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AN OPTIMAL CONTROL FOR A HYBRID HYPERBOLIC DYNAMIC SYSTEM

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Abstract. In this paper, we are concerned with a hybrid hyperbolic dynamic system formulated by partial differential equations with initial and boundary conditions. An optimal energy control of the system is investigated. First, the system is transformed to an abstract evolution system in an appropriate Hilbert space, and then semigroup generation of the system operator is discussed. Finally, an optimal energy control problem is proposed and it is shown that an optimal energy control can be obtained by a finite dimensional approximation.

Keywords: optimal control; hybrid hyperbolic dynamic system; partial differential equations.

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1. Introduction

In this paper, we are concerned with the following general hyperbolic system with static boundary condition in one space variable in normal form studied in [1] and [2]:

$$(1)\qquad \begin{cases} \dfrac{\partial}{\partial t}\begin{bmatrix} u(x,t) \\ v(x,t) \end{bmatrix} + K(x)\dfrac{\partial}{\partial x}\begin{bmatrix} u(x,t) \\ v(x,t) \end{bmatrix} + C(x)\begin{bmatrix} u(x,t) \\ v(x,t) \end{bmatrix} = 0, 0<x<1, t>0, \\ v(1,t)=Du(1,t), u(0,t)=Ev(0,t) \end{cases}$$

where

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(H1) $K(x) = \text{diag}\{\lambda_1(x), \lambda_2(x), \cdots, \lambda_m(x), \mu_1(x), \mu_2(x), \cdots, \mu_k(x)\}$ is a diagonal $n \times n, (n = m+k)$, matrix with real entries $\lambda_j(x), \mu_j(x) \in C^1[0,1], \lambda_j(x) > 0, \mu_i(x) < 0, \forall\, x \in [0,1], i = 1,2,\cdots,k, j = 1,2,\cdots,m$.

(H2) $C(x) = \text{diag}\{c_1(x), c_2(x), \cdots, c_n(x)\}$ is an $n \times n$ diagonal matrix with continuous entries in $x \in [0,1]$;

(H3) $u(x) = [u_1(x), u_2(x), \cdots, u_m(x)]^\top$ is a column vector in $\mathbb{R}^m$ (or $\mathbb{C}^m$) and $v(x) = [v_1(x), v_2(x), \cdots, v_k(x)]^\top$ is a column vector in $\mathbb{R}^k$ (or $\mathbb{C}^k$);

(H4) D, E, F and G are real (or complex) constant matrices of appropriate size.

In this paper, our goal is to investigate an optimal energy control of the system. First, we transfer the system to an abstract Cauchy problem in an appropriate Hilbert space, and then discuss the semigroup generation of the system operator. Finally, we propose an optimal energy control problem and show that the optimal energy control exists and it can be obtained by a finite dimensional approximation.

2. Semigroup Generation of the System

Consider the system (1.1) in the underlying Hilbert space $\mathscr{H} = \left(L^2(0,1)\right)^2$. Define the operator $\mathscr{A} : D(\mathscr{A})(\subset \mathscr{H}) \to \mathscr{H}$ by

$$(1)\quad \begin{cases} \mathscr{A}\begin{bmatrix} u(x) \\ v(x) \end{bmatrix} = -K(x)\dfrac{\partial}{\partial x}\begin{bmatrix} u(x) \\ v(x) \end{bmatrix} - C(x)\begin{bmatrix} u(x) \\ v(x) \end{bmatrix}, \\ D(\mathscr{A}) = \{[u,v]^\top \in \left(H^1(0,1)\right)^m \times \left(H^1(0,1)\right)^k, u(0) = Ev(0), v(1) = Du(1)\}. \end{cases}$$

Then the system (2.1) can be written an an evolution equation in $\mathscr{H}$:

$$(2)\quad \frac{dW(t)}{dt} = \mathscr{A}W(t), t > 0$$

with $W(t) = [u(\cdot,t), v(\cdot,t)]^\top$.

Lemma 2.1 The operator $\mathscr{A}$ definde by (2.2) has compact resolvent and hence $\sigma(\mathscr{A})$ consists only isolated eigenvalues.

Proof. Given $(f, g, b) \in X$, we solve

$(\lambda-\mathscr{A})(u,v,d)=(f,g,b)$

that is,

$$(3)\quad \begin{cases} \frac{\partial}{\partial t}\begin{bmatrix} u(x,t) \\ v(x,t) \end{bmatrix} = -K^{-1}(x)[\lambda+C(x)]\begin{bmatrix} u(x,t) \\ v(x,t) \end{bmatrix} + K^{-1}(x)\begin{bmatrix} f(x) \\ g(x) \end{bmatrix}, 0<x<1, t>0, \\ v(1,t)=Du(1,t), u(0,t)=Ev(0,t) \end{cases}$$

Denote by $M(x,y,\lambda)$ the fundamental matrix of the system

$$(4)\qquad \frac{d}{dx}\begin{bmatrix} u(x) \\ v(x) \end{bmatrix} = -K^{-1}(x)[\lambda+C(x)]\begin{bmatrix} u(x) \\ v(x) \end{bmatrix}$$

It follows from (2.3) that

$$(5)\qquad \begin{bmatrix} u(x) \\ v(x) \end{bmatrix} = M(x,0,\lambda)\begin{bmatrix} \mathrm{E} \\ \mathrm{I} \end{bmatrix} v(0) + \int_0^x M(x,y,\lambda)K^{-1}(y)\begin{bmatrix} f(y) \\ g(y) \end{bmatrix} \mathrm{dy}$$

On the other hand, we see from the boundary condition in (2.4) that

$$b=(-\lambda D-F,\lambda-G)\begin{bmatrix} u(1) \\ v(1) \end{bmatrix} = (\lambda D-F,\lambda-G)M(1,0,\lambda)\begin{bmatrix} E \\ I \end{bmatrix} v(0)$$

$$+(\lambda D-F,\lambda-G)\int_0^1 M(1,y,\lambda)K^{-1}(y)\begin{bmatrix} f(y) \\ g(y) \end{bmatrix} \mathrm{dy}$$

Consequently,

$$(6)\qquad H(\lambda)v(0)=b+\int_0^1 (\lambda D+F,G-\lambda)M(1,y,\lambda)K^{-1}(y)\begin{bmatrix} f(y) \\ g(y) \end{bmatrix} \mathrm{dy}$$

where

$$H(\lambda)=-(\lambda D+F,G-\lambda)M(1,0,\lambda)\begin{bmatrix} E \\ I \end{bmatrix}$$

Defining $h(\lambda)=detH(\lambda)$, we see that $\lambda\in\sigma(\mathscr{A})$ if and only if λ is a zero of the entire function $h(\lambda)$. When $h(\lambda)\neq 0$, $\lambda\in\rho(\mathscr{A})$ and $R(\lambda,\mathscr{A})(f,g,b)$=$(u,v,d)$ where (u,v) is given by (2.5) with $v(0)$ determined by (2.6) and $d=v(1)-Du(1)$. It can be seen from (2.5) that $R(\lambda,\mathscr{A})$ is compact for any $\lambda\in\rho(\mathscr{A})$.

Theorem 2.2. *The operator $\mathscr{A}$ defined by (2.1) generates a C_0-semigroup $T(t)$ on $\mathscr{H}$.*

Proof. We need only to prove the assertion for the case $C \equiv 0$ because is a bounded operator by assumption $(H2)$, and bounded perturbations do not affect C_0-semigroup generations. For the sake of simplicity, we assume that $\mathscr{H}$ is real. The idea is to define an equivalent norm on $\mathscr{H}$ by properly choosing some positive weighting functions $f_i(x), 1 \le i \le N$ and $g_i(x), N+1 \le i \le n$, namely, define the norm on $\mathscr{H}$ as

$$\|(u,v,d)\|^p = \sum_{i=1}^{N}\int_0^1 f_i(x)|u_i(x)|^p \mathrm{dx} + \sum_{\mathrm{j=N+1}}^{\mathrm{n}}\int_0^1 \mathrm{g_j(x)|v_j(x)|^p dx} + \sum_{\mathrm{j=N+1}}^{\mathrm{n}} |\mathrm{d_j}|^{\mathrm{p}} \tag{2.3}$$

It is easily verified that $\mathscr{H}^*$, the dual space of $\mathscr{H}$, consisting of all elements (u^*,v^*,d^*) with

$$\begin{aligned}
u_i^*(x) &= \|((u,v,d)\|^{2-p}|u_i(x)|^{\frac{p}{q}}\mathrm{sign(u_i(x))}, \quad 1 \le \mathrm{i} \le \mathrm{N}, \\
v_j^*(x) &= \|((u,v,d)\|^{2-p}|v_j(x)|^{\frac{p}{q}}\mathrm{sign(v_j(x))}, \quad \mathrm{N}+1 \le \mathrm{j} \le \mathrm{n}, \\
d_j^*(x) &= \|((u,v,d)\|^{2-p}|d_j|^{\frac{p}{q}}\mathrm{sign(d_j)}, \quad \mathrm{N}+1 \le \mathrm{j} \le \mathrm{n}.
\end{aligned}$$

where q denotes the conjugate number of p, which satisfies $\frac{1}{p} + \frac{1}{q} = 1$.

For any $(u,v,d) \in D(A)$, $(u,v,d) \neq 0$ and any $(u^*,v^*,d^*) \in F((u,v,d)) \subset \mathscr{H}$, where F denotes the duality set. A direct calculation shows that

$$\begin{aligned}
&\|(u,v,d)\|^{p-2}\langle (u^*,v^*,d^*), A(u,v,d)\rangle \\
= \;& \sum_{i=1}^{N}\int_0^1 -\lambda_i(x)f_i(x)\frac{\mathrm{d}}{}dx|u_i(x)|^p \mathrm{dx} + \sum_{\mathrm{j=N+1}}^{\mathrm{n}}\int_0^1 -\mu_{\mathrm{j}}(\mathrm{x})\mathrm{g_j(x)}\frac{\mathrm{d}}{\mathrm{dx}}|\mathrm{v_j(x)}|^{\mathrm{p}}\mathrm{dx} \\
&+\langle Fu(1)+Gv(1), [v(1)-Du(1)]''\rangle \\
= \;& -\sum_{i=1}^{N}\lambda_i(1)f_i(1)|u_i(1)|^p - \sum_{j=N+1}^{n}\mu_j(1)g_j(1)|v_j(1)|^p \\
&+\sum_{i=1}^{N}\lambda_i(0)f_i(0)|u_i(0)|^p + \sum_{j=N+1}^{n}\mu_j(0)g_j(0)|v_j(0)|^p \\
&+\sum_{i=1}^{N}\int_0^1 |u_i(x)|^p\frac{\mathrm{d}}{\mathrm{d}x}[\lambda_i(x)f_i(x)]\mathrm{dx} + \sum_{\mathrm{j=N+1}}^{\mathrm{n}}\int_0^1 |\mathrm{v_j(x)}|^{\mathrm{p}}\frac{\mathrm{d}}{\mathrm{dx}}[\mu_{\mathrm{j}}(\mathrm{x})\mathrm{g_j(x)}]\mathrm{dx} \\
&+\langle Fu(1)+Gv(1), [v(1)-Du(1)]''\rangle = I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

We estimate I_i separately. It is clear from the expression of I_3 that

$$I_3 \le C_0\|(u,v,d)\|^p \tag{2.4}$$

where $C_0 = \max i, j \max_{x\in[0,1]}\{\frac{\mathrm{d}}{\mathrm{dx}}[\lambda_i(x)f_i(x)], \frac{\mathrm{d}}{\mathrm{dx}}[\mu_j(x)g_j(x)]\mathrm{dx}]\}$.
Nothing that $u_i(0) = \sum_{j=N+1}^{n} e_{ij}v_j(0)$, we see that

$$\begin{aligned} I_2 &= \sum_{i=1}^{N}\lambda_i(0)f_i(0)|u_i(0)|^p + \sum_{j=N+1}^{n}\mu_j(0)g_j(0)|v_j(0)|^p \\ &\le \sum_{j=N+1}^{n}[\mu_j(0)g_j(0) + \sum_{i=1}^{N}\lambda_i(0)f_i(0)(\sum_{k=N+1}^{n}|e_{ik}|^q)^{\frac{p}{q}}]|v_j(0)|^p. \end{aligned} \tag{2.5}$$

Because $\lambda_i(0) > 0$ and $\mu_j(0) < 0$ from (H1) , we can always find $g_j(0) > 0$ and $f_i(0) > 0$ such that

$$\mu_j(0)g_j(0) + \sum_{i=1}^{N}\lambda_i(0)f_i(0)(\sum_{k=N+1}^{n}|e_{ik}|^q)^{\frac{p}{q}}] \le 0, \quad N+1 \le j \le n \tag{2.6}$$

holds, which implies that $I_2 \le 0$.

We now estimate I_4 by means of the inequalities $(|a|+|b|)^p \le 2^p(|a|^p + |b|^p)$ and $|a|^{\frac{1}{p}}|b|^{\frac{1}{q}} \le \frac{|a|}{p} + \frac{|b|}{q}$ which hold for any real a and b, we have

$$\begin{aligned} I_4 &\le \sum_{j=N+1}^{n}|\sum_{i=1}^{N}f_{ji}u_i(1) + \sum_{i=N+1}^{n}g_{ji}v_i(1)||v_j(1) - \sum_{i=1}^{N}d_{ji}u_i(1)|^{\frac{p}{q}} \\ &\le \frac{1}{p}\sum_{j=N+1}^{n}|\sum_{i=1}^{N}f_{ji}u_i + \sum_{i=N+1}^{n}g_{ji}v_i(1)|^p + \frac{1}{q}\sum_{j=N+1}^{n}|v_j(1) - \sum_{i=1}^{N}d_{ji}u_i(1)|^p \\ &\le \frac{2^p}{p}\sum_{j=N+1}^{n}\left[|\sum_{i=1}^{N}f_{ji}u_i(1)|^p + |\sum_{i=N+1}^{n}g_{ji}v_i(1)|^p\right] + \frac{1}{q}\|(u,v,d)\|^p \\ &\le \frac{2^p}{p}\sum_{j=N+1}^{n}\left(\sum_{i=1}^{N}|f_{ji}|^p\right)^{\frac{p}{q}}\sum_{i=1}^{N}|u_i(1)|^p + \frac{2^p}{p}\sum_{j=N+1}^{n}\left(\sum_{i=1}^{N}|g_{ji}|^p\right)^{\frac{p}{q}}\sum_{i=N+1}^{n}|v_i(1)|^p + \frac{1}{q}\|(u,v,d)\|^p \\ &= \sum_{i=1}^{N}\alpha_i|u_i(1)|^p + \sum_{j=N+1}^{n}\beta_j|v_j(1)|^p + \frac{1}{q}\|(u,v,d)\|^p \end{aligned}$$

with α_i and β_j denoting the obvious constants. Finally, it can be seen that

$$
\begin{aligned}
& I_1 + I_4 - \tfrac{1}{q}\|(u,v,d)\|^p \\
\le\ & \sum_{i=1}^{N}[-\lambda_i(1)f_i(1)+\alpha_i]|u_i(1)|^p + \sum_{j=N+1}^{n}[\beta_j - \mu_j(1)g_j(1)]|v_j(1)|^p \\
\le\ & \sum_{i=1}^{N}[-\lambda_i(1)f_i(1)+\alpha_i]|u_i(1)|^p + 2^p\sum_{j=N+1}^{n}|\beta_j - \mu_j(1)g_j(1)||v_j(1) - \sum_{i=1}^{N} d_{ji}u_i(1)|^p \\
& +2^p\sum_{j=N+1}^{n}|\beta_j - \mu_j(1)g_j(1)||\sum_{i=1}^{N} d_{ji}u_i(1)|^p \\
\le\ & \sum_{i=1}^{N}[-\lambda_i(1)f_i(1)+\alpha_i]|u_i(1)|^p + \sum_{j=N+1}^{n}2^p|\beta_j - \mu_j(1)g_j(1)||v_j(1) - \sum_{i=1}^{N} d_{ji}u_i(1)|^p \\
& +2^p\sum_{j=N+1}^{n}|\beta_j - \mu_j(1)g_j(1)|\left(\sum_{i=1}^{N}|d_{ji}|^q\right)^{\frac{p}{q}}\sum_{i=1}^{N}|u_i(1)|^p \\
=\ & \sum_{i=1}^{N}\left[-\lambda_i(1)f_i(1)+\alpha_i + 2^p\sum_{j=N+1}^{n}|\beta_j - \mu_j(1)g_j(1)|\left(\sum_{i=1}^{N}|d_{ji}|^q\right)^{\frac{p}{q}}\right]|u_i(1)|^p \\
& + \sum_{j=N+1}^{n}2^p|\beta_j - \mu_j(1)g_j(1)||v_j(1) - \sum_{i=1}^{N} d_{ji}u_i(1)|^p.
\end{aligned}
$$

If we choose $f_i(1) > 0,\ g_j(1) > 0$ such that

$$
\begin{cases}
-\lambda_i(1)f_i(1)+\alpha_i + 2^p\sum_{j=N+1}^{n}|\beta_j - \mu_j(1)g_j(1)|\left(\sum_{i=1}^{N}|d_{ji}|^q\right)^{\frac{p}{q}} \le 0 \\
2^p|\beta_j - \mu_j(1)g_j(1)| \le C
\end{cases}
\tag{2.7}
$$

for any $1 \le i \le N$ and $N+1 \le j \le n$, then

$$
I_1 + I_4 \le (C + \frac{1}{q})\|(u,v,d)\|^p.
$$

The estimations of I_i above show that there exists a constant M such that

$$
\langle (u^*, v^*, d^*), A(u,v,d)\rangle \le M\|(u,v,d)\|^2 \tag{2.8}
$$

Now we choose a weighting functions $f_i(x)$ and $g_i(x)$ such that they satisfy (2.6) and (2.7) and then define a norm in $\mathscr{H}$ according to (2.3).

Because $\mathscr{A} - M$ is dissipative and $\mathscr{A}$ has the properties stated in the Lemma 2.1, we can conclude from the standard argument in [6] that $\mathscr{A}$ generates a C_0-semigroup on $\mathscr{H}$ □

3. An Optimal Energy Control

In this section, we will discuss an optimal control problem of the hyperbolic system (2.2):

$$\begin{aligned} &\tfrac{dW}{dt} = \mathscr{A}\mathscr{W}(\sqcup) + \mathscr{B}\sqcap(\mathscr{W}(\sqcup), \sqcup) \\ &W(0) = W_0 \end{aligned} \tag{3.1}$$

where both state space $\mathscr{H}$ and control space $\mathscr{U}$ are Hilbert spaces, the state function $W(t)$ on $[0,T]$ is valued in $\mathscr{H}$, $\mathscr{A}$ is the infinitesimal generator of a C_0-semigroup $S(t)$, $t \geq 0$. B is a bounded linear operator from $L^2([0,T];\mathscr{U})$ to $L^2([0,T];\mathscr{U})$, $u(W(t),t)$ is a control of the system.

In this section, we shall discuss a specific optimal control, that is, the minimum energy control of the system (3.1). We know that the minimum energy control in an abstract space is, in general, the minimum norm control. So, from mathematics point of view, the existence and uniqueness of the optimal control are essential. If these are true, then how to obtain the optimal control is a significant problem. The main content of this paper is to solve these essential and significant issue.

From the theory of operator semigroup, we see that for every control element $u(W(\cdot),\cdot) \in L^2([0,T],\mathscr{U})$, the system (3.1) has an unique mild solution

$$W(t) = S(t)W_0 + \textstyle\int_0^t S(t-s)B(u(W(s),s))ds \tag{3.2}$$

let $\varphi(\cdot)$ be an arbitrary element in $C([0,T];\mathscr{H})$, and

$$\rho = inf_{u \in L^2([0,T];\mathscr{U})} \|\varphi(t) - S(t)W_0 - \int_0^t S(t-s)Bu(W(s),s)ds\|,$$

define the admissible control set of the system (3.1) as follows

$$\mathscr{U}_{ad} = \{u \in L^2([0,T];\mathscr{U}) : \|\varphi(t) - S(t)W_0 - \textstyle\int_0^t S(t-s)Bu(W(s),s)\| \leq \rho + \varepsilon\} \tag{3.3}$$

where ε is any positive number.

It can be seen from (3.3) that $\mathscr{U}_{ad}$ is not empty and contains infinitely many elements related to φ and ε. The minimum energy control problem is actually to find the element u, satisfying

$$\|u_0\| = min\{\|u\| : u \in \mathscr{U}_{ad}\} \tag{3.4}$$

where u_0 is said to be a minimum energy control element.

Lemma 3.1 The admissible control set $\mathscr{U}_{ad}$ defined by (3.3) is a closed convex set in Hilbert space $L^2([0,T];\mathscr{U})$.

Proof. Convexity. For any u_1, $u_2 \in \mathscr{U}_{ad}$ and a real number λ, $0<\lambda<1$, it is easy to see from (3.3) that

$$\|\varphi(t)-S(t)W_0-\int_0^t S(t-s)Bu_i(W(s),s))\| \;\leq\; \rho+\varepsilon, \quad i=1,2 \tag{3.5}$$

and hence

$$\begin{aligned} & \|\varphi(t)-S(t)W_0-\int_0^t S(t-s)B(\lambda u_1(W(s),s)+(1-\lambda)u_2(W(s),s))ds\| \\ \leq\; & \lambda\|\varphi(t)-S(t)W_0-\int_0^t S(t-s)Bu_1(W(s),s))ds\| \\ & +(1-\lambda)\|\varphi(t)-S(t)W_0-\int_0^t S(t-s)Bu_2(W(s),s)ds\|. \end{aligned} \tag{3.6}$$

Since $\lambda u_1+(1-\lambda)u_2 \in L^2([0,T];\mathscr{U})$, it follows that $\lambda u_1+(1-\lambda)u_2 \in \mathscr{U}_{ad}$, this implies that $\mathscr{U}_{ad}$ is a convex subset of $L^2([0,T];\mathscr{U})$.

Closedness. Suppose $\{u_n\}\subset \mathscr{U}_{ad}$, and $\lim_{n\to\infty}\|u_n-u^*\|=0$. It can be shown that $u^*\in\mathscr{U}_{ad}$. In fact, from the definition of $\mathscr{U}_{ad}$ we see that

$$\|\varphi(t)-S(t)W_0-\int_0^t S(t-s)Bu_n(W(s),s)ds\|\leq\rho+\varepsilon, \quad n=1,2,\cdots$$

Since $S(t)$, $t\geq 0$ is a C_0-semigroup in Hilbert space $\mathscr{H}$, there is a constant $M>0$ such that $\sup_{0\leq t\leq T}\|S(t)\|\leq M$. On the other hand, since $W(s)$ is differentiable on $[0,T]$, it is continuous on $[0,T]$, and hence $\{W(s): s\in[0,T]\}$ is a bounded set in $L^2([0,T];\mathscr{U})$. Thus there is a constant $N>0$ such that $\|Bu(W(s),s)\|\leq N$ $(0\leq s\leq T)$ and

$$\begin{aligned} & \|\varphi(t)-S(t)W_0-\int_0^t S(t-s)Bu^*(W(s),s)ds\| \\ \leq\; & \|\varphi(t)-S(t)y_0-\int_0^t u_n(W(s),s)Bu(W(s),s)ds\| \\ & +\|\int_0^t S(t-s)B[u_n(W(s),s)-u^*(W(s),s)]\| \\ \leq\; & \rho+\varepsilon+M\|u_n-u^*\|\cdot NT \end{aligned} \tag{3.7}$$

Letting $n\to\infty$ leads to

$$\|\varphi(t)-S(t)W_0-\int_0^t S(t-s)Bu^*(W(s),s)ds\|\leq\rho+\varepsilon\,.$$

Thus, $u^*\in\mathscr{U}_{ad}$, and $\mathscr{U}_{ad}$ is a closed set. The proof is complete.

Theorem 3.2 There exists an unique minimum energy control element in the admissible control set $\mathscr{U}_{ad}$ for the system (3.1)

Proof. Since $L^2([0,T],\mathscr{U})$ is a Hilbert space, it is naturally a strict convex Banach Space. From the preceding Lemma, we have seen that $\mathscr{U}_{ad}$ is a closed convex set in $L^2([0,T],\mathscr{U})$, it follows from [2] that there is an unique element $u_0 \in \mathscr{U}_{ad}$ such that

$$\|u_0\| = min \ \{\|u\| : u \in \mathscr{U}_{ad}\}$$

According to the definition (3.4), u_0 is just the desired minimum energy control element of the system (3.1). The proof is complete.

Finally, we shall show that the minimum energy control element can be approached.

Theorem 3.3 Suppose that u_0 is the minimum energy control element of the system (1.1), then there exists a sequence $\{u_n\}$ of $\mathscr{U}_{ad}$ such that $\{u_n\}$ converges strongly to u_0 in $L^2([0,T];\mathscr{U})$, namely,

$$\lim_{n\to\infty} \|u_n - u_0\| = 0$$

Proof. Let $\{u_n\}$ be a minimized sequence in the admissible control set $\mathscr{U}_{ad}$, then it follows that

$$\|u_{n+1}\| \le \|u_n\|, \quad n = 1,2,\cdots \tag{3.8}$$

and

$$\lim_{n\to\infty} \|u_n\| = inf\{\|u\| : u \in \mathscr{U}_{ad}\} \tag{3.9}$$

It is obvious that $\{u_n\}$ is a bounded sequence in $L^2([0,T];\mathscr{U})$, and so there is a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $\{u_{n_k}\}$ weakly converges to an element $\widetilde{u}$ in $L^2([0,T];\mathscr{U})$ (see [3]).

Since $\mathscr{U}_{ad}$ is a closed convex set in $L^2([0,T];\mathscr{U})$ (see Lemma 3.1), we see from Mazur's Theorem that $\mathscr{U}_{ad}$ is a weakly closed set in $L^2([0,T];\mathscr{U})$, thus $\tilde{u} \in \mathscr{U}_{ad}$. Combining (3.2) and employing the properties of limits of weakly convergent sequence on norm yield

$$\begin{aligned} inf\{\|u\| : u \in \mathscr{U}_{ad}\} \le \|\tilde{u}\| &\le \underline{\lim}_{k\to\infty} \|u_{n_k}\| \\ &= \lim_{n_k\to\infty} \|u_{n_k}\| = \lim_{n\to\infty} \|u_n\| = \inf\{\|u\|; u \in \mathscr{U}_{ad}\}. \end{aligned} \tag{3.10}$$

Thus, we have

$$\lim_{n\to\infty}\|u_n\| = \|\tilde{u}\| \tag{3.11}$$

and

$$\|\tilde{u}\| = inf\{\|u\| : u \in \mathscr{U}_{ad}\}. \tag{3.12}$$

Since $\{u_{n_k}\}$ is weakly convergent to $\tilde{u}$, it follows from (3.3) that $\{u_{n_k}\}$ converges to $\tilde{u}$. Therefore, we see in terms of Theorem 3.2 and (3.4) that $\tilde{u} = u_0$, namely, $\tilde{u}$ is the minimum energy control element. Thus, $\{u_{n_k}\}$ strongly converges to the minimum energy control element in $L^2([0,T];\mathscr{U})$. Without loss of generality, we can rewrite $\{u_{n_k}\}$ by $\{u_n\}$, then the conclusion of theorem is now obtained.

The Theorem 3.2 points out that the minimum energy control element can be approached by a weakly convergent sequence in the control space, which provides the theoretical basis of approximate computation for finding the minimum energy control element.

Conflict of Interests

The authors declare that there is no conflict of interests.

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