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IDENTITIES OF FIBONACCI – LIKE SEQUENCE

V. K. GUPTA[1], YASHWANT K. PANWAR[2,*] AND NIRMALA GUPTA[1]

[1]Department of Mathematics, Govt. Madhav Science College, Ujjain, India

[2]Department of Mathematics, Mandsaur Institute of Technology, Mandsaur, India

Abstract: In this paper, we present identities of Fibonacci-Like sequence. The Fibonacci-Like Sequence is defined by recurrence relation $S_k = S_{k-1} + S_{k-2}$, $k \geq 2$ *with* $S_0 = 2,\ S_1 = 2$. This was introduced by Singh, Sikhwal and Bhatnagar in 2010. Also we describe and derive connection formulae and negation formula.

Keywords: Fibonacci sequence, Lucas sequence, Fibonacci-Like sequence.

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1. Introduction

Many authors have generalized second order recurrence sequences by preserving the recurrence relation and alternating the first two terms of the sequence and some authors have generalized these sequences by preserving the first two terms of the sequence but altering the recurrence relation slightly.

Kalman and Mena [6] generalize the Fibonacci sequence by

$$F_n = aF_{n-1} + bF_{n-2},\ n \geq 2 \quad with \quad F_0 = 0,\ F_1 = 1 \qquad (1.1)$$

Horadam [2] defined generalized Fibonacci sequence { H_n } by

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*Corresponding author

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$$H_n = H_{n-1} + H_{n-2},\ n \geq 3\ with\ H_1 = p,\ H_2 = p + q, \tag{1.2}$$

where p and q are arbitrary integers

L. R. Natividad [8], Deriving a Formula in solving Fibonacci-Like sequence. He found missing terms in Fibonacci-Like sequence and solved by standard formula.

V. K. Gupta, Y. K. Panwar and O. Sikhwal [11], defined generalized Fibonacci sequences and derived its identities connection formulae and other results.

B. Singh, O. Sikhwal, and S. Bhatnagar [5], defined Fibonacci-Like sequence by recurrence relation $S_k = S_{k-1} + S_{k-2}$, $k \geq 2\, with\ S_0 = 2,\ S_1 = 2$

The associated initial conditions S_0 and S_1 are the sum of the Fibonacci and Lucas sequences respectively, i.e. $S_0 = F_0 + L_0$ and $S_1 = F_1 + L_1$.

In this paper, we introduce identities of the Fibonacci-Like sequence and we describe and derive the connection formulae and negation formula.

2. Preliminaries

Before presenting our main theorems, we will need to introduce some known results and notations.

The sequence of Fibonacci numbers F_n, [10], is defined by

$$F_n = F_{n-1} + F_{n-2},\ n \geq 2\ with\ F_0 = 0,\ F_1 = 1 \tag{2.1}$$

The sequence of Lucas numbers L_n, [10], is defined by

$$L_n = L_{n-1} + L_{n-2},\ n \geq 2\ with\ L_0 = 2,\ L_1 = 1 \tag{2.2}$$

The sequence of Fibonacci-Like numbers S_k, [5], is defined by

$$S_k = S_{k-1} + S_{k-2},\ k \geq 2\ with\ S_0 = 2,\ S_1 = 2 \tag{2.3}$$

The Binet's formula for Fibonacci-Like sequence is given by

$$S_k = 2\left(\frac{\Re_1^{k+1} - \Re_2^{k+1}}{\Re_1 - \Re_2}\right) \tag{2.4}$$

Where $\Re_1$ & $\Re_2$ are the roots of the characteristic equation $x^2 = x + 1$ and

$\Re_1 = \frac{1+\sqrt{5}}{2}$ *and* $\Re_2 = \frac{1-\sqrt{5}}{2}$.

3. Main results

Fibonacci-Like sequence is similar to the other second order classical sequences. In this section we present identities of the Fibonacci-Like sequence (2.3) and we describe and derive the connection formulae and negation formula. We shall use the Induction method and Binet's formula for derivation.

The few terms of the Fibonacci-Like sequence are 2, 2, 4, 6, 10, 16 and 26 and so on.

Theorem 3.1. *If* S_k *is Fibonacci-Like sequence then*

$$(i) \quad 2\left(S_0 + S_3 + S_6 + S_9 + \ldots + S_{3k}\right) = 2\sum_{k=o}^{n} S_{3k} = S_{3k+2} \tag{3.1}$$

$$(ii) \quad 2\left(S_3 + S_6 + S_9 + \ldots + S_{3k}\right) = 2\sum_{k=1}^{n} S_{3k} = S_{3k+2} + 4 \tag{3.2}$$

$$(iii) \quad 2\left(S_1 + S_4 + S_7 + S_{10} + \ldots + S_{3k-2}\right) = 2\sum_{k=1}^{n} S_{3k-2} = S_{3k} - 2 \tag{3.3}$$

$$(iv) \quad 2\left(S_2 + S_5 + S_8 + S_{11} + \ldots + S_{3k-1}\right) = 2\sum_{k=1}^{n} S_{3k-1} = S_{3k+1} - 2 \tag{3.4}$$

$$(v) \quad 2\left[S_0 + 2\left(S_3 + S_6 + S_9 + \ldots + S_{3k}\right)\right] = 2\left\{S_0 + 2\sum_{k=1}^{n} S_{3k}\right\} = S_{3k+2} + 2 \tag{3.5}$$

$$(vi) \quad 2\left(S_1 - S_2 + S_4 - S_5 + S_7 - S_8 + \ldots + S_{3k-2} - S_{3k-1}\right) = 2\sum_{k=1}^{n}\left(S_{3k-2} - S_{3k-1}\right) = S_{3k} - S_{3k+1} \tag{3.6}$$

The identities from (3.1) to (3.4) can be derived by induction method.

If we adding (3.1) term wise from (3.2), we get the identity (3.5) and if we subtract (3.3) term wise from (3.4), we get the identity (3.6).

Theorem 3.2. *Prove that* $3F_{k+3} - L_{k+3} = S_k$ (3.7)

Proof. We shall use mathematical induction method over k. For $k=0$,

$3F_{0+3}-L_{0+3}=3\times 2-4=2=S_0$

For $k=1$, $3F_{1+3}-L_{1+3}=3\times 3-7=2=S_1$, which is also true for $k=1$.

Assume that the result is true for $k=n$, then $3F_{n+3}-L_{n+3}=S_n$, Now

$$3F_{(n+1)+3}-L_{(n+1)+3}=3F_{e+3}-L_{e+3}\ ; \quad where\ e=(n+1)$$

$$=3(F_{e+2}+F_{e+1})-(L_{e+2}+L_{e+1})$$

$$=(3F_{e+2}-L_{e+2})+(3F_{e+1}-L_{e+1})$$

$$=S_{e-1}+S_{e-2} \qquad \text{(By induction hypothesis)}$$

$$=S_e = S_{n+1}$$

Therefore, $3F_{(n+1)+3}-L_{(n+1)+3}=S_{n+1}$, which is also true for $k=n+1$.

Hence, the result is true for all k.

Theorem 3.3. *If* S_k *is Fibonacci-Like sequence then*

$$S_{2k-4}\,S_{2k+1}-S_{2k-2}\,S_{2k-1}=F_6=8, \quad k>1 \tag{3.8}$$

Theorem 3.4. *If* S_k *is Fibonacci-Like sequence then*

$$S_{2k-3}\,S_{2k+2}-S_{2k-1}\,S_{2k}=-F_6=-8, \quad k>1 \tag{3.9}$$

Theorem 3.5. *If* S_k *is Fibonacci-Like sequence then*

$$S_{2k}\,S_{2k+3}-S_{2k+1}\,S_{2k+2}=L_3=4, \quad k\ge 0 \tag{3.10}$$

Theorem 3.6. *If* S_k *is Fibonacci-Like sequence then*

$$2S_{2k-1}\,S_{2k}-S_{2k-2}\,S_{2k+3}-S_{2k-2}\,S_{2k+1}=L_3=4, \quad k\ge 1 \tag{3.11}$$

Theorem 3.7. *If* S_k *is Fibonacci-Like sequence then*

$$S_{2k-1}^2 - S_{2k-3}\,S_{2k+1} = L_3 = 4, \quad k > 1 \tag{3.12}$$

Theorem 3.8. *If* S_k *is Fibonacci-Like sequence then*

$$S_{2k}^2 - S_{2k-2}\,S_{2k+2} = -L_3 = -4, \quad k \geq 1 \tag{3.13}$$

Theorem 3.9. *If* S_k *is Fibonacci-Like sequence then*

$$\left(S_{2k}\,S_{2k+4} - S_{2k+2}^2 + S_{2k+4}\,S_{2k} - S_{2k+1}\,S_{2k+3}\right) + 3\left(S_{2k}\,S_{2k+4} - S_{2k+2}^2\right) = L_3 = 4, \quad k \geq 0 \tag{3.14}$$

Theorem 3.10. *If* S_k *is Fibonacci-Like sequence then*

$$S_{2k+2}\,S_{2k+3} - S_{2k}\,S_{2k+3} + 4\left(S_{2k}\,S_{2k+4} - S_{2k+2}^2\right) + 3\left(S_{2k-1}\,S_{2k+4} - S_{2k+2}\,S_{2k+1}\right) = L_5 + 1 = 12, \quad k \geq 1 \tag{3.15}$$

Theorem 3.11. *If* S_k *is Fibonacci-Like sequence then*

$$S_{2k-4}\,S_{2k+2} + S_{2k-3}\,S_{2k+1} - S_{2k-2}\,S_{2k} - S_{2k-1}^2 = F_6 = 8, \quad k > 1 \tag{3.16}$$

Theorem 3.12. *If* S_k *is Fibonacci-Like sequence then*

$$3S_{2k-3} - S_{2k-4} = 13F_{2k} - 5F_{2k} \tag{3.17}$$

Theorem 3.13. *If* S_k *is Fibonacci-Like sequence then*

$$S_k^3 + S_{k-1}^3 + 3S_{k-1}\,S_k^2 = 8F_{3k+2}\ , \quad k \geq 1 \tag{3.18}$$

Proof of the theorems (3.3, …, 3.13), It can be proved same as Theorem: 3.14

Theorem 3.14. *If* S_k *is Fibonacci-Like sequence then*

$$S_k^3 - S_{k-1}^3 + 3S_k\,S_{k-1}^2 = 8F_{3k+1}\ , \quad k \geq 1 \tag{3.19}$$

Proof. *By Binet's formula (2.4), we have*

$$S_k^3 - S_{k-1}^3 + 3S_k\, S_{k-1}^2$$

$$= \frac{8}{(\Re_1-\Re_2)^3}\left\{\left(\Re_1^{k+1}-\Re_2^{k+1}\right)^3 - \left(\Re_1^{k}-\Re_2^{k}\right)^3\right\} + \frac{24}{(\Re_1-\Re_2)^3}\left\{\left(\Re_1^{k+1}-\Re_2^{k+1}\right)\left(\Re_1^{k}-\Re_2^{k}\right)^2\right\}$$

$$= \frac{8}{(\Re_1-\Re_2)^2}\left\{2\left(\frac{\Re_1^{3k+1}-\Re_2^{3k+1}}{\Re_1-\Re_2}\right) + 3(-1)^k\left(\frac{\Re_1^{k+1}-\Re_2^{k+1}}{\Re_1-\Re_2}\right) + 3(-1)^k\left(\frac{\Re_1^{k}-\Re_2^{k}}{\Re_1-\Re_2}\right)\right\}$$

$$+ \frac{24}{(\Re_1-\Re_2)^2}\left\{\left(\frac{\Re_1^{3k+1}-\Re_2^{3k+1}}{\Re_1-\Re_2}\right) - 2(-1)^k\left(\frac{\Re_1^{k+1}-\Re_2^{k+1}}{\Re_1-\Re_2}\right) + (-1)^k\left(\frac{\Re_1^{k-1}-\Re_2^{k-1}}{\Re_1-\Re_2}\right)\right\}$$

$$= \frac{16}{5}F_{3k+1} + \frac{24}{5}(-1)^k F_{k+2} + \frac{24}{5}F_{3k+1} - \frac{48}{5}(-1)^k F_{k+1} + \frac{24}{5}(-1)^k F_{k-1}$$

$$= 8F_{3k+1} + \frac{24}{5}(-1)^k\left[F_{k+2} - 2F_k - F_{k-1}\right]$$

$$= 8F_{3k+1} + \frac{24}{5}(-1)^k\left[F_{k+1} - F_{k+1}\right]$$

$$= 8F_{3k+1}$$

This completes the proof.

Theorem 3.15. ***Prove that*** $S_{-k} = (-1)^{k+2} S_k$ (3.20)

Proof. *By Binet's formula (2.4), we have*

$$S_{-k} = \frac{2}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{-k-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{-k-1}\right\}$$

$$= \frac{2}{\sqrt{5}}\left\{\left(\frac{2}{1+\sqrt{5}}\right)^{k+1} - \left(\frac{2}{1-\sqrt{5}}\right)^{k+1}\right\}$$

$$= \frac{2}{\sqrt{5}}\left[\left\{-\left(\frac{1-\sqrt{5}}{2}\right)\right\}^{k+1} - \left\{-\left(\frac{1+\sqrt{5}}{2}\right)\right\}^{k+1}\right]$$

$$= \frac{2}{\sqrt{5}}(-1)^{k+2}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+1}\right\}$$

$$S_{-k} = (-1)^{k+2} S_k$$

This completes the proof.

4. Conclusion

In this paper, we have stated and derived identities for Fibonacci-Like sequence. Also described and derived connection formulae and negation formula for Fibonacci-Like sequence.

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