Available online at http://scik.org
J. Math. Comput. Sci. 10 (2020), No. 1, 110-135
https://doi.org/10.28919/jmcs/4006
ISSN: 1927-5307

# A STUDY ON APPROXIMATE SOLUTIONS AND ACCURACY OF PARALLEL ALGORITHMS FOR SOLVING SYSTEM OF ODES

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**Abstract.** The purpose of this paper is, to study the numerical computation of system of ordinary differential equations of first order with initial value problems and all steps of parallel algorithms of R.K. methods are tested in MATLAB (2009a) [2]. In this part, we use four classical RK2 order methods to introduce the basic ideas associated with initial value problems for solving system of ODEs of two or more equations. At the end we make a comparison between these numerical methods for approximate solutions and numerical errors. Numerical examples are given to illustrate the computational accuracy and robustness of these parallel algorithms**.**

**Keywords:** system of ODEs; RK2 methods; Matlab; scientific computation; accuracy and efficiency; stability analysis.

**2010 AMS Subject Classification:** 34A45.

## 1. INTRODUCTION

Numerical Analysis is the area of mathematics and computer science that creates algorithm and implements numerical methods for solving the problems of continuous mathematics. In the field of Engineering and Science, we come across physical and natural phenomena which, when represented by mathematical models happen to be differential

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Received January 23, 2019

equations. Solving system of linear simultaneous differential equations is one of the most important and challenging problems in science and engineering applications. It arises in a wide variety of practical applications in Physics, Chemistry, Biosciences, Engineering, etc. System of linear differential equations arises in various theoretical research fields as well as applications in science and engineering. After the availability of computers, we go to numerical methods which are suited for computer operations. For example, system of Lorentz equations and system of coupled equations, deflection of a beam, etc. are represented by differential equations. Hence solution of differential equation is a necessity in such studied. There are number of differential equations which we studied in calculus to get closed form solutions. But all differential equations do not possess closed form solutions or finite form solutions. Even they possess closed form solutions; we do not know the method of getting it. In such situations depending upon the need of the hour, we go in for numerical solutions of differential equations. In researches, especially after the advent of computer, the numerical solutions of the differential equations have becomes easy for manipulation. The dynamic behavior of systems is an important subject. A mechanical system involves displacements, velocities, and accelerations. An electric or electronic system involves voltages, currents, and time derivatives of these quantities. An equation that involves one or more derivatives of the unknown function is called an ordinary differential equation, abbreviated as Ode. The problems of solving an ode are classified into initial-value problems and boundary value problems, depending on how the conditions at the endpoints of the domain are specified. All the conditions of an initial-value problem are specified at the initial point. On the other hand, the problem becomes a boundary-value problem if the conditions are needed for both initial and final points. The ode in the time domain are initial-value problems, so all the conditions are specified at the initial time, such as $x = 0$. For notations, we use $x$ as an independent variable. It is important to note that our focus here is on the practical use of numerical methods in order to solve some typical problems, not to present any consistent theoretical background. Today there are numerous methods that produce numerical approximations to the solution of differential equations. There are many excellent and exhaustive texts on these subjects that may be consulted. [3-7]. Our purpose is here, to compare the accuracy of various algorithms and how we can solve the systems of ODEs numerically and also checks the stability analysis using Matlab [2].

**1.1. DEFINITION AND NOTATION.** The system of ordinary differential equations considers has the form:

$$\frac{dy}{dx} = f(x, y(x)), \quad y(x_0) = y_0 , \quad x \in [x_0, x_n ] \qquad \text{(i)}$$

Here y(x) and f(x, y) are vector -valued functions

$$y(x) = (y_1(x) , y_2(x), y_3(x), \dots \dots \dots \dots . y_m(x)) ,$$

$$f(x, y) = (f_1(x, y) , f_2(x, y), f_3(x, y), \dots \dots \dots \dots . f_m(x, y)) ,$$

so that we are dealing with m simultaneous first-order equations.

**Definition:** The function $f(x, y_1, y_2 \dots . y_m)$, defined on the set D = {(x, $y_1$,$y_2$, . . . , $y_m$) | a ≤ x ≤ b and − ∞ < $y_i$ < ∞, for each i = 1, 2,. . . , m} is said to satisfy a Lipschitz condition on D in the variables $y_1$, $y_2$, . . . , $y_m$ if a constant L > 0 exists with

$$\left|f(x, y_1, y_2 \dots . y_m) - f\left(x, z_1, z_2 \dots . z_m \right) \right| \le L \sum_{j=1}^{j=m} \left|y_j - z_j\right| \qquad \text{(ii)}$$

for all (x, $y_1$, . . . , $y_m$) and (x, $z_1$, . . . , $z_m$) in D. By using the Mean Value Theorem, it can be shown that if f and its first partial derivatives are continuous on D and if

$$\partial f (x, y_1, y_2 . . . , y_m) / \partial y_i \le L, \quad \text{for each } i = 1, 2, . . . , m$$

and all (x, $y_1$, . . . , $y_m$) in D, then f satisfies a Lipschitz condition on D with Lipschitz constant L [3]. A basic existence and uniqueness theorem follows.

**Theorem:** Suppose that D = {(x, $y_1$, $y_2$, . . ., $y_m$) | a ≤ x ≤ b and − ∞ < $y_i$ < ∞, for each i = 1, 2,. . . , m}, and let $f_i$(x, $y_1$, . . . , $y_m$), for each i = 1, 2, . . . , m, be continuous and satisfy a Lipschitz condition on D. The system of first-order differential equations (1), subject to the initial conditions (1), has a unique solution $y_1$(x), . . . , $y_m$(x), for a ≤ x ≤ b.

Methods to solve systems of first-order differential equations are generalizations of the methods for a single first-order equation presented.[3]

## 2. MATERIALS AND METHODS

Numerical methods are commonly used for solving mathematical problems that are formulated in science and engineering where it is difficult or even impossible to obtain exact solutions. Only a limited number of differential equations can be solved analytically. Numerical methods, on the other hand, can give an approximate solution to (almost) any equation. Literal implementation of this procedure results in Euler's method, which is, however, not recommended for any practical use. There are other methods more sophisticated than Euler's.

Among one of them is Runge-Kutta methods. Now, we are interested to talk about Runge Kutta methods. In the differential equation $y' = f(x, y)$ on the interval [$x_j$, $x_{j+1}$] we get [1]

**Second Order Explicit Runge Kutta Methods:** Consider the following Second Order Runge kutta methods with two slopes

$$y_{j+1} = y_j + [w_1 k_1 + w_2 k_2] \quad (2)$$

$$k_1 = h\, f\,(x_j\,,\ y_j)$$

$$k_2 = h\, f\,(x_j + c_2 h,\ y_j + a_{21} k_1)$$

where $k_1$ and $k_2$ are mention above. Thus we have four parameters $c_2$, $a_{21}$, $w_1$ and $w_2$ are chosen to make $y_{j+1}$ closer to $y\,(x_{j+1})$.and to be determined. The values of $c_2$, $a_{21}$, $w_1$ and $w_2$ are evaluated by setting the second order equation to Taylor series expansion to the second order term. Now Taylor series expansion about $x_j$ gives

$$y(x_{j+1}) = y(x_j) + hy'(x_j) + \frac{h^2}{2!} y''(x_j) + \frac{h^3}{3!} y'''(x_j) \dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$y(x_{j+1}) = y(x_j) + hf(x_j\,, y(x_j)) + \frac{h^2}{2!}(f_x + f f_y) + \frac{h^3}{3!}\,[f_{xx} + 2f f_{xy} + f^2 f_{yy} + f_y(f_x + f f_y)] + \cdots \dots\dots\dots.. \quad (3)$$

We also have $k_1 = h\, f_j$

$$k_2 = h\, f\,(x_j + c_2 h\,,\ y_j + a_{21} h f_j)$$

$$k_2 = h[f_j + h(c_2 f_x + a_{21} f f_y)_{xj} + \frac{h^2}{2!}(c^2\, f_{xx}\ + 2c_2 a_{21}\, f f_{xy} + {a_{21}}^2\, f^2 f_{yy}\,) + \dots\dots\dots.]$$

Putting the values of $k_1$ and $k_2$ in (2) we get

$$y_{j+1} = y_j + (w_1 + w_2)hf_j + h^2(w_2 c_2 f_x + w_2 a_{21} f f_y) + \frac{h^3}{2} w_2({c_2}^2 f_{xx} + 2c_2 a_{21} f f_{xy} + {a_{21}}^2 f^2 f_{yy}) + \cdots .... \quad (4)$$

Comparing the coefficients of h and $h^2$ in (3) and (4) we obtain

$$w_1 + w_2 = 1$$

$$w_2 c_2 = \frac{1}{2}$$

$$w_2 a_{21} = \frac{1}{2}$$

Thus we get three equations of four unknowns and their solution is

$$,\ a_{21} = c_2\ ,\quad w_2 = \frac{1}{2c_2}\ , and \quad w_1 = 1 - \frac{1}{2c_2}$$

Where $c_2$ is not equal to zero. It is not possible the compare of $h^3$ as there are five terms in (3) and only three terms in (4). Therefore Runge-Kutta method using two evaluations of f is

$$y_{j+1} = y_j + (1 - \frac{1}{2c_2})k_1 + \frac{1}{2c_2}k_2 \quad (5)$$

Where $k_1 = h\, f\,(x_j\,,\; y_j)$

$k_2 = h\, f\,(x_j+c_2h,\; y_j+c_2k_1)$

The free parameter $c_2$ is usually taken between 0 and 1. Sometimes $c_2$ is chosen such that one of the $w_i$'s in the formula (2) is zero. For example, the choice $c_2 = 1/2$ makes $w_1=0$

Case (i) If $c_2=1/2$, we get

$k_1 = h\, f\,(x_j\,,\; y_j)$

$k_2 = h\, f\,(x_j+h/2,\; y_j+k_1/2)$

$$y_{j+1} = y_j + k_2 \quad (6)$$

which is called Mid-point method. It reduces to the mid-point quadrature rule when f(x,y) is independent of y.

Case (ii). If $c_2=1$, we get $k_1 = h\, f\,(x_j\,,\; y_j)$

$k_2 = h\, f\,(x_j+h\,,\, y_j+k_1)$

$$y_{j+1} = y_j + \frac{1}{2}[k_1 + k_2] \quad (7)$$

Which is called Heun's Method?

Case (iii). If $c_2=2/3$ we get $k_1 = h\, f\,(x_j\,,\; y_j)$

$k_2 = h\, f\,(x_j+2/3\, h\,,\, y_j+2/3\, k_1)$

$$y_{j+1} = y_j + \frac{1}{4}[k_1 + 3k_2] \quad (8)$$

Which is called nearly Optimal method. It may be noted that the explicit Runge Kutta methods using two evaluations of f have one arbitrary parameter and have produced second order methods.

Case (iv). If $c_2=3/4$, we get $k_1 = h\, f\,(x_j,\, y_j)$

$k_2 = h\, f\,(x_j+3/4\, h\,,\, y_j+3/4\, k_1)$

$$y_{j+1} = y_j + \frac{1}{3}[k_1 + 2k_2] \quad (9)$$

Which is called Raltson's Method and have two functions of evaluations. Ralston (1962) and Ralston and Rabinowitiz (1978) determined that choosing ω2 = 2/3 provides a minimum bound on the truncation error for the second order RK algorithms.[7]

**STABILITY ANALYSIS:[1]** Consider the initial value problem for the test equation as

$$y' = \lambda y\;, \quad y(x_0) = y_0 \quad (10)$$

Where λ may be real or complex number. The exact solution of the initial value problem is

$$y(x) = y(x_0)e^{\lambda(x-x_0)} = y_0\, e^{\lambda(x-x_0)} \tag{11}$$

Substituting $x = x_{j+1}$ and $x = x_j$ in (11) and dividing, we get

$$y(x_{j+1}) = y(x_j)e^{\lambda(x_{j+1}-x_j)} = y(x_j)e^{\lambda h} = E(\lambda h)y(x_j) \tag{12}$$

Thus the numerical method is defined by above equation. Now applying the test equation in RK method of order $\quad y_{j+1} = y_j + (1 - \frac{1}{2c_2})k_1 + \frac{1}{2c_2}k_2$

Where $\quad k_1 = hf(x_j\,, y_j) = \lambda h y_j$

$$k_2 = \text{h f}\left(x_j + c_2\text{h}\,,\ y_j + c_2 k_1\right) = \lambda h\left(y_j + c_2 k_1\right) = \lambda h[1 + c_2\lambda h]y_j$$

$$y_{j+1} = y_j + (1 - \frac{1}{2c_2})\lambda h y_j + \frac{1}{2c_2}\lambda h[1 + c_2\lambda h]y_j$$

$$y_{j+1} = [1 + (1 - \frac{1}{2c_2})\lambda h + \frac{1}{2c_2}\lambda h(1 + c_2\lambda h)]y_j$$

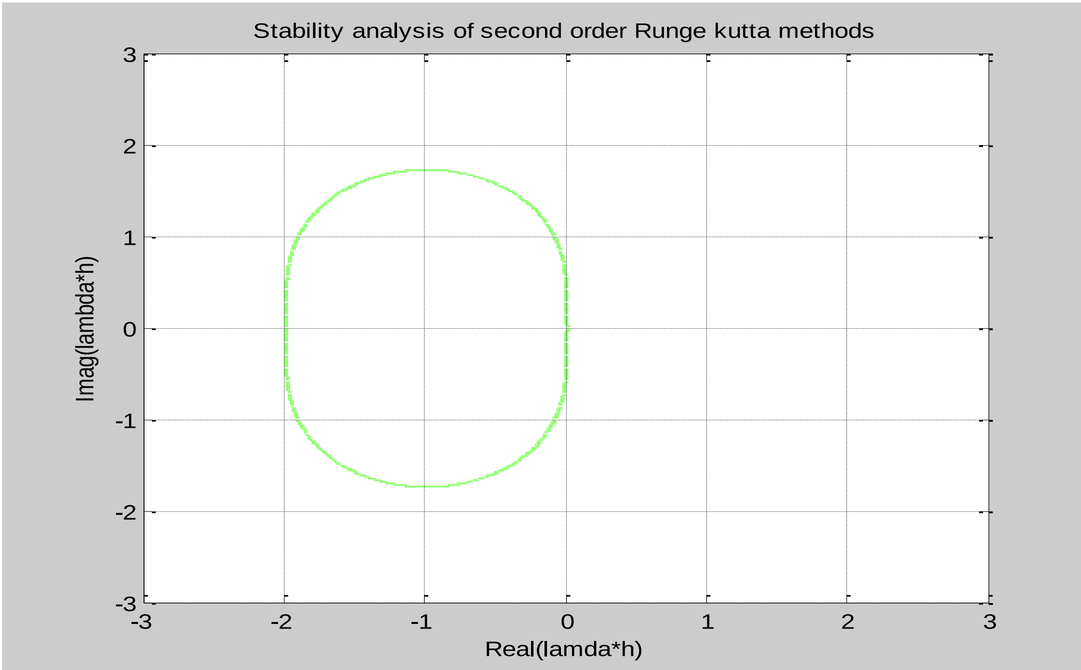
$$y_{j+1} = \left[1 + \lambda h + \frac{1}{2}(\lambda h)^2\right]y_j = E(\lambda h)y_j \tag{13}$$

Hence the propagation factor $E(\lambda h)$ is independent of the parameter $c_2$ .Therefore the stability intervals or regions of all second order Runge kutta methods is same. Now for $\lambda$ real and $\lambda < 0$, the condition

$$|E(\lambda h)| = \left|1 + \lambda h + \frac{1}{2}(\lambda h)^2\right| < 1$$

$$|E(\lambda h)| = \left|1 + z + \frac{1}{2}z^2\right| < 1$$

is satisfied when $\lambda h \in (-2,0)$ Hence the region of stability of second order is $(-2, 0)$. The plots of the stability regions for the second -order Runge-Kutta algorithms is shown in below Figure. This stability region is larger than those of multi-step methods. In particular, the stability regions of the multi-stage schemes grow with increasing accuracy while the stability regions of multi-step methods decrease with increasing accuracy. When analyzing multi-step methods, the next step would be to determine the locations in the λh-plane of the stability boundary (i.e. where |w| = 1). This however is not easy for Runge-Kutta methods and would require the solution of a higher-order polynomial for the roots. Instead, the most common approach is to simply rely on a contour plotter in which the λh-plane is discretized into a finite set of points and |w| is evaluated at these points.[5]




Then, the |w| = 1 contour can be plotted. Above Figure is Stability boundaries for second-order Runge-Kutta algorithms (stable within the boundaries).

**2.5. MATLAB CODES:** According to [10-11]. Write all programs in M-file and save it as midpoint.m, heun.m, Optimal.m, Raltson.m,

```
function [x,y]=systMidpoint(g,x0,xN,N,y0)
h=(xN-x0)/(N); x=[x0:h:xN]; y=zeros(length(y0),length(x)); y(:,1)=y0;
for n=1:1:N
k1=h*g(x(n),y(:,n));
k2=h*g(x(n)+h/2,y(:,n)+k1/2);
y(:,n+1)=y(:,n)+k2;
end
end

function [x,y]=systHeun(g,x0,xN,N,y0)
h=(xN-x0)/(N); x=[x0:h:xN]; y=zeros(length(y0),length(x)); y(:,1)=y0;
for n=1:1:N,
k1=h*g(x(n),y(:,n));
k2=h*g(x(n)+h,y(:,n)+k1);
y(:,n+1)=y(:,n)+1/2*(k1+k2);
```

```
end,
end

function [x,y]=systOptimal(g,x0,xN,N,y0)
h=(xN-x0)/(N); x=[x0:h:xN]; y=zeros(length(y0),length(x)); y(:,1)=y0;
for n=1:1:N
k1=h*g(x(n),y(:,n));
k2=h*g(x(n)+2/3*h,y(:,n)+2/3*k1);
y(:,n+1)=y(:,n)+1/4*(k1+3*k2);
end
end

function [x,y]=systRaltson(g,x0,xN,N,y0)
h=(xN-x0)/(N); x=[x0:h:xN]; y=zeros(length(y0),length(x)); y(:,1)=y0;
for n=1:1:N
k1=h*g(x(n),y(:,n));
k2=h*g(x(n)+3/4*h,y(:,n)+3/4*k1);
y(:,n+1)=y(:,n)+1/3*(k1+2*k2);
end
end
```

Call functions for example 1

```
g = @(x,y)[-2*y(1)+y(2)+2*sin(x);y(1)-2*y(2)+2*(cos(x)-sin(x))]; [x,y1]=systHeun(g,0,2,10,[2
3]);                                          [x,y2]=systMidpoint(g,0,2,10,[2 3]);
[x,y3]=systOptimal(g,0,2,10,[2 3]);
[x,y4]=systRaltson(g,0,2,10,[2 3]);
gexact = @(x,y)[2*exp(-x)+sin(x);2*exp(-x)+cos(x)];
xe=[0:0.2:2];
ye=gexact(xe);
```

Call functions for example 2

```
g = @(x,y)[y(2);-2*y(1)-2*exp(x)+1;-y(1)-exp(x)+1]; [x,y1]=systHeun(g,0,2,10,[1 0 1]);
[x,y2]=systMidpoint(g,0,2,10,[1 0 1]);
[x,y3]=systOptimal(g,0,2,10,[1 0 1]);
[x,y4]=systRaltson(g,0,2,10,[1 0 1]);
gexact = @(x,y)[cos(x)+sin(x)-exp(x)+1;-sin(x)-exp(x)+cos(x);sin(x)+cos(x)];
xe=[0:0.2:2];
ye=gexact(xe);
```

## 3. CONVERGENCE ANALYSIS OF METHODS

The numerical solutions $y_i$ will contain errors. We shall be concerned with the effect of these errors on the solutions, To finding the numerical solution of ordinary differential equations, two types of errors occurs. Round-off errors and Truncation errors. Rounding errors originate from the fact that computer can only represent numbers using a fixed and limited number of significant figures. Thus, such numbers cannot be represented exactly in computer memory. The discrepancy introduced by this limitation is called Round-off error. Truncation error in numerical analysis arises when approximations are used to estimate some quantity. The accuracy of the solution will depend on how small we take the step size $h$. A numerical method is said to be convergent if the numerical solution approaches the exact solution as the step size $h$ goes to 0. A method is convergent if, as more grid points are taken or step size is decreased.[8]

## 4. NUMERICAL RESULTS AND COMPARATIVE DISCUSSION

In this section, we employ the different techniques, obtained in this paper to solve system of simultaneous the ordinary differential equations with initial value problems and compare them. We use the stopping criteria up to fifteen decimal places. We have $T_{i+1} = y(t_{i+1}) - y_{i+1}, i = 0,1, \dots \dots \dots, N-1$ for computer programs (2010) [9]. All programs are written in Matlab 2009a. Let us consider the initial value

Problem1.

$$\frac{dy_1}{dx} = -2y_1 + y_2 + 2\sin(x), \qquad y_1(0) = 2, \qquad x \in [0,2]$$

$$\frac{dy_2}{dx} = y_1 - 2y_2 + 2(\cos(x) - \sin(x)), \quad y_2(2) = 3, \quad x \in [0,2]$$

The exact solution for this problem is $y_1(x) = 2e^{-x} + (\sin(x))$

$$y_2(x) = 2e^{-x} + (\cos(x))$$

Problem2.

$$\frac{dy_1}{dx} = y_2 , \qquad y_1(0) = 1 , \qquad x \in [0, 2]$$

$$\frac{dy_2}{dx} = -y_1 - 2e^x + 1, \quad y_2(0) = 0, \qquad x \in [0, 2]$$

$$\frac{dy_3}{dx} = -y_1 - e^x + 1, \qquad y_3(0) = 1 , \qquad x \in [0, 2]$$

The exact solution for this problem is $y_1(x) = \cos(x) + \sin(x) - e^x + 1$

$$y_2(x) = -\sin(x) - e^x + \cos(x)$$

$$y_3(x) = -\sin(x) + \cos(x)$$

**Table1. Numerical and Exact solutions of various techniques at grid points when h=0.2**

| Numerical Solution of Heun method | Numerical Solution of Midpoint method | Numerical Solution of Optimal method | Numerical Solution of Raltson method | Exact Solution $y(x_i)$ |
|---|---|---|---|---|
| 5.00000000000000 | 5.00000000000000 | 5.00000000000000 | 5.00000000000000 | 5.0000000000000 |
| 4.45800166611121 | 4.45800166611121 | 4.45733728161025 | 4.45700562078294 | 4.45365892094823 |
| 3.99849329874779 | 3.99849329874779 | 3.99730764948530 | 3.99671687973316 | 3.99175952045409 |
| 3.59295508996922 | 3.59295508996922 | 3.59139108295707 | 3.59061340670400 | 3.58522463268082 |
| 3.21914662409217 | 3.21914662409217 | 3.21734503526659 | 3.21645134925520 | 3.21137865671557 |
| 2.86047595068996 | 2.86047595068996 | 2.85857291132017 | 2.85763155645204 | 2.85329105536181 |
| 2.50541663590127 | 2.50541663590127 | 2.50354074640011 | 2.50261607574080 | 2.49917368809271 |
| 2.14694686270981 | 2.14694686270981 | 2.14521614930759 | 2.14436702241676 | 2.14180472865513 |
| 1.78199262237311 | 1.78199262237311 | 1.78051154030880 | 1.77978984579992 | 1.77796015271884 |
| 1.41086413351980 | 1.41086413351980 | 1.40972080373036 | 1.40917010330129 | 1.40784108907145 |
| 1.03668084652855 | 1.03668084652855 | 1.03594469135723 | 1.03559906750869 | 1.03449172322499 |

**Table2. Numerical and Exact solutions of various techniques at grid points when h=0.1**

| Numerical Solution of Heun method | Numerical Solution of Midpoint method | Numerical Solution of Optimal method | Numerical Solution of Raltson method | Exact Solution $y(x_i)$ |
|---|---|---|---|---|
| 5.00000000000000 | 5.00000000000000 | 5.00000000000000 | 5.00000000000000 | 5.00000000000000 |
| 4.71450041652780 | 4.71475005207899 | 4.71466679010517 | 4.71462517574829 | 4.71418725406869 |
| 4.45417990961681 | 4.45465297106592 | 4.45449509639074 | 4.45441622527195 | 4.45365892094823 |
| 4.21477245912148 | 4.21544275737837 | 4.21521892213185 | 4.21510714963255 | 4.21412957851382 |
| 3.99245545892654 | 3.99329687310565 | 3.99301571637767 | 3.99287538772289 | 3.99175952045409 |
| 3.78382593597781 | 3.78481248069112 | 3.78448260606167 | 3.78431804509913 | 3.78313073934511 |
| 3.58587846412102 | 3.58698437381846 | 3.58661432117698 | 3.58642981631125 | 3.58522463268082 |
| 3.39598443409984 | 3.39718426186642 | 3.39678247437810 | 3.39658226209089 | 3.39540108968782 |
| 3.21187238065068 | 3.21314110889103 | 3.21271589596161 | 3.21250414234584 | 3.21137865671557 |
| 3.03160910515718 | 3.03292226562990 | 3.03248176271098 | 3.03226254344110 | 3.03121551686054 |
| 2.85358136789842 | 2.85491516860513 | 2.85446729466870 | 2.85424457382278 | 2.85329105536181 |
| 2.67647795761876 | 2.67780941410731 | 2.67736182760094 | 2.67713943574105 | 2.67628781627933 |
| 2.49927197802095 | 2.50057904672969 | 2.50013910080233 | 2.49992071270855 | 2.49917368809271 |
| 2.32120322087432 | 2.32246493222466 | 2.32203962998691 | 2.32182874242720 | 2.32118418617783 |
| 2.14176052375749 | 2.14295711279568 | 2.14255306334821 | 2.14235297325054 | 2.14180472865513 |
| 1.96066403702832 | 1.96177706952456 | 1.96140044545314 | 1.96121422882896 | 1.96075282886548 |
| 1.77784734943060 | 1.77885984146367 | 1.77851633846034 | 1.77834683040809 | 1.77796015271884 |
| 1.59343944479802 | 1.59433597398629 | 1.59403077321240 | 1.59388054930804 | 1.59355441236788 |
| 1.40774648358630 | 1.40851329027065 | 1.40825102403001 | 1.40812238338747 | 1.40784108907145 |
| 1.22123342243688 | 1.22185849927696 | 1.22164322051605 | 1.22153817077445 | 1.22128499771445 |
| 1.03450550263294 | 1.03497867124403 | 1.03481382734141 | 1.03473407180845 | 1.03449172322499 |

**Table3. Numerical and Exact solutions of various techniques at grid points when h=0.05**

| Numerical Solution of Heun method | Numerical Solution of Midpoint method | Numerical Solution of Optimal method | Numerical Solution of Raltson method | Exact Solution $y(x_i)$ |
|---|---|---|---|---|
| 5.00000000000000 | 5.00000000000000 | 5.00000000000000 | 5.00000000000000 | 5.00000000000000 |
| 4.85368751301975 | 4.85371875162757 | 4.85370833719121 | 4.85370313049291 | 4.85364712766850 |
| 4.71426109239270 | 4.71432196864596 | 4.71430167061457 | 4.71429152369691 | 4.71418725406869 |
| 4.58114211588606 | 4.58123102898421 | 4.58120137829343 | 4.58118655762692 | 4.58104111610987 |
| 4.45378139283064 | 4.45389674251688 | 4.45385827006933 | 4.45383904205239 | 4.45365892094823 |
| 4.33165833346314 | 4.33179852058147 | 4.33175175706099 | 4.33172838792707 | 4.33151951325079 |
| 4.21428015419435 | 4.21444358141167 | 4.21438905704247 | 4.21436181273972 | 4.21412957851382 |
| 4.10118111555321 | 4.10136618823652 | 4.10130443249058 | 4.10127357853559 | 4.10102287917768 |
| 3.99192178973038 | 3.99212691696910 | 3.99205845822510 | 3.99202425953249 | 3.99175952045409 |
| 3.88608835481380 | 3.88631195057726 | 3.88623731573431 | 3.88620003642377 | 3.88592524295100 |
| 3.78329191297290 | 3.78353239739305 | 3.78345211144267 | 3.78341201462507 | 3.78313073934511 |
| 3.68316783000824 | 3.68342363077771 | 3.68333821633711 | 3.68329556388144 | 3.68301099251211 |
| 3.58537509383864 | 3.58564464771421 | 3.58555462452439 | 3.58550967680707 | 3.58522463268082 |
| 3.48959568964967 | 3.48987744405139 | 3.48978332844316 | 3.48973634408141 | 3.48945331132916 |
| 3.39553398957456 | 3.39582640427076 | 3.39572870860200 | 3.39567994417287 | 3.39540108968782 |
| 3.30291615492250 | 3.30321770379096 | 3.30311693585835 | 3.30306664560503 | 3.30279383986121 |
| 3.21148954910889 | 3.21179872196456 | 3.21169538439293 | 3.21164381991992 | 3.21137865671557 |
| 3.12102215957807 | 3.12133746405781 | 3.12123205367032 | 3.12117946362915 | 3.12092327882018 |
| 3.03130202714171 | 3.03162199063661 | 3.03151499780890 | 3.03146162757584 | 3.03121551686054 |
| 2.94213668128460 | 2.94245985291061 | 2.94235176091199 | 2.94229785225936 | 2.94206268807126 |
| 2.85335258011492 | 2.85367753271249 | 2.85356881703728 | 2.85351460779992 | 2.85329105536181 |
| 2.76479455375764 | 2.76511988591142 | 2.76501101360344 | 2.76495673734178 | 2.76474526993037 |
| 2.67632525010809 | 2.67664958817769 | 2.67654101715083 | 2.67648690281213 | 2.67628781627933 |
| 2.58782458197725 | 2.58814658213033 | 2.58803876048817 | 2.58798503206734 | 2.58779845866089 |
| 2.49918917477169 | 2.49950752501086 | 2.49940089036821 | 2.49934776656962 | 2.49917368809271 |
| 2.41033181395897 | 2.41064523613400 | 2.41054021494308 | 2.41048790884482 | 2.41032616919162 |
| 2.32118089167348 | 2.32148814347069 | 2.32138515035479 | 2.32133386907667 | 2.32118418617783 |
| 2.23167985191871 | 2.23197972881960 | 2.23187916591677 | 2.23182911029347 | 2.23169108750327 |
| 2.14178663391961 | 2.14207797112081 | 2.14198022744029 | 2.14193159170090 | 2.14180472865513 |
| 2.05147311327216 | 2.05175478755940 | 2.05166023835319 | 2.05161320980842 | 2.05149691278015 |
| 1.96072454062848 | 1.96099547219712 | 1.96090447834917 | 1.96085923708746 | 1.96075282886548 |
| 1.86953897774221 | 1.86979813195738 | 1.86971103939281 | 1.86966775798638 | 1.86957048729942 |
| 1.77792673078286 | 1.77817311987248 | 1.77809025898894 | 1.77804910221116 | 1.77796015271884 |
| 1.68590978090755 | 1.68614246558176 | 1.68606415070522 | 1.68602527526008 | 1.68594377413020 |
| 1.59352121215521 | 1.59373930314601 | 1.59366583201296 | 1.59362938627795 | 1.59355441236788 |
| 1.50080463680113 | 1.50100729631642 | 1.50093894958448 | 1.50090507336764 | 1.50083566502623 |
| 1.40781361837907 | 1.40800006146545 | 1.40793710225430 | 1.40790592656646 | 1.40784108907145 |
| 1.31461109264394 | 1.31478058845313 | 1.31472326191740 | 1.31469490876049 | 1.31463362140530 |
| 1.22126878681021 | 1.22142065976415 | 1.22136919269995 | 1.22134377487220 | 1.22128499771445 |
| 1.12786663745963 | 1.12800026830964 | 1.12795486879637 | 1.12793248971514 | 1.12787516999864 |
| 1.03449220756693 | 1.03460703434280 | 1.03456789142153 | 1.03454864496443 | 1.03449172322499 |

**Table4.Numerical and Exact solutions of various techniques at grid points when h=0.025**

| Numerical Solution of Heun method | Numerical Solution of Midpoint method | Numerical Solution of Optimal method | Numerical Solution of Raltson method | Exact Solution $y(x_i)$ |
|---|---|---|---|---|
| 5.00000000000000 | 5.00000000000000 | 5.00000000000000 | 5.00000000000000 | 5.00000000000000 |
| 4.92592968790689 | 4.92593359380086 | 4.92593229178723 | 4.92593164079666 | 4.92592456030375 |
| 4.85365693845579 | 4.85366465137591 | 4.85366208021540 | 4.85366079470114 | 4.85364712766850 |
| 4.78310654083759 | 4.78311796191733 | 4.78311415447857 | 4.78311225090756 | 4.78309247069916 |
| 4.71420517968410 | 4.71422021006119 | 4.71421519921366 | 4.71421269405243 | 4.71418725406869 |
| 4.64688140752003 | 4.64689994834097 | 4.64689376695353 | 4.64689067666742 | 4.64686001095294 |
| 4.58106561785900 | 4.58108757028490 | 4.58108025122382 | 4.58107659227604 | 4.58104111610987 |
| 4.51669001891542 | 4.51671528412930 | 4.51670686025583 | 4.51670264910621 | 4.51666275957536 |
| 4.45368860790528 | 4.45371708712061 | 4.45370759128815 | 4.45370284439173 | 4.45365892094823 |
| 4.39199714590922 | 4.39202874037979 | 4.39201820543057 | 4.39201293923596 | 4.39196534423820 |
| 4.33155313327216 | 4.33158774430322 | 4.33157620306462 | 4.33157043401214 | 4.33151951325079 |
| 4.27229578551441 | 4.27233331447502 | 4.27232079975542 | 4.27231454427504 | 4.27226062748422 |
| 4.21416600972985 | 4.21420635806641 | 4.21419290265070 | 4.21418617715978 | 4.21412957851382 |
| 4.15710638144745 | 4.15714945069835 | 4.15713508734303 | 4.15712790824390 | 4.15706892684011 |
| 4.10106112193301 | 4.10110681374440 | 4.10109157517234 | 4.10108395884963 | 4.10102287917768 |
| 4.04597607590875 | 4.04602429205136 | 4.04600821094617 | 4.04600017376404 | 4.04593726616225 |
| 3.99179868966894 | 3.99184933205593 | 3.99183244105601 | 3.99182399935518 | 3.99175952045409 |
| 3.93847798957041 | 3.93853096027623 | 3.93851329196849 | 3.93850446206307 | 3.93843865521685 |
| 3.88596456087746 | 3.88601976215761 | 3.88600134907088 | 3.88599214724509 | 3.88592524295100 |
| 3.83421052694123 | 3.83426786125296 | 3.83424873585108 | 3.83423917835579 | 3.83417139466220 |
| 3.78316952869419 | 3.78322889871807 | 3.78320909339269 | 3.78319919644184 | 3.78313073934511 |
| 3.73279670444120 | 3.73285801310352 | 3.73283756016663 | 3.73282733993349 | 3.73275840376400 |
| 3.68304866992886 | 3.68311182042480 | 3.68309075210102 | 3.68308022471431 | 3.68301099251211 |
| 3.63388349867571 | 3.63394839449333 | 3.63392674291187 | 3.63391592445180 | 3.63384656833212 |
| 3.58526070254639 | 3.58532724749128 | 3.58530504467775 | 3.58529395117187 | 3.58522463268082 |
| 3.53714121255333 | 3.53720931077394 | 3.53718658864188 | 3.53717523606088 | 3.53710610652164 |
| 3.48948735987009 | 3.48955691588374 | 3.48953370622595 | 3.48952211047931 | 3.48945331132916 |
| 3.44226285704124 | 3.44233377576075 | 3.44231011024040 | 3.44229828717198 | 3.44222995029034 |
| 3.39543277937400 | 3.39550496613483 | 3.39548087627641 | 3.39546884165987 | 3.39540108968782 |
| 3.34896354649742 | 3.34903690708536 | 3.34901242426543 | 3.34900019379968 | 3.34893314045096 |
| 3.30282290407553 | 3.30289734475494 | 3.30287250019273 | 3.30286008949717 | 3.30279383986121 |
| 3.25697990566137 | 3.25705533320380 | 3.25703015795161 | 3.25701758256140 | 3.25695223339840 |
| 3.21140489467914 | 3.21148121639245 | 3.21145574132598 | 3.21144301668714 | 3.21137865671557 |
| 3.16606948652269 | 3.16614661028044 | 3.16612086608896 | 3.16610800755350 | 3.16604471773022 |
| 3.12094655075836 | 3.12102438502972 | 3.12099840220609 | 3.12098542502703 | 3.12092327882018 |
| 3.07601019342155 | 3.07608864730134 | 3.07606245613197 | 3.07604937545831 | 3.07598843911339 |
| 3.03123573939595 | 3.03131472263510 | 3.03128835318981 | 3.03127518406145 | 3.03121551686054 |
| 2.98659971486576 | 2.98667913790193 | 2.98665262002370 | 2.98663937736636 | 2.98658103188090 |
| 2.94207982983092 | 2.94215960381936 | 2.94213296711404 | 2.94211966573410 | 2.94206268807126 |
| 2.89765496067639 | 2.89773499752096 | 2.89770827134691 | 2.89769492592621 | 2.89763935596923 |

| | | | | |
|---|---|---|---|---|
| 2.85330513278666 | 2.85338534517099 | 2.85335855862867 | 2.85334518371939 | 2.85329105536181 |
| 2.80901150319736 | 2.80909180461612 | 2.80906498653767 | 2.80905159655702 | 2.80899893793127 |
| 2.76475634327603 | 2.76483664806627 | 2.76480982700502 | 2.76479643623008 | 2.76474526993037 |
| 2.72052302142481 | 2.72060324479730 | 2.72057644901731 | 2.72056307157979 | 2.72051341487955 |
| 2.67629598579804 | 2.67637604386867 | 2.67634930133424 | 2.67633595121525 | 2.67628781627933 |
| 2.63206074702837 | 2.63214055684944 | 2.63211389521468 | 2.63210058623947 | 2.63205398033116 |
| 2.58780386095518 | 2.58788334054674 | 2.58785678714507 | 2.58784353297787 | 2.58779845866089 |
| 2.54351291134980 | 2.54359197973076 | 2.54356556156454 | 2.54355237570332 | 2.54350883103905 |
| 2.49917649263215 | 2.49925506985137 | 2.49922881358150 | 2.49921570935281 | 2.49917368809271 |
| 2.45478419257407 | 2.45486219974122 | 2.45483613167683 | 2.45482312223064 | 2.45478261400412 |
| 2.41032657498472 | 2.41040393430097 | 2.41037808038918 | 2.41036517869379 | 2.41032616919162 |
| 2.36579516237422 | 2.36587179716274 | 2.36584618297845 | 2.36583340181536 | 2.36579587296872 |
| 2.32118241859163 | 2.32125825332776 | 2.32123290406358 | 2.32122025602249 | 2.32118418617783 |
| 2.27648173143404 | 2.27655669177528 | 2.27653163223150 | 2.27651912970529 | 2.27648449379503 |
| 2.23168739522391 | 2.23176140803951 | 2.23173666261420 | 2.23172431779402 | 2.23169108750327 |
| 2.18679459335194 | 2.18686758675217 | 2.18684317943140 | 2.18683100430177 | 2.18679914823113 |
| 2.14179938078330 | 2.14187128414839 | 2.14184723849658 | 2.14183524483055 | 2.14180472865513 |
| 2.09669866652540 | 2.09676941053402 | 2.09674574968438 | 2.09673394903882 | 2.09670473566352 |
| 2.05149019605556 | 2.05155971271293 | 2.05153645935808 | 2.05152486306894 | 2.05149691278015 |
| 2.00617253370743 | 2.00624075637294 | 2.00621793275566 | 2.00620655193340 | 2.00617982254714 |
| 1.96074504501529 | 1.96081190842966 | 1.96078953633383 | 1.96077838185888 | 1.96075282886548 |
| 1.91520787901564 | 1.91527331932757 | 1.91525142006921 | 1.91524050258752 | 1.91521607929299 |
| 1.86956195050581 | 1.86962590529810 | 1.86960449971667 | 1.86959382963537 | 1.86957048729942 |
| 1.82380892225979 | 1.82387133057486 | 1.82385043902471 | 1.82384002650781 | 1.82381771447873 |
| 1.77795118720136 | 1.77801198956632 | 1.77799163190830 | 1.77798148687255 | 1.77796015271884 |
| 1.73199185053556 | 1.73205098898650 | 1.73203118457980 | 1.73202131669063 | 1.73200090632967 |
| 1.68593471183902 | 1.68599212994479 | 1.68597289763903 | 1.68596331630655 | 1.68594377413020 |
| 1.63978424711066 | 1.63983988999600 | 1.63982124812345 | 1.63981196249864 | 1.63979323149588 |
| 1.59354559078420 | 1.59359940515211 | 1.59358137152019 | 1.59357239049113 | 1.59355441236788 |
| 1.54722451770402 | 1.54727645185755 | 1.54725904374156 | 1.54725037592981 | 1.54723309122584 |
| 1.50082742506670 | 1.50087742892994 | 1.50086066306600 | 1.50085231682310 | 1.50083566502623 |
| 1.45436131433024 | 1.45440933946860 | 1.45439323204688 | 1.45438521545093 | 1.45436913510847 |
| 1.40783377309356 | 1.40787977273332 | 1.40786433939144 | 1.40785666024390 | 1.40784108907145 |
| 1.36125295694902 | 1.36129688599608 | 1.36128214181286 | 1.36127480763537 | 1.36125968262298 |
| 1.31462757131091 | 1.31466938636877 | 1.31465534585819 | 1.31464836388957 | 1.31463362140530 |
| 1.26796685322308 | 1.26800651260999 | 1.26799318971543 | 1.26798656690886 | 1.26797214279983 |
| 1.22128055314918 | 1.22131801691445 | 1.22130542500321 | 1.22129916802345 | 1.22128499771445 |
| 1.17457891674897 | 1.17461414668852 | 1.17460229854659 | 1.17459641376742 | 1.17458243235696 |
| 1.12787266664470 | 1.12790562631563 | 1.12789453414283 | 1.12788902764461 | 1.12787516999864 |
| 1.08117298418144 | 1.08120363891583 | 1.08119331432111 | 1.08118819188847 | 1.08117439273169 |
| 1.03449149118554 | 1.03451980810330 | 1.03451026210047 | 1.03450552922016 | 1.03449172322499 |

**Table 5. Numerical and Exact solutions of various methods at grid points when h=0.0125**

| Numerical Solution of Heun method | Numerical Solution of Midpoint method | Numerical Solution of Optimal method | Numerical Solution of Raltson method | Exact Solution $y(x_i)$ |
|---|---|---|---|---|
| 5.00000000000000 | 5.00000000000000 | 5.00000000000000 | 5.00000000000000 | 5.00000000000000 |
| 4.96273339845022 | 4.96273388672034 | 4.96273372396210 | 4.96273364258349 | 4.96273275247448 |
| 4.92592582412688 | 4.92592679452560 | 4.92592647105349 | 4.92592630931951 | 4.92592456030375 |
| 4.88956769402675 | 4.88956914041257 | 4.88956865827098 | 4.88956841720486 | 4.88956583983235 |
| 4.85364954540027 | 4.85365146163170 | 4.85365082286507 | 4.85365050349006 | 4.85364712766850 |
| 4.81816203486401 | 4.81816441479964 | 4.81816362145240 | 4.81816322479172 | 4.81815907979638 |
| 4.78309593752379 | 4.78309877502233 | 4.78309782913891 | 4.78309735621578 | 4.78309247069916 |
| 4.74844214610843 | 4.74844543502877 | 4.74844433865358 | 4.74844379049120 | 4.74843819249256 |
| 4.71419167011372 | 4.71419540431497 | 4.71419415949239 | 4.71419353711390 | 4.71418725406869 |
| 4.68033563495642 | 4.68033980829803 | 4.68033841707234 | 4.68033772150086 | 4.68033078024999 |
| 4.64686528113818 | 4.64686988748001 | 4.64686835189541 | 4.64686758415398 | 4.64686001095294 |
| 4.61377196341906 | 4.61377699662149 | 4.61377531872203 | 4.61377447983360 | 4.61376630036149 |
| 4.58104715000045 | 4.58105260392449 | 4.58105078575404 | 4.58104987674148 | 4.58104111610987 |
| 4.54868242171719 | 4.54868829022460 | 4.54868633382683 | 4.54868535571288 | 4.54867603847457 |
| 4.51666947123873 | 4.51667574819216 | 4.51667365561048 | 4.51667260941774 | 4.51666275957536 |
| 4.48500010227906 | 4.48500678154218 | 4.48500455481968 | 4.48500344157057 | 4.48499308258513 |
| 4.45366622881517 | 4.45367330425283 | 4.45367094543226 | 4.45366976614902 | 4.45365892094823 |
| 4.42265987431399 | 4.42266733979239 | 4.42266485091607 | 4.42266360662071 | 4.42265229760730 |
| 4.39197317096746 | 4.39198102035432 | 4.39197840346410 | 4.39197709517840 | 4.39196534423820 |
| 4.36159835893560 | 4.36160658610033 | 4.36160384323755 | 4.36160247198299 | 4.36159030049301 |
| 4.33152778559742 | 4.33153638441134 | 4.33153351761672 | 4.33153208441448 | 4.33151951325079 |
| 4.30175390480941 | 4.30176286914592 | 4.30175988045952 | 4.30175838633044 | 4.30174543587598 |
| 4.27226927617144 | 4.27227859990624 | 4.27227549136741 | 4.27227393733194 | 4.27226062748422 |
| 4.24306656429998 | 4.24307624131129 | 4.24307301495856 | 4.24307140203674 | 4.24305775221541 |
| 4.21413853810826 | 4.21414856227706 | 4.21414522014807 | 4.21414354935949 | 4.21412957851382 |
| 4.18547807009344 | 4.18548843530369 | 4.18548497943514 | 4.18548325179888 | 4.18546897841512 |
| 4.15707813563040 | 4.15708883576929 | 4.15708526819683 | 4.15708348473145 | 4.15706892684011 |
| 4.12893181227210 | 4.12894284123029 | 4.12893916398843 | 4.12893732571193 | 4.12892250089492 |
| 4.10103227905624 | 4.10104363072816 | 4.10103984585023 | 4.10103795377996 | 4.10102287917768 |
| 4.07337281581825 | 4.07338448410232 | 4.07338059362032 | 4.07337864877297 | 4.07336334109129 |
| 4.04594680251013 | 4.04595878130906 | 4.04595478725360 | 4.04595279064518 | 4.04593726616225 |
| 4.01874771852525 | 4.01876000174634 | 4.01875590614656 | 4.01875385879228 | 4.01873813336536 |
| 3.99176914202887 | 3.99178172358429 | 3.99177752846771 | 3.99177543138200 | 3.99175952045409 |
| 3.96500474929412 | 3.96501762310120 | 3.96501333049367 | 3.96501118469010 | 3.96499510329657 |
| 3.93844831404343 | 3.93846147402498 | 3.93845708595056 | 3.93845489244178 | 3.93843865521685 |
| 3.91209370679515 | 3.91210714687975 | 3.91210266536062 | 3.91210042515832 | 3.91208404634144 |
| 3.88593489421525 | 3.88594860833762 | 3.88594403539392 | 3.88594174950880 | 3.88592524295100 |
| 3.85996593847398 | 3.85997992057526 | 3.85997525822505 | 3.85997292766672 | 3.85995630683682 |

| | | | | |
|---|---|---|---|---|
| 3.83418099660727 | 3.83419524063537 | 3.83419049089449 | 3.83418811667143 | 3.83417139466220 |
| 3.80857431988278 | 3.80858881979274 | 3.80858398467467 | 3.80858156779420 | 3.80856475732840 |
| 3.78314025317044 | 3.78315500292475 | 3.78315008444053 | 3.78314762590872 | 3.78313073934511 |
| 3.75787323431736 | 3.75788822788634 | 3.75788322804444 | 3.75788072886604 | 3.75786377820524 |
| 3.73276779352690 | 3.73278302488904 | 3.73277794569524 | 3.73277540687368 | 3.73275840376400 |
| 3.70781855274192 | 3.70783401588424 | 3.70782885934151 | 3.70782628187878 | 3.70780923762195 |
| 3.68302022503182 | 3.68303591395027 | 3.68303068205863 | 3.68302806695525 | 3.68301099251211 |
| 3.65836761398356 | 3.65838352268339 | 3.65837821743981 | 3.65837556569473 | 3.65835847169078 |
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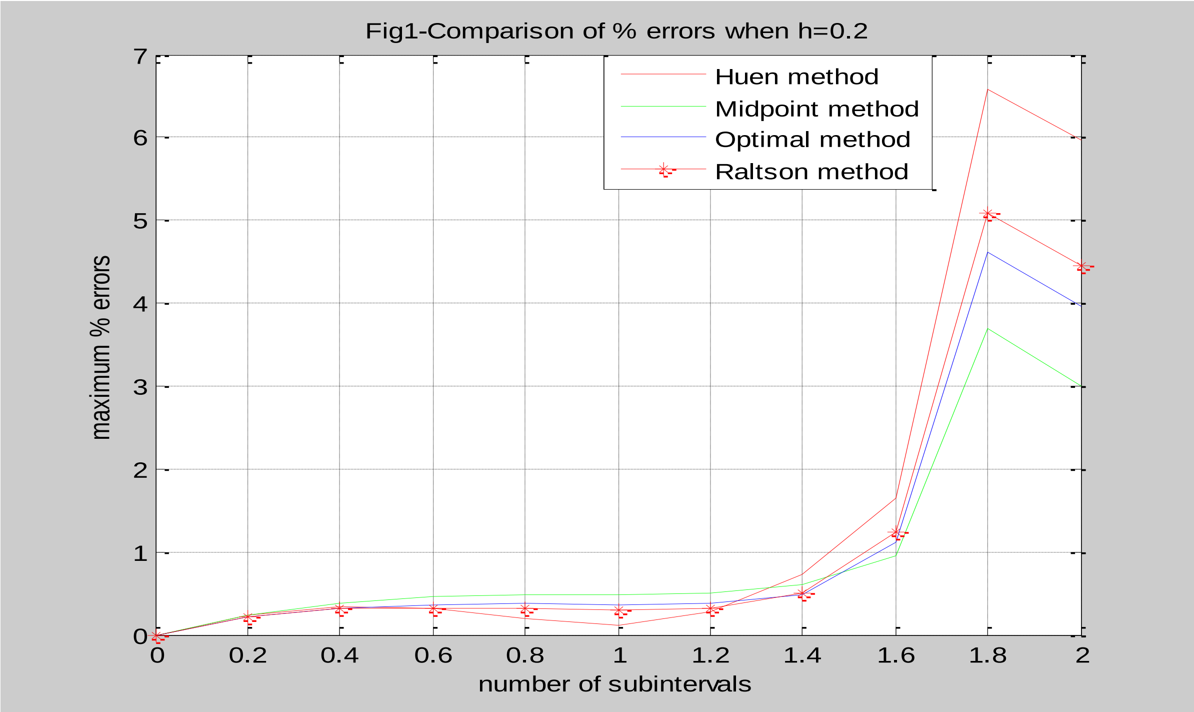
**Table.6. Estimation of average % errors by various methods at different step size h**

| Step size h | Average % Error in Midpoint method | Average % Error in Heun method | Average %Error in Optimalmethod | Average %Error in Raltson method |
|---|---|---|---|---|
| h=0.2 | p2=9.804732502 677809e-001 | p1=1.490367120 678143e+000 | p3=1.1060551209701 10e+000 | p4=1.1871343958564 42e+000 |
| h=0.1 | p2=3.47504795215030 8e-001 | p1=5.2851196716503 65e-001 | p3=3.9833419508330 45e-001 | p4=4.2723398234604 53e-001 |
| h=0.05 | p2=7.54606238799641 5e-002 | p1=1.0985600082081 12e-001 | p3=8.4892578515347 21e-002 | p4=9.0347788915291 19e-002 |
| h=0.025 | p2=2.41242827674237 2e-002 | p1=3.5632497169804 07e-002 | p3=2.7478122509415 96e-002 | p4=2.9327572298309 16e-002 |
| h=0.0125 | p2=5.80408547079670 7e-003 | p1=8.4994213086670 43e-003 | p3=6.5872554942438 05e-003 | p4=7.0196919845357 70e-003 |



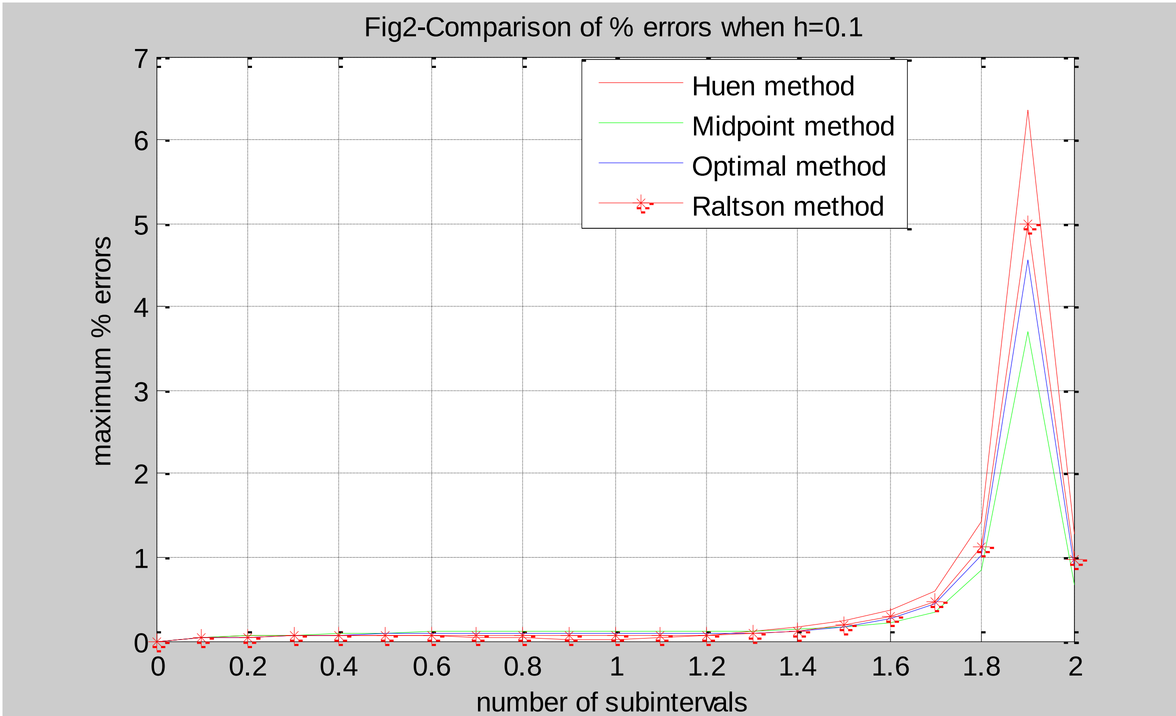


Fig2-Comparison of % errors when h=0.1
Huen method
Midpoint method
Optimal method
Raltson method
maximum % errors
number of subintervals
0
1
2
3
4
5
6
7
0.2
0.4
0.6
0.8
1.2
1.4
1.6
1.8

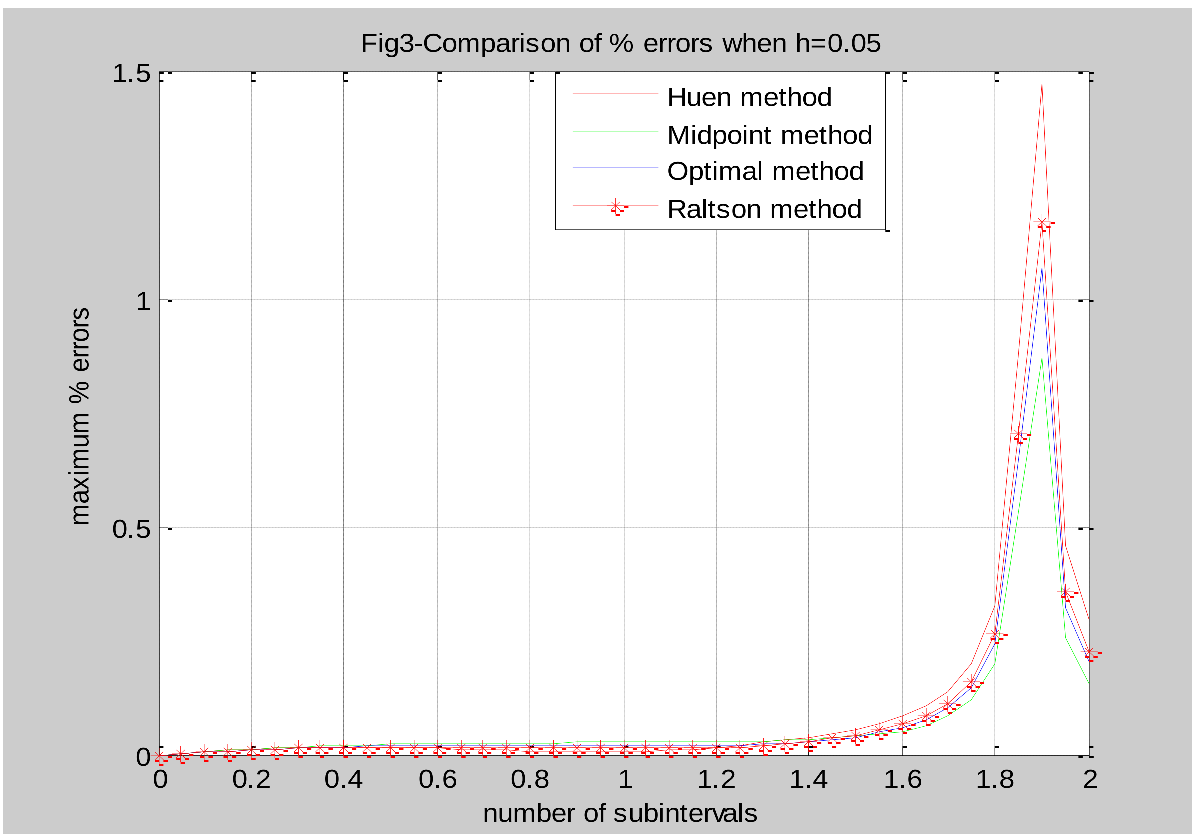


Fig3-Comparison of % errors when h=0.05
Huen method
Midpoint method
Optimal method
Raltson method
maximum % errors
number of subintervals
0
0.5
1
1.5
0.2
0.4
0.6
0.8
1.2
1.4
1.6
1.8
2

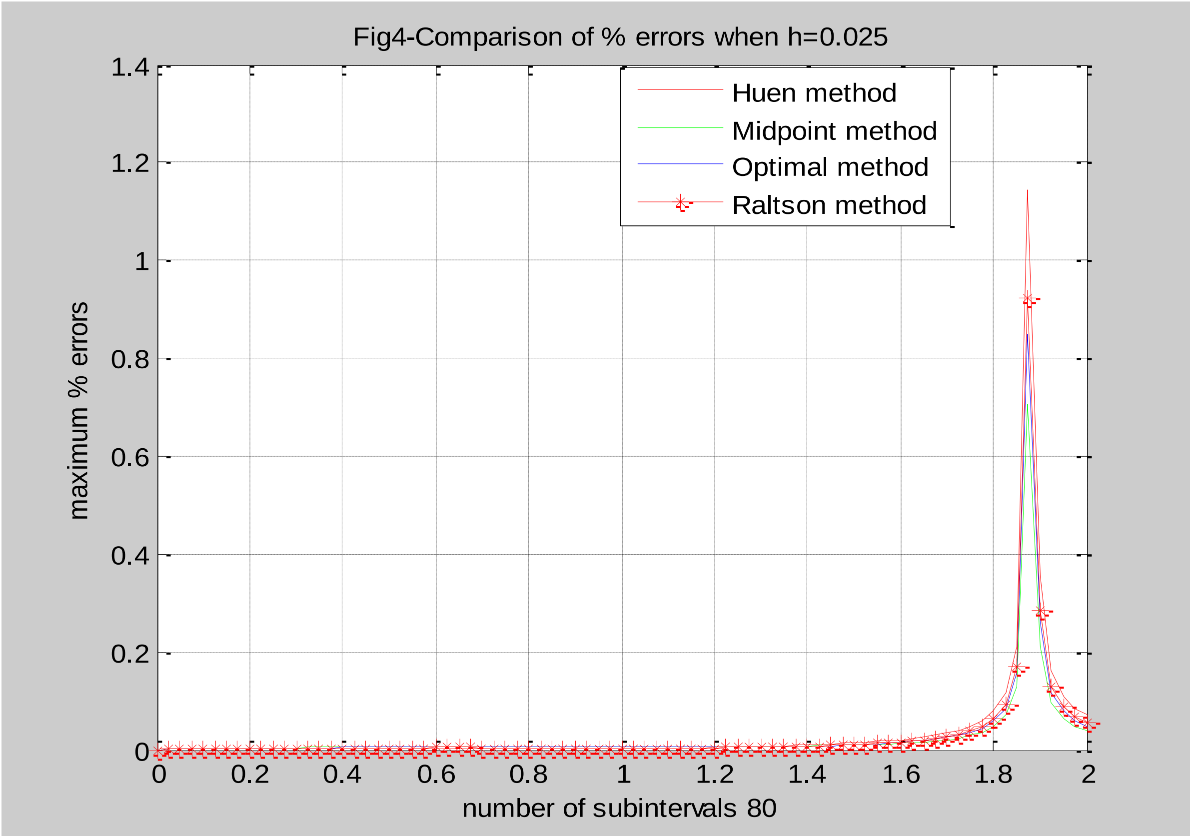


Fig4-Comparison of % errors when h=0.025
maximum % errors
number of subintervals 80
Huen method
Midpoint method
Optimal method
Raltson method
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2

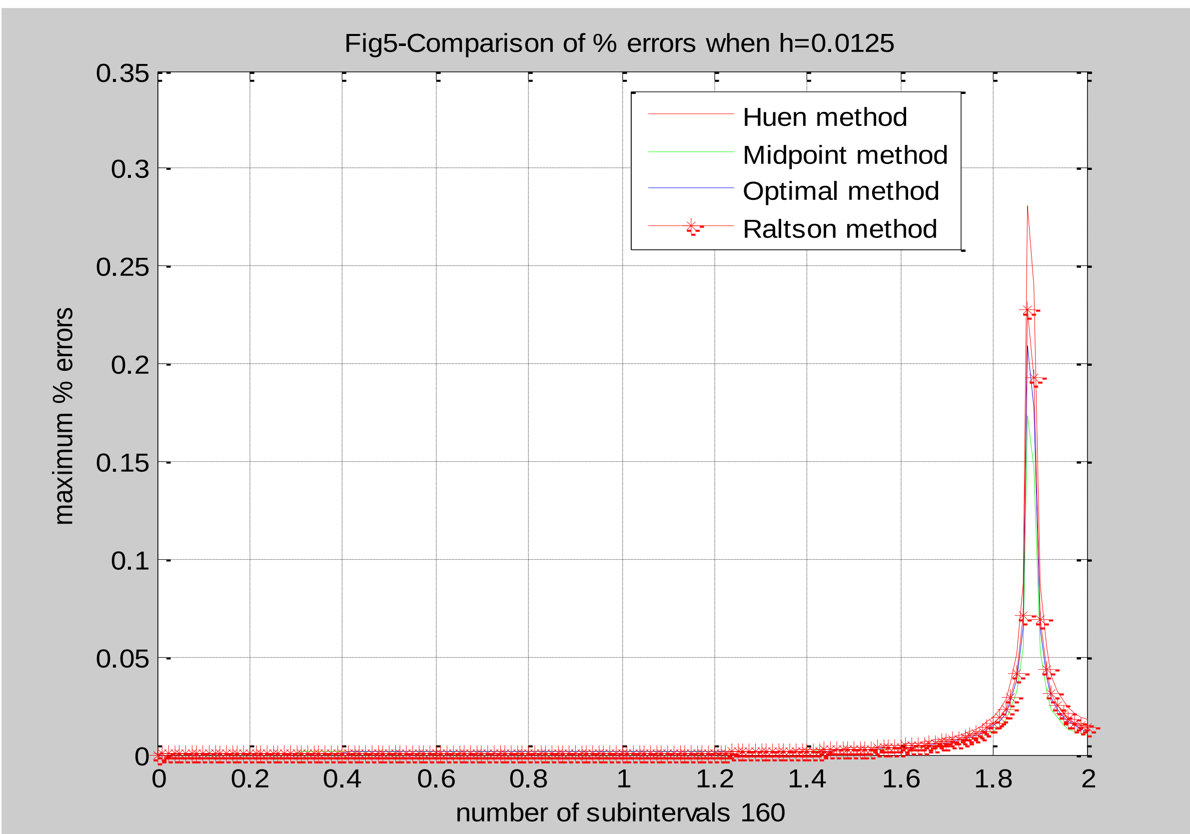
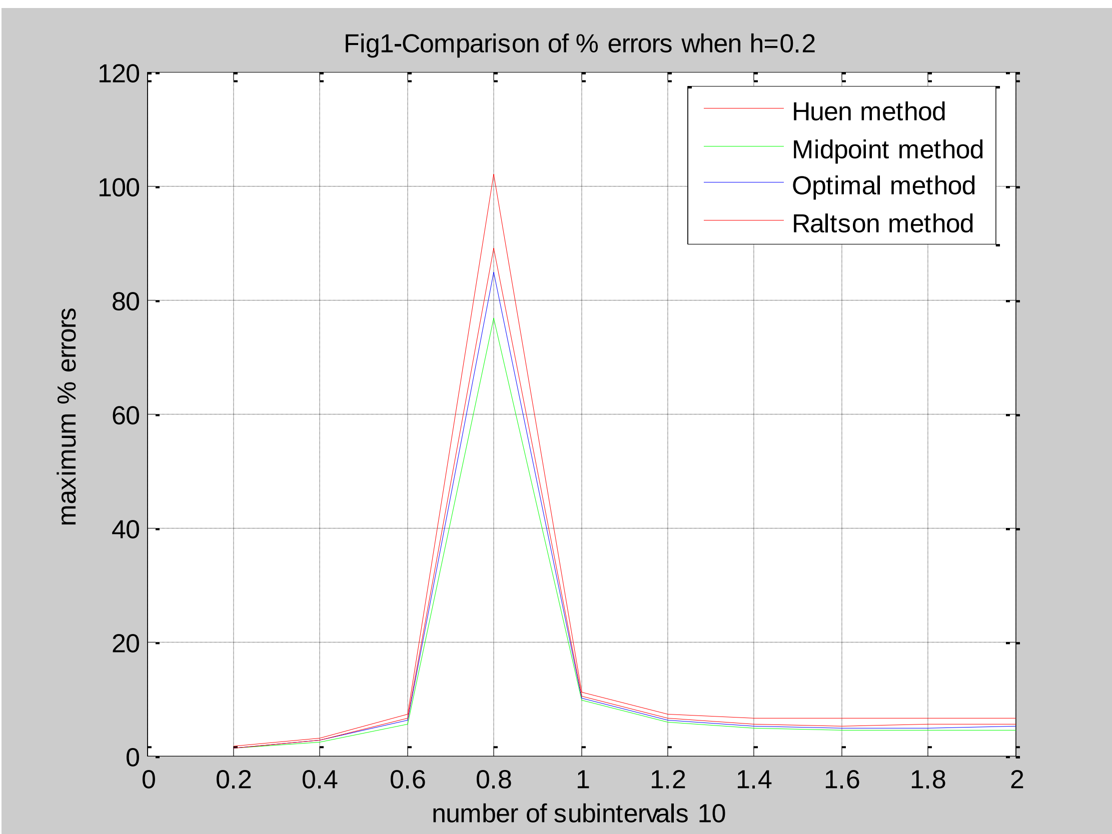


Fig5-Comparison of % errors when h=0.0125
maximum % errors
number of subintervals 160
Huen method
Midpoint method
Optimal method
Raltson method
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2

**Problem 2.**

**Table 7. Estimation of % errors by various methods at end point with step size h**

| Step size h | Average % Error in Midpoint method | Average % Error in Heun method | Average % Error in Optimal method | Average % Error in Raltson method |
|---|---|---|---|---|
| h=0.2 | p2 =4.540740607540969e+000 | p1=6.561985920100095e+000 | p3=5.110977640457402e+000 | p4= 5.467626711817050e+000 |
| h=0.1 | p2 = 1.147795897214805e+000 | p1 = 1.647311763461657e+000 | p3= 1.301168211503721e+000 | p4= 1.386979510516571e+000 |
| h=0.05 | p2= 2.883504446231504e-001 | p1 = 4.123111731385671e-001 | p3= 3.280777118834293e-001 | p4= 3.490481222284070e-001 |
| h=0.025 | p2= 7.224800320544049e-002 | p1 = 1.031082950998461e-001 | p3= 8.235421372031196e-002 | p4= 8.753191155711636e-002 |
| h=0.0125 | p2= 1.808099509046013e-002 | p1 = 2.577878432869392e-002 | p3= 2.062940553985026e-002 | p4= 2.191540843967385e-002 |



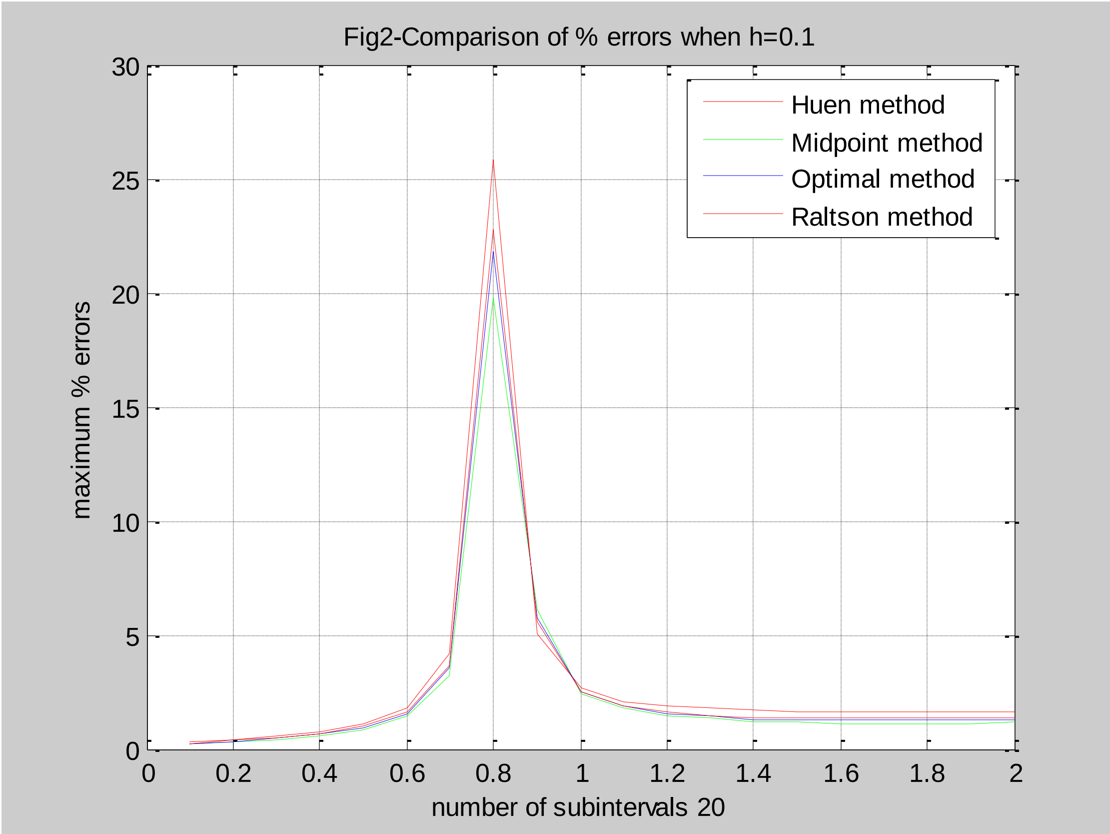


Fig2-Comparison of % errors when h=0.1
maximum % errors
number of subintervals 20
Huen method
Midpoint method
Optimal method
Raltson method
0
5
10
15
20
25
30
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2

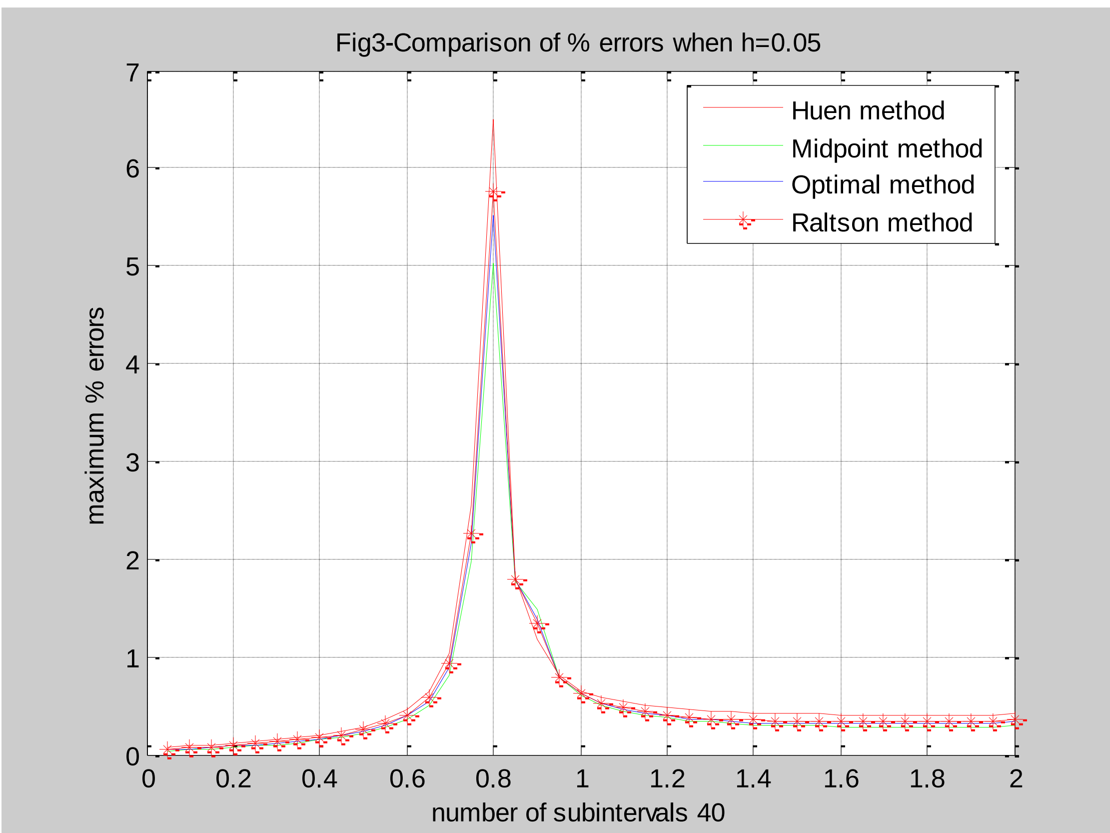


Fig3-Comparison of % errors when h=0.05
maximum % errors
number of subintervals 40
Huen method
Midpoint method
Optimal method
Raltson method
0
1
2
3
4
5
6
7
0.2
0.4
0.6
0.8
1.2
1.4
1.6
1.8

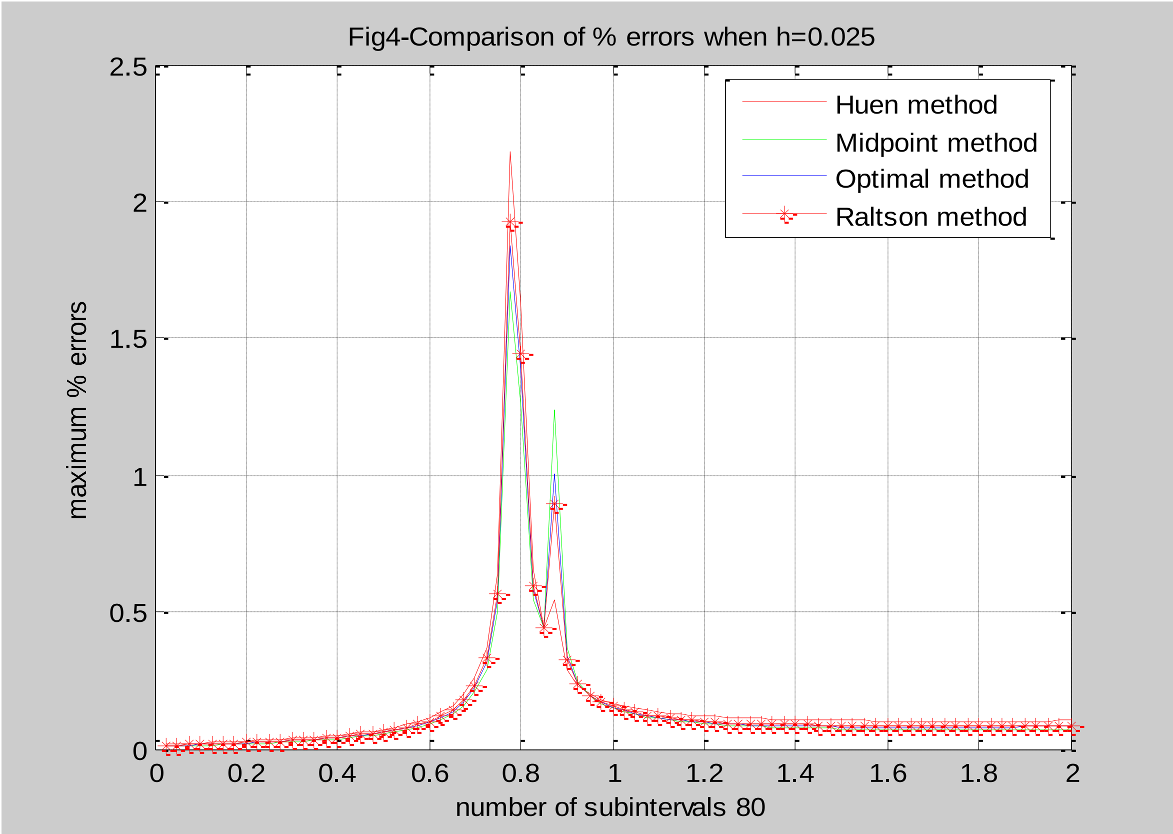


Fig4-Comparison of % errors when h=0.025
Huen method
Midpoint method
Optimal method
Raltson method
maximum % errors
0
0.5
1
1.5
2
2.5
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
number of subintervals 80

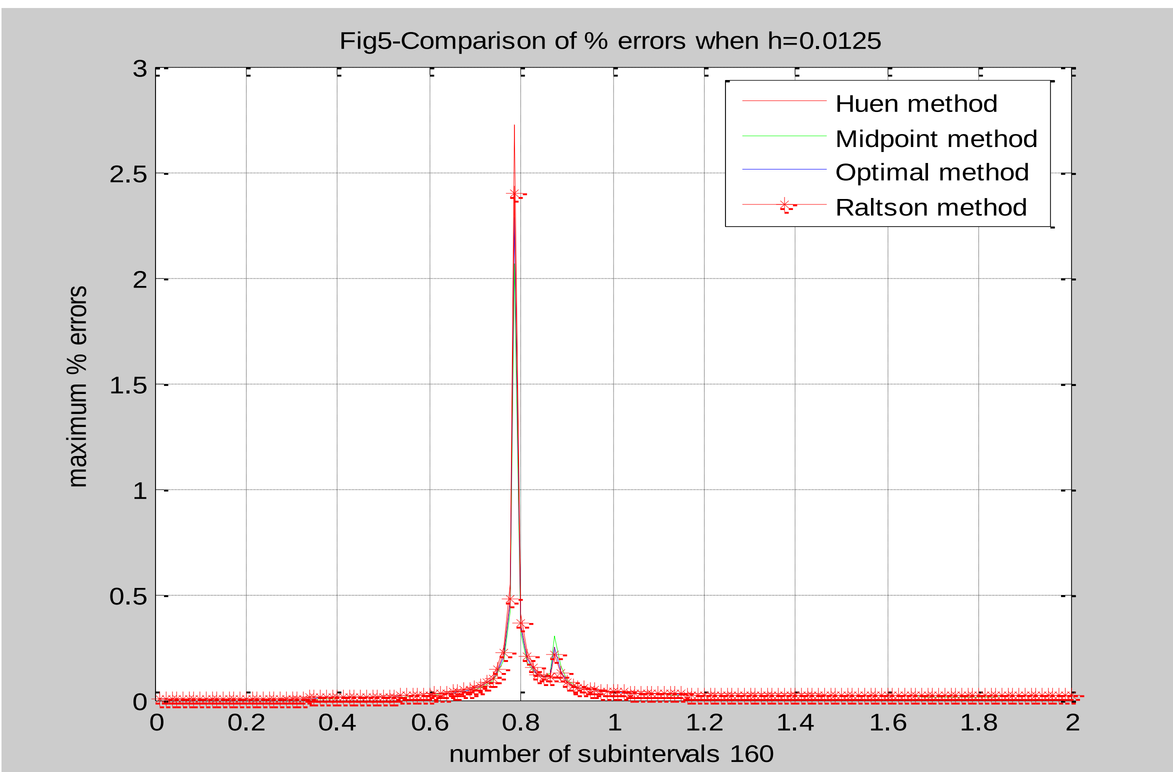


Fig5-Comparison of % errors when h=0.0125
Huen method
Midpoint method
Optimal method
Raltson method
maximum % errors
0
0.5
1
1.5
2
2.5
3
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
number of subintervals 160

**Table8. Ratios of percentage error between different values of h**

| Step Size h | Ratios % Error in midpoint Method | | Ratios % Error in Huen Method | | Ratios % Error in Optimal Method | | Ratios % Error in Raltson Method | |
|---|---|---|---|---|---|---|---|---|
| | Problm1 | Problm2 | Prob1 | Prob2 | Prob1 | Prob2 | Prob1 | Prob2 |
| 0.2 | | | | | | | | |
| | 2.8215 | 3.9561 | 2.8199 | 3.9835 | 2.7767 | 3.9280 | 2.7787 | 3.9421 |
| 0.1 | | | | | | | | |
| | 4.6051 | 3.9806 | 4.8110 | 3.9953 | 4.6922 | 3.9660 | 4.7288 | 3.9736 |
| 0.05 | | | | | | | | |
| | 3.1280 | 3.9911 | 3.0830 | 3.9988 | 3.0895 | 3.9837 | 3.0806 | 3.9877 |
| 0.025 | | | | | | | | |
| | 4.1564 | 3.9958 | 4.1923= | 3.9997 | 4.1714 | 3.9921 | 4.1779 | 3.9941 |
| 0.0125 | $= 2^2$ | $= 2^2$ | $2^2$ | $= 2^2$ | $= 2^2$ | $= 2^2$ | $= 2^2$ | $= 2^2$ |

From the Table6 and Table7.The calculated results are displayed for average percentage errors at different step size. Reasonably good results are obtained even for a large step size and the approximation can be improved by decreasing the step size. According to the results Midpoint method give good result when compare with exact solution. Since midpoint method has minimum error as compared to other methods. From the Table8 we see that when we reduce the step size h/2 the ratios of % errors are equivalent to $2^2$. This verifies that each method is second order RK method. Thus the accuracy of Midpoint method is good among them. Hence we conclude that Midpoint method is most effective scheme overall.

## 5. CONCLUSIONS

The RK2 methods are simple methods of solving first-order system of ODE, particularly suitable for quick programming because of their great simplicity, although their accuracy is not high. First, the midpoint method is more accurate than other RK2 methods. Second, it is more stable. The above plots shows the results obtained from different algorithms at different step size. One can easily adopt this paper as needed for a different type of problems for solving system ordinary differential equations for more variables first order linear equations. Inner working of the numerical methods will be clear, especially for the students. In using numerical procedure, such as Midpoint method, one must always keep in mind the question of whether the results are accurate enough to be useful. In the preceding examples, the accuracy of the numerical results could be ascertained directly by a comparison with the solution obtained analytically. Thus from the numerical computations we see that accuracy and efficiency of midpoint method is most effective.

## CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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