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# LINK BETWEEN WRONSKIAN CONDITIONS AND GRAMMIAN CONDITIONS

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**Abstract.** In this paper, the link between Wronskian conditions and Grammian conditions for nonlinear evolution equations is firstly found. By using the link mentioned, we obtained Grammian solutions of some nonlinear evolution equations.

**Keywords**: nonlinear evolution equations; Wronskian condition; Grammian condition; Grammian solution .

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## 1. Introduction

The direct method proposed by Hirota becomes a powerful tool for constructing multi-soliton solutions to integrable NLEEs [1]. The general idea of the method is first to make a transformation into new variables, so that in these new variables multi-soliton solutions appear in a particularly simple form. The method turned out to be very effective and was quickly shown to give $N$-soliton solutions to some nonlinear equations. Further, the solution obtained by Hirota's method can commonly be written in terms of a determinant. Since differentiation of an $N$th order determinant usually lead to the sum of $N$

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determinants, it is difficult to get the derivatives of the $N$-soliton solutions. To avoid this difficulty, an alternative formulation is called for. Another determinant form for soliton solutions is the Grammian [2-4] which can be expressed by means of a Pfaffian and consequently the proof of the Grammian solving the bilinear equations can easily be completed by virtue of Pfaffian properties. It is a common feature that many NLEEs admit Grammian solutions. As we know, in the process of constructing Grammian solutions, the main difficulty lies in looking for the linear differential conditions, which the functions in the Grammian determinant should satisfy.

In this papaer, we use the link to derive Grammian conditions and solutions of NLEEs. As an application, the construction problems of Grammian conditions to the following equations are treated:

$(2+1)$-dimensional KP equation [5]

$$u_t + 6uu_x + u_{xxx} + 3\partial^{-1}u_{yy} = 0, \tag{1.1}$$

$(2+1)$-dimensional Korteweg-de Vries (KdV) system [6,7]

$$u_t + u_{xxx} - 3(uv)_x = 0, \tag{1.2a}$$

$$u_x = v_y, \tag{1.2b}$$

Eq (1.1) is a $(2+1)$ dimensions generalization of the KdV equation. Kadomtsev and Petviashvili discovered the equation when they relaxed the restriction that the waves are strictly one-dimensional. The KP equation is used to model shallow water waves with weakly nonlinear restoring forces and waves in ferromagnetic media. System (1.2) was originally derived by the idea of the weak Lax pair [7] and can be obtained from the Kadomtsev-Petviashvili (KP) equation using inner parameter-dependent symmetry constraint [8]. It has been shown that in Ref. [9] such a system (1.2) admits the painlevé property. Obviously, it can be reduced to the well-known $(1+1)$-dimensional KdV equation if $y = x$, which was initially used to describe competition between week nonlinearity and weak disperson in shallow water. Based on the bilinear method and bilinear BT of System (1.2), the main goal of our work is to obtain Wronskian conditions and solutions for System (1.2) by applying the balance method. Our results will show that these equations have generalized Wronskian determinant solutions under different linear differential conditions.

The structure of this paper is as follows. In Section 2, the link between wronskian conditions and Grammian conditions is simply introduced. In Section 3, we construct and prove Grammian solutions for (1.1)-(1.2). Finally, we have the summary in section 4.

## 2. Preliminaries

We consider a general form of a partial differential equation

$$F(u_t, u_x, u_y, u_{tt}, u_{tx}, u_{ty}, u_{xx}, u_{xy}, u_{yy}, \cdots) = 0, \tag{2.1}$$

where $u = u(x, y, t)$, $F$ is a polynomial about $u$ and its derivatives. By the transformation $u = T(f(x, y, t))$, (2.1) can be converted into the bilinear form

$$G(D_x, D_y, D_t) f \cdot f = 0, \tag{2.2}$$

where $G(D_x, D_y, D_t)$ is the operator polynomial and $D_x$, $D_y$, $D_t$ are defined by[9]

$$D_x^m D_y^n D_t^k a \cdot b = (\partial_x - \partial_{x'})^m (\partial_y - \partial_{y'})^n (\partial_t - \partial_{t'})^k a(x, y, t) b(x', y', t')|_{x'=x, y'=y, t'=t}. \tag{2.3}$$

If (2.2) has the solution in the Wronskian form

$$f = W(\phi_1, \phi_2, \cdots, \phi_N) = \begin{vmatrix} \phi_1^{(0)} & \phi_1^{(1)} & \cdots & \phi_1^{(N-1)} \\ \phi_2^{(0)} & \phi_2^{(1)} & \cdots & \phi_2^{(N-1)} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_N^{(0)} & \phi_N^{(1)} & \cdots & \phi_N^{(N-1)} \end{vmatrix}, \tag{2.4}$$

where $\phi_i^{(m)}$ is defined by $\phi_i^{(m)} = \phi_{i,mx}$, and $\phi_i = \phi_i(x, y, t)$ $(i = 1, 2, \cdots, N)$ in $t \geq 0$, $-\infty < x, y < +\infty$ has continuous derivative up to any order. For a convenient notation, we use the Freeman and Nimmos suppression

$$\begin{vmatrix} \phi_1^{(0)} & \phi_1^{(1)} & \cdots & \phi_1^{(N-1)} \\ \phi_2^{(0)} & \phi_2^{(1)} & \cdots & \phi_2^{(N-1)} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_N^{(0)} & \phi_N^{(1)} & \cdots & \phi_N^{(N-1)} \end{vmatrix} = |\phi, \phi^{(1)}, \cdots, \phi^{(N-1)}| = |\widehat{N-1}|. \tag{2.5}$$

$\phi_i$ needs satisfy the Wronskian conditions

$$\phi_{i,t} = \alpha_1 \phi_{i,n_1 x} + \alpha_2 \phi_{i,n_2 x}, \quad \phi_{i,y} = \beta \phi_{j,mx}, \tag{2.6}$$

where $\alpha_1, \alpha_2, \beta$ are undetermined constants. Then, we can suppose bilinear equation (2.2) has the following Grammian solutions:

$$f_N = det|a_{ij}|_{1 \le i \le j \le N}, \quad a_{ij} = \delta_{ij} + \int^x \phi_i \psi_j dx, \tag{2.7}$$

where $\delta_{ij}$ are arbitrary constants. The functions $\phi_i = \phi_i(x, y, t)$, $\psi_j = \psi_j(x, y, t)$ satisfy the two sets of conditions

$$\phi_{i,t} = \alpha_1 \phi_{i,n_1 x} + \alpha_2 \phi_{i,n_2 x}, \qquad \phi_{i,y} = \beta(t) \phi_{i,mx}, \tag{2.8$a$}$$

$$\psi_{j,t} = (-1)^{n_1+1} \alpha_1 \psi_{j,n_1 x} + (-1)^{n_2+1} \alpha_2 \psi_{j,n_2 x}, \quad \psi_{j,y} = (-1)^{m+1} \beta \psi_{j,mx}. \tag{2.8$b$}$$

## 3. Main results

**Theorem 3.1.** The equation (1.1) has the following Grammian solutions:

$$f_N = det|a_{ij}|_{1 \le i \le j \le N}, \quad a_{ij} = \delta_{ij} + \int^x \phi_i \psi_j dx, \tag{3.1}$$

where $\delta_{ij}$ are arbitrary constants. The functions $\phi_i = \phi_i(x, y, t)$, $\psi_j = \psi_j(x, y, t)$ satisfy the two sets of conditions

$$\phi_{i,t} = -4\phi_{i,xxx} - 3\beta^2 \phi_{i,x}, \quad \phi_{i,y} = \beta \phi_{i,x}, \tag{3.2$a$}$$

$$\psi_{j,t} = -4\psi_{j,xxx} - 3\beta^2 \psi_{j,x}, \quad \psi_{j,y} = \beta \psi_{j,x}. \tag{3.2$b$}$$

**Proof.** By the dependent variable transformation

$$u(x, y, t) = 2(\ln f)_{xx}, \tag{3.3}$$

(1.1) can be represented through the bilinear form

$$[D_x D_t + D_x^4 + 3D_y^2] f \cdot f = 0, \tag{3.4}$$

and the nonlinear partial differential equation

$$f_{tx} f + f f_{xxxx} + 3f_{yy} f - f_t f_x - 4f_x f_{xxx} + 3f_{xx}^2 - 3f_y^2 = 0. \tag{3.5}$$

We have known that (1.1) has the generalized Wronskian conditions

$$\phi_{i,t} = -4\phi_{i,xxx} - 3\beta^2 \phi_{i,x}, \quad \phi_{i,y} = \beta\phi_{i,x}. \tag{3.6}$$

We first consider a differential of the determinant $f_N$. It is expressed by means of a Pfaffian as

$$f_N = (1, 2, \cdots, N, N^*, \cdots, 2^*, 1^*), \tag{3.7}$$

$$a_{ij} = (i, j^*) = \delta_{ij} + \int^x \phi_i \psi_j dx, \tag{3.8}$$

$$(i, j) = (i^*, j^*) = 0. \tag{3.9}$$

Next let us introduce Pfaffians $(m, n = 0, 1, 2, \cdots, N)$ defined by

$$(d_n, j^*) = \frac{\partial^n}{\partial x^n}\psi_j, \quad (d_m, d_n^*) = 0, \tag{3.10}$$

$$(d_n^*, i) = \frac{\partial^n}{\partial x^n}\phi_i, \quad (d_n, i) = (d_m^*, j^*) = 0. \tag{3.11}$$

By virtue of the above Pfaffians, differentials of the elements $a_{ij}$ $(i = 1, 2, \cdots, n; j = 1, 2, \cdots, n)$ are expressed as follows:

$$\frac{\partial}{\partial x} a_{ij} = \phi_i \psi_j = (d_0, d_0^*, i, j^*), \tag{3.12$a$}$$

$$\frac{\partial}{\partial y} a_{ij} = \int^x (\phi_{i,y}\psi_j + \phi_i \psi_{j,y}) dx = \int^x (\beta\phi_{i,x}\psi_j + \beta\phi_i\psi_{j,x}) dx = \beta\phi_i\psi_j = \beta(d_0, d_0^*, i, j^*), \tag{3.12$b$}$$

$$\frac{\partial}{\partial t} a_{ij} = \int^x (\phi_{i,t}\psi_j + \phi_i\psi_{j,t}) dx = 4[(d_1, d_1^*, i, j^*) - (d_0, d_2^*, i, j^*) - (d_2, d_0^*, i, j^*)] - 3\beta^2 (d_0, d_0^*, i, j^*). \tag{3.12$c$}$$

If we denote $f_N = (1, 2, \cdots, N, N^*, \cdots, 2^*, 1^*) = (\bullet)$, then we have the following differential formulaes for $f_N$:

$$f_{N,x} = (d_0, d_0^*, \bullet), \tag{3.13a}$$

$$f_{N,xx} = (d_1, d_0^*, \bullet) + (d_0, d_1^*, \bullet), \tag{3.13b}$$

$$f_{N,y} = \beta f_{N,x}, \quad f_{N,yy} = \beta^2 f_{N,xx}, \tag{3.13c}$$

$$f_{N,xxx} = (d_2, d_0^*, \bullet) + 2(d_1, d_1^*, \bullet) + (d_0, d_2^*, \bullet), \tag{3.13d}$$

$$f_{N,xxxx} = (d_3, d_0^*, \bullet) + 3(d_2, d_1^*, \bullet) + 3(d_1, d_2^*, \bullet) + 2(d_0, d_0^*, d_1, d_1^*, \bullet) + (d_0, d_3^*, \bullet), \tag{3.13e}$$

$$f_{N,t} = 4[(d_1, d_1^*, \bullet) - (d_0, d_2^*, \bullet) - (d_2, d_0^*, \bullet)] - 3\beta^2(d_0, d_0^*, \bullet), \tag{3.13f}$$

$$f_{N,xt} = 4[(d_0, d_0^*, d_1, d_1^*, \bullet) - (d_0, d_3^*, \bullet) - (d_3, d_0^*, \bullet)] - 3\beta^2[(d_1, d_0^*, \bullet) + (d_0, d_1^*, \bullet)]. \tag{3.13g}$$

Using the identities of determinant, we can easily get

$$[(d_0, d_1^*, \bullet) - (d_1, d_0^*, \bullet)]^2$$

$$= [(d_3, d_0^*, \bullet) + (d_0, d_3^*, \bullet) + 2(d_0, d_0^*, d_1, d_1^*, \bullet) - (d_1, d_2^*, \bullet) - (d_2, d_1^*, \bullet)](\bullet). \tag{3.14}$$

Substituting the above Pfaffians into (3.4), after some calculations, we obtain

$$[D_x D_t + D_x^4 + 3D_y^2] f \cdot f$$

$$= f_{tx} f + f f_{xxxx} + 3 f_{yy} f - f_t f_x - 4 f_x f_{xxx} + 3 f_{xx}^2 - 3 f_y^2$$

$$= 12[(d_0, d_0^*, d_1, d_1^*, \bullet)(\bullet) - (d_0, d_0^*, \bullet)(d_1, d_1^*, \bullet) + (d_1, d_0^*, \bullet)(d_0, d_1^*, \bullet)]. \tag{3.15}$$

We can find that (3.15) is the Jacobi identity for the determinant, so it equals to zero. This shows that the Grammian determinant $f_N$ solves (1.1). This completes the proof.

**Theorem 3.2.** The equation (1.2) has the following Grammian solutions:

$$f_N = det|a_{ij}|_{1 \le i \le j \le N}, \quad a_{ij} = \delta_{ij} + \int^x \phi_i \psi_j dx, \tag{3.16}$$

where $\delta_{ij}$ are arbitrary constants. The functions $\phi_i = \phi_i(x, y, t)$, $\psi_j = \psi_j(x, y, t)$ satisfy the two sets of conditions

$$\phi_{i,y} = \beta \phi_{i,x}, \quad \phi_{i,t} = -4\phi_{i,xxx}, \tag{3.17}$$

$$\psi_{j,y} = \beta \psi_{j,x}, \quad \psi_{j,t} = -4\psi_{j,xxx}. \tag{3.18}$$

**Proof.** By the dependent variable transformation

$$u(x,y,t) = -2(\ln f)_{xy}, \quad v(x,y,t) = -2(\ln f)_{xx}, \tag{3.19}$$

(1.2) can be represented through the bilinear form

$$[D_yD_t + D_x^3D_y^2]f \cdot f = 0. \tag{3.20}$$

It is equal to

$$f_{xxxy}f + f_{yt}f - f_tf_y - f_{xxx}f_y + 3f_{xx}f_{xy} - 3f_{xxy}f_x = 0. \tag{3.21}$$

We have known that (1.2) has the generalized Wronskian conditions

$$\phi_{i,y} = \beta\phi_{i,x}, \quad \phi_{i,t} = -4\phi_{i,xxx}. \tag{3.22}$$

By virtue of (3.7-3.11), differentials of the elements $a_{ij}$ $(i = 1,2,\cdots,n; j = 1,2,\cdots,n)$ are expressed as follows

$$\frac{\partial}{\partial x}a_{ij} = \phi_i\psi_j = (d_0, d_0^*, i, j^*), \tag{3.23$a$}$$

$$\frac{\partial}{\partial y}a_{ij} = \int^x(\phi_{i,y}\psi_j + \phi_i\psi_{j,y})dx = \int^x(\beta\phi_{i,x}\psi_j + \beta\phi_i\psi_{j,x})dx = \beta\phi_i\psi_j = \beta(d_0, d_0^*, i, j^*), \tag{3.23$b$}$$

$$\frac{\partial}{\partial t}a_{ij} = \int^x(\phi_{i,t}\psi_j + \phi_i\psi_{j,t})dx = 4[(d_1, d_1^*, i, j^*) - (d_0, d_2^*, i, j^*) - (d_2, d_0^*, i, j^*)]. \tag{3.23$c$}$$

If we denote $f_N= (1,2,\cdots,N,N^*,\cdots,2^*,1^*)= (\bullet)$, then we have the following differential formulaes for $f_N$:

$$f_{N,x} = (d_0, d_0^*, \bullet), \quad f_{N,xx} = (d_1, d_0^*, \bullet) + (d_0, d_1^*, \bullet), \tag{3.24$a$}$$

$$f_{N,xxx} = (d_2, d_0^*, \bullet) + 2(d_1, d_1^*, \bullet) + (d_0, d_2^*, \bullet), \tag{3.24$b$}$$

$$f_{N,y} = \beta f_{N,x}, f_{N,xy} = \beta f_{N,xx}, f_{N,xxy} = \beta f_{N,xxx}, \tag{3.24$c$}$$

$$f_{N,xxxy} = \beta[(d_3, d_0^*, \bullet) + 3(d_2, d_1^*, \bullet) + 3(d_1, d_2^*, \bullet) + 2(d_0, d_0^*, d_1, d_1^*, \bullet) + (d_0, d_3^*, \bullet)], \tag{3.24$d$}$$

$$f_{N,t} = 4[(d_1, d_1^*, \bullet) - (d_0, d_2^*, \bullet) - (d_2, d_0^*, \bullet)], \tag{3.24$e$}$$

$$f_{N,yt} = 4\beta[(d_0, d_0^*, d_1, d_1^*, \bullet) - (d_0, d_3^*, \bullet) - (d_3, d_0^*, \bullet)]. \tag{3.24$f$}$$

Substituting the above Pfaffians into (3.20), after some calculations, we obtain

$$[D_yD_t + D_x^3D_y^2]f \cdot f$$

$$= f_{xxxy}f + f_{yt}f - f_tf_y - f_{xxx}f_y + 3f_{xx}f_{xy} - 3f_{xxy}f_x$$

$$= 12\beta[(d_0, d_0^*, d_1, d_1^*, \bullet)(\bullet) - (d_0, d_0^*, \bullet)(d_1, d_1^*, \bullet) + (d_1, d_0^*, \bullet)(d_0, d_1^*, \bullet)]. \tag{3.25}$$

We can find that (3.25) is the Jacobi identity for the determinant, so it equals to zero. This shows that the Grammian determinant $f_N$ solves (1.2).This completes the proof.

**Corollary 3.3.** In summary, by using of the link between Wronskian conditions and Grammian conditions, we have found the (2+1)-dimensional KP equation and the (2+1)-dimensional KdV system admit Grammian solutions. The method can also be easily applied to other NLEEs for diverse Grammian conditions and solutions. Of course, there should exist other more general conditions involving combined equations for Grammian solutions of high-dimensional NLEEs. The work in this direction is in progress.

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