqrt{k}w_x + kw_t$ for the corresponding finite problem with $i = 0, 1, 2, ..., n$, where n is the number of moments μ_i. We consider the basis $\phi_i(x,t) \quad i = 0, 1, 2, ..., n$ obtained by applying the Gram-Schmidt orthonormalization process on $H_i(x,t) \quad i = 0, 1, 2, ..., n$.
We approximate the solution $-\sqrt{k}w_x + kw_t$ with [4] and generalized in [3] y [5]:

$$p_n(x,t) = \sum_{i=0}^{n} \lambda_i \phi_i(x,t) \quad \text{donde} \quad \lambda_i = \sum_{j=0}^{i} C_{ij}\mu_j \quad i = 0, 1, 2, ..., n$$

And the coefficients C_{ij} verify

$$C_{ij} = \left(\sum_{k=j}^{i-1} (-1) \frac{\langle H_i(x,t) \mid \phi_k(x,t) \rangle}{\|\phi_k(x,t)\|^2} C_{kj} \right) . \|\phi_i(x,t)\|^{-1} \ 1 < i \le n; 1 \le j < i$$

The terms of the diagonal are

$$C_{ii} = \|\phi_i(x,t)\|^{-1} \qquad i = 0, 1, ..., n.$$

The proof of the following theorem is in [6, 7]. In [7] the demonstration is made for b_2 finite. If $b_2 = \infty$ instead of taking the Legendre polynomials we take the Laguerre polynomials. En [8] the demonstration is made for the one-dimensional case.

This Theorem gives a measure about the accuracy of the approximation.

Theorem. Let $\{\mu_i\}_{i=0}^n$ be a set of real numbers and suppose that $f(x,t) \in L^2((a_1,b_1)\times(a_2,b_2))$ for two positive numbers ε and M verify:

$$\sum_{i=0}^{n}\left|\int_{a_2}^{b_2}\int_{a_1}^{b_1} H_i(x,t)f(x,t)dxdt - \mu_i\right|^2 \leq \varepsilon^2$$

$$\int_{a_2}^{b_2}\int_{a_1}^{b_1}((b_1-a_1)^2 f_x^2 + (b_2-a_2)^2 f_t^2)dxdt \leq M^2 \tag{14}$$

then

$$\int_{a_2}^{b_2}\int_{a_1}^{b_1}|f(x,t)|^2 dxdt \leq \min_i \left\{ \left\|CC^T\right\|\varepsilon^2 + \frac{M^2}{8(i+1)^2};\ i=0,1,...,n \right\}$$

where C it is a triangular matrix with elements C_{ij} $\quad (1 < i \leq n;\, 1 \leq j < i)$ and

$$\int_{a_2}^{b_2}\int_{a_1}^{b_1}|p_n(x,t)-f(x,t)|^2 dxdt \leq \left\|CC^T\right\|\varepsilon^2 + \frac{M^2}{8(n+1)^2} \tag{15}$$

If b_2 it is not finite then (14) change by

$$\int_{a_2}^{b_2}\int_{a_1}^{b_1}(xf_x^2 + tf_t^2)dxdt \leq M^2 \tag{16}$$

And it must be fulfilled that

$$t^i f(x,t) \longrightarrow 0 \quad \text{if} \quad t \longrightarrow \infty \quad \forall i \varepsilon N$$

So we have an equation in first order partial derivatives of the form

$$-\sqrt{k}w_x(x,t) + kw_t(x,t) = p_n(x,t)$$

that is, it can be written as

$$A_1(x,t)w_x(x,t) + A_2(x,t)w_t(x,t) = p_n(x,t)$$

where $A_1(x,t) = -\sqrt{k}$ and $A_2(x,t) = k$.

It is resolved as in [9], that is, we can prove that solving this equation is equivalent to solving the integral equation

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2} K(m,r,x,t)w(x,t)dtdx = \varphi_2(m,r) \tag{17}$$

with $K(m,r,x,t) = u(m,r,x,t)(m_1\sqrt{k}(m+1) - m_2k(r+1))$

where now it is taken as an auxiliary function

$$u(m,r,x,t) = e^{-m_1(m+1)(x+1)}e^{-m_2(r+1)(t+1)}$$

The values of m_1 and m_2 are chosen in a convenient way to avoid discontinuities.

and

$$\varphi_2(m,r) = \int_{a_1}^{b_1} u(m,r,x,b_2)kw(x,b_2) - u(m,r,x,a_2)kw(x,a_2)dx -$$

$$-\int_{a_2}^{b_2} u(m,r,b_1,t)\sqrt{k}w(b_1,t) - u(m,r,a_1,t)\sqrt{k}w(a_1,t)dt - \int_{a_2}^{b_2}\int_{a_1}^{b_1} p_n(x,t)udxdt$$

Again we take a base:

$$\psi_{ij}(m,r) = m^i r^j e^{-(m+r)} \qquad i = 0,1,...,n_1 \quad j = 0,1,2,...,n_2$$

and we multiply both members of (17) by $\psi_{ij}(m,r)$ and we integrate with respect to m and r

We have then the generalized moments problem

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2} w(x,t)H_{ij}(x,t) = \mu_{ij} \tag{18}$$

where

$$\mu_{ij} = \int_{a_1}^{b_1}\int_{a_2}^{b_2} \varphi_2(m,r)\psi_{ij}(m,r)dmdr$$

$$H_{ij}(x,t) = \int_{a_1}^{b_1}\int_{a_2}^{b_2} K(m,r,x,t)\psi_{ij}(m,r)dmdr$$

We apply the truncated expansion method and find a numerical approximation for $w(x,t)$.

3.1. Numerical examples.

Example 1. We consider the equation

$$w_{xx} + w_{tt} = \frac{120}{(3+t+2x)^2} \qquad \text{in} \quad (0,1)\times(0,1)$$

Conditions:

$$w(0,t) = \frac{4}{(3+t)^2} \quad w(1,t) = \frac{4}{(5+t)^2}$$
$$w(x,0) = \frac{4}{(3+2x)^2} \quad w(x,1) = \frac{4}{(4+2x)^2}$$

The solution is : $w(x,t) = \dfrac{4}{(3+2x+t)^2}$.

For the first step we take $n = 5$ moments and we approximate $-w_x(x,t) + w_t(x,t) = G(x,t)$ with accuracy

$$\int_0^1 \int_0^1 (p_5(x,t) - G(x,t))^2 dtdx = 0.014211$$

In the Figure 1 we show $p_5(x,t)$ and $G(x,t)$ overlapping.

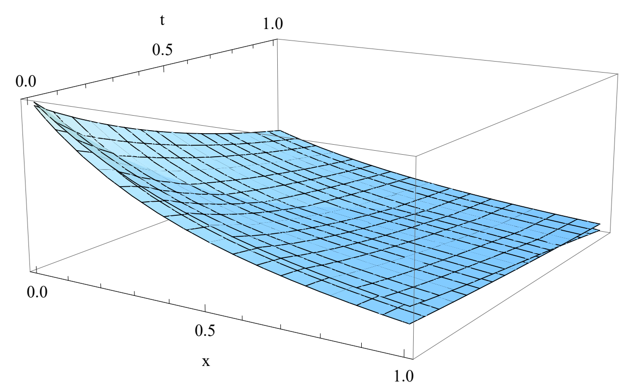

FIGURE 1. $p_5(x,t)$ and $G(x,t)$.

For the second step we take $m_1 = 1$ and $m_2 = 2$. We also consider $n_1 = 3$ and $n_2 = 2$, that is 6 moments.

We approximate $w(x,t)$ with accuracy

$$\int_0^1 \int_0^1 (p_6(x,t) - w(x,t))^2 dtdx = 0.0380442$$

In the Figure 2 we show $p_6(x,t)$ and $w(x,t)$ overlapping.

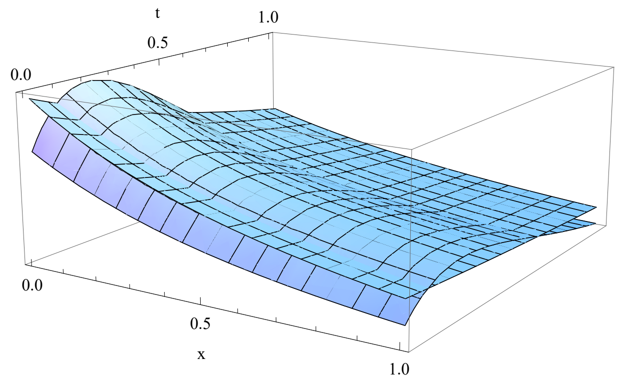

FIGURE 2. $p_6(x,t)$ and $w(x,t)$.

Ejemplo 2. We consider the equation

$$w_{xx} + w_{tt} = 2e^{-1-x-t} \qquad \text{in} \quad (0,2)\times(0,\infty)$$

Conditions:

$$w(0,t) = e^{-1-t} \quad w(2,t) = e^{-3-t} \qquad w(x,0) = e^{-1-x}$$

The solution is : $w(x,t) = e^{-1-x-t}$.

For the first step we take $n = 5$ moments and we approximate $-\sqrt{2}w_x(x,t) + 2w_t(x,t) = G(x,t)$ with accuracy

$$\int_0^2 \int_0^\infty (p_5(x,t) - G(x,t))^2 dt dx = 0.0121825$$

In this example we take $k = 2$,since otherwise $G(x,t) = 0$.

In the Figure 3 we show $p_5(x,t)$ and $G(x,t)$ overlapping.

For the second step we take $m_1 = 1$ and $m_2 = 2$. We also consider $n_1 = 3$ y $n_2 = 2$, that is 6 moments.

We approximate $w(x,t)$ with accuracy

$$\int_0^2 \int_0^\infty (p_6(x,t) - w(x,t))^2 dt dx = 0.0427058$$

In the Figure 4 we show $p_6(x,t)$ and $w(x,t)$ overlapping.

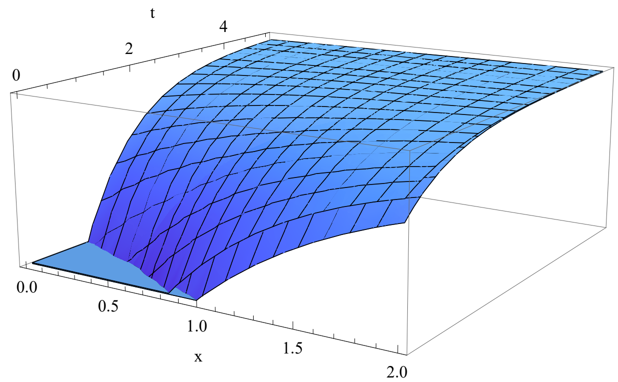

FIGURE 3. $p_5(x,t)$ and $G(x,t)$.

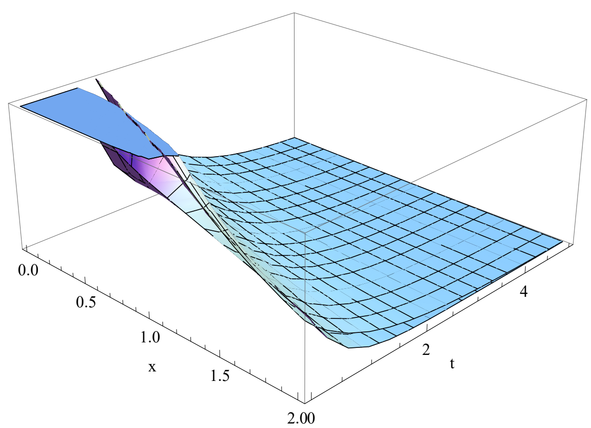

FIGURE 4. $p_6(x,t)$ and $w(x,t)$.

Ejemplo 3. In general, the method could be applied to any region that can be written simultaneously as

$$E = \{(x,t); \quad a_1 < x < b_1; \quad g_1(x) < t < g_2(x)\}$$

and

$$E = \{(x,t); \quad h_1(t) < x < h_2(t); \quad a_2 < t < b_2\}$$

We can apply the above to a circular region:

We consider the equation

$$w_{xx} + w_{tt} = 5e^{-1-x-2t} \qquad \text{en} \quad E = \{(x,t); -1 < x < 1; -\sqrt{1-x^2} < t < \sqrt{1-x^2}\}$$

Conditions:we must know $w(x,t)$ on the edge of E

$$w(-\sqrt{1-t^2},t) \quad w(\sqrt{1-t^2},t) \qquad w(x,-\sqrt{1-x^2}) \quad w(x,\sqrt{1-x^2})$$

The solution is : $w(x,t) = e^{-1-x-2t}$.

The integrals are solved in numerical form without making change to polar coordinates using the Mathematica software.

For the first step we take $n = 5$ moments and we approximate $-w_x(x,t) + w_t(x,t) = G(x,t)$ with accuracy

$$\iint_E (p_5(x,t) - G(x,t))^2 dtdx = 0.459748$$

In this example we take $k = 1$.

In the Figure 5 we show $p_5(x,t)$ and $G(x,t)$ overlapping.

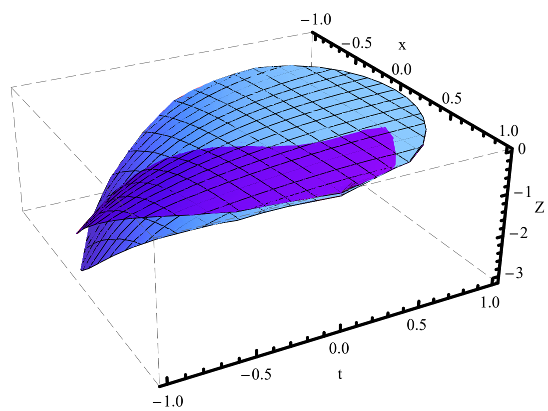

FIGURE 5. $p_5(x,t)$ and $G(x,t)$.

For the second step we take $m_1 = 1$ and $m_2 = 2$. We also consider $n_1 = 3$ y $n_2 = 2$, that is 6 moments.

We approximate $w(x,t)$ with accuracy

$$\iint_E (p_6(x,t) - w(x,t))^2 dtdx = 0.176225$$

In the Figure 6 we show $p_6(x,t)$ and $w(x,t)$ overlapping.

Ejemplo 4. We consider the equation

$$w_{xx} + w_{tt} = 5e^{-1-x-2t} \qquad \text{on} \quad E = \{(x,t); \quad 0 < x < 2 \quad ; \quad 2x < t < x^2\}$$

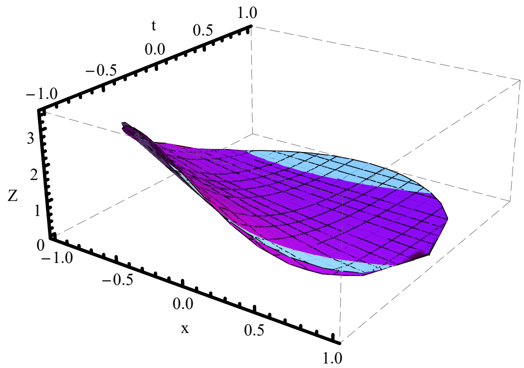

FIGURE 6. $p_6(x,t)$ and $w(x,t)$.

or

$$w_{xx} + w_{tt} = 5e^{-1-x-2t} \qquad \text{on} \quad E = \{(x,t); \quad \frac{t}{2} < x < \sqrt{t} \quad ; \quad 0 < t < 4\}$$

Conditions:we must know $w(x,t)$ on the edge of E

$$w(2x,t) \quad w(x^2,t) \qquad w(x,\frac{t}{2}) \quad w(x,\sqrt{t})$$

The solution is : $w(x,t) = e^{-1-x-2t}$.

For the first step we take $n = 5$ moments and we approximate $-w_x(x,t) + w_t(x,t) = G(x,t)$ with accuracy

$$\iint_E (p_5(x,t) - G(x,t))^2 dt dx = 0.00435357$$

In this example we take $k = 1$.

In the Figure 7 we show $p_5(x,t)$ and $G(x,t)$ overlapping.

For the second step we take $m_1 = 1$ and $m_2 = 2$. We also consider $n_1 = 3$ y $n_2 = 2$, that is 6 moments.

We approximate $w(x,t)$ with accuracy

$$\iint_E (p_6(x,t) - w(x,t))^2 dt dx = 0.00137451$$

In the Figure 8 we show $p_6(x,t)$ and $w(x,t)$ overlapping.

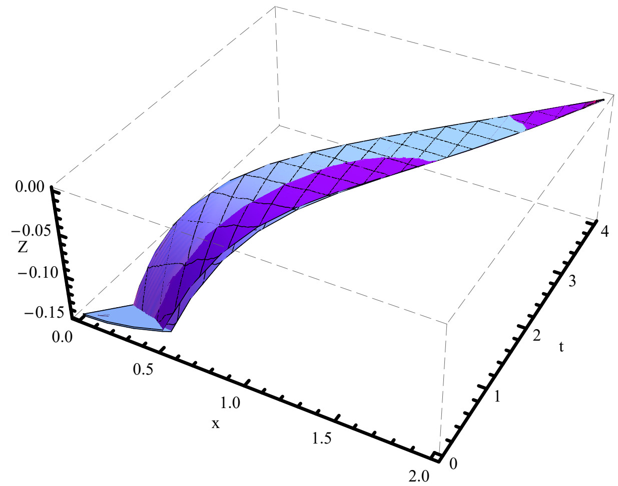

FIGURE 7. $p_5(x,t)$ and $G(x,t)$.

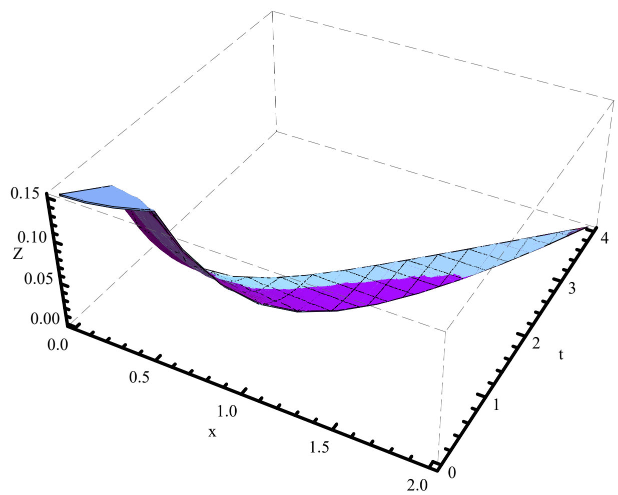

FIGURE 8. $p_6(x,t)$ and $w(x,t)$.

4. CONCLUSIONS

An equation in partial Poisson derivatives of the form $w_{xx}+w_{tt}=f(x,t)$ where the unknown function $w(x,t)$ is defined in $E=(a_1,b_1)\times(a_2,b_2)$ or $E=(a_1,b_1)\times(a_2,\infty)$ or E a region that can be written simultaneously as

$$E=\{(x,t);\quad a_1<x<b_1;\quad g_1(x)<t<g_2(x)\}$$

and

$$E=\{(x,t);\quad h_1(t)<x<h_2(t);\quad a_2<t<b_2\}$$

under the conditions of Dirichlet can be solved numerically by applying inverse problem techniques of moments in two steps:

(1) first we consider the integral equation

$$\iint_E u(-\sqrt{k}w_x + kw_t)dA = \varphi_1(r)$$

we can solve it numerically as a inverse moments problem, and we get an approximate solution for $-\sqrt{k}w_x(x,t) + kw_t(x,t)$.

(2) as a second step we consider the integral equation

$$\iint_E K(m,r,x,t)w(x,t)dtdx = \varphi_2(m,r)$$

and again it can be solved numerically by applying inverse moments problem techniques, and we get an approximate solution for $w(x,t)$.

The function $f(x,t)$ it is not used in calculations, but it is implicitly considered in the boundary conditions.

Acknowledgments

I appreciate the support of the Faculty of Engineering and the Faculty of Exact Sciences of the National University of La Plata, Argentina.

Conflict of Interests

The authors declare that there is no conflict of interests.

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