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LIMIT CYCLES FOR PLANAR DIFFERENTIAL SYSTEMS WITH QUASI-HOMOGENEOUS NONLINEARITIES

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Abstract. In this paper we study the estimate for the limit cycles of the planar differential systems of $(p,q)-$quasi-homogeneous polynomials nonlinearities. We give a condition on the polynomials to let the system have at most two limit cycles by introducing a new form of transformation. The tool that we mainly use is generalized polar coordinates and a fact introduced by Gasull and Llibre in estimating the number of limit cycles. This is achieved by estimating the number of isolated periodic solutions of Abel equation.

Keywords: quasi-homogeneous polynomial; limit cycles; planar differential systems.

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1. Introduction

In this paper we study a significant problem in the qualitative study theory of real planar differential systems which is to control the number of limit cycles for a given class of polynomial systems, the quasi-homogeneous polynomial differential systems. This problem is originated from the Hilbert's 16th problem. These kind of systems have been studied from many different points of view. In this paper we restrict our study to the number of limit cycles surrounding the

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origin for the planar system with quasi-homogeneous nonlinearities

$$
\begin{aligned}
\dot{x} &= ax + xP_{n,(p,q)}(x,y) \\
\dot{y} &= ay + yQ_{m,(p,q)}(x,y)
\end{aligned}
\tag{1.1}
$$

where $P_{n,(p,q)}(x,y), Q_{n,(p,q)}(x,y)$ are $(p,q)-$quasi-homogeneous polynomials of weight degree n. The system is called $(p,q)-$quasi-homogeneous of quasi-degree n differential system. Notice that homogeneous polynomials of degree n are quasi-homogeneous of quasi-degree n and weight $(1,1)$.

The quasi-homogeneous (and in general nonhomogeneous) polynomial function is defined as follows,

Definition 1. *Let p,q,n be positive integers. The polynomial $P_{n,(p,q)}(x,y)$ is called a $(p,q)-$quasi-homogeneous polynomial of weight degree n if*

$$P_{n,(p,q)}(\lambda^p x, \lambda^q y) = \lambda^n P_{n,(p,q)}(x,y)$$

for all real number λ. For instance see [15] and references therein.

Notice that $P_{n,(p,q)}(x,y)$ can be written as

$$P_{n,(p,q)}(x,y) = \sum_{ip+jq=n} p_{ij} x^i y^j$$

Or

$$
\begin{aligned}
P_{n,(p,q)}(x,y) &= \sum_{k=1}^{r} p_{i_k j_k} x^{i_k} y^{j_k}, \qquad i_k p + j_k q = n \\
&= (x^{i_1}, \ldots, x^{i_r}) \begin{pmatrix} p_{i_1 j_1} & & & & \\ & . & 0 & & \\ & & . & & \\ & 0 & & . & \\ & & & & p_{i_r j_r} \end{pmatrix} \begin{pmatrix} y^{j_1} \\ . \\ . \\ . \\ y^{j_r} \end{pmatrix}
\end{aligned}
$$

One of the particularities of these kinds of systems is that each limit cycle surrounding the origin can be expressed in polar coordinates as $r = r(\theta)$ with $r(\theta)$ being a smooth periodic function. This provides us an opportunity to consider the Helbert's 16th problem in a simple way.

The used technique to study system (1.1) is by transforming it to Abel equation

$$\frac{dx}{dt} = a_3(t)x^3 + a_2(t)x^2 + a_1(t)x \tag{1.2}$$

where $x \in R$ and $a_i \in C^\infty([0,2\pi])$, $i = 1,2,3$, applying a new form of transformation.

A solution of $x(t)$ of 1.2 is called periodic solution, if it is defined in $[0,2\pi]$ with $x(0) = x(2\pi)$. Moreover, the isolated periodic solution $x = x(t)$ of 1.2 in the strip $[0,2\pi] \times R$ is corresponding to limit cycle for system 1.1. That means the problem of estimating the number of limit cycles of system 1.1 is reduced to the estimating the number of isolated periodic solution of 1.2. Lins-Neto [8] and Lloyd [9-11] proved that equation 1.2 has at most one (resp. two) periodic solution if $a_3 = a_2 \equiv 0$ (resp. $a_3 \equiv 0$). In [5] the authors proved that for the case $a_2 \neq 0$ keeps the sign, the number of non-zero isolated periodic solutions of 1.2 is at most two. Our result is built upon this fact.

The quasi-homogeneous polynomial differential systems have been studied from many different point of view, one of these studies is the centre, see for instance [1], [2], [4]. But up to now there was not an algorithm for constructing all the quasi-homogeneous polynomial differential systems for a given degree. Our result extends the homogeneous case as a particular case.

2. Main Results

We consider a class of differential system given in 1.1. In fact, in a generalized polar coordinates

$$\begin{aligned} x &= r^p \cos\theta \\ y &= r^q \sin\theta \end{aligned} \tag{2.1}$$

where p, q are positive integers $p \neq q \geq 0,\ p, q \leq n$, system 1.1 can be written in the form.

$$\begin{aligned} \dot{r} &= \frac{1}{p\cos^2\theta + q\sin^2\theta}[ar + r^{n+1}\varphi(\theta)] \\ \dot{\theta} &= \frac{1}{p\cos^2\theta + q\sin^2\theta}[a(p-q)\sin\theta\cos\theta + r^n\psi(\theta)] \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} \varphi(\theta) &= \cos^2\theta P(\cos\theta, \sin\theta) + \sin^2\theta Q(\cos\theta, \sin\theta) \\ \psi(\theta) &= -q\sin\theta\cos\theta P(\cos\theta, \sin\theta) + p\sin\theta\cos\theta Q(\cos\theta, \sin\theta) \end{aligned} \tag{2.3}$$

$$\dot{r} = \frac{dr}{dt}, \quad \text{and} \quad \dot{\theta} = \frac{d\theta}{dt} \tag{2.4}$$

The limit cycles surrounding the origin do not intersect the curve $\dot{\theta} = 0$. In other words, the limit cycles surrounding the origin do not intersect the curve

$$a(p-q)\sin\theta\cos\theta + r^n\psi(\theta) = 0$$

Therefore the limit cycles can be investigated by the differential equation

$$\frac{dr}{d\theta} = \frac{ar + r^{n+1}\varphi(\theta)}{a(p-q)\sin\theta\cos\theta + r^n\psi(\theta)} \tag{2.5}$$

Furthermore, we introduce a new transformation

$$\rho = \frac{r^n}{a(p-q)\sin\theta\cos\theta + r^n\psi(\theta)}, \tag{2.6}$$

This transformation is a modified transformation of a transformation introduced by [5]. The modification is made according to our need. Equation 2.5 becomes an Abel equation

$$\frac{d\rho}{d\theta} = a_3(\theta)\rho^3 + a_2(\theta)\rho^2 + a_1(\theta)\rho \tag{2.7}$$

where

$$\begin{aligned} a_3(\theta) &= \frac{n\psi^2(\theta)}{(p-q)\sin\theta\cos\theta} - n\varphi(\theta)\psi(\theta)\ldots\ldots\ldots\ldots\ldots\ldots \\ a_2(\theta) &= \frac{-2n\psi}{(p-q)\sin\theta\cos\theta} + \frac{\cos^2\theta - \sin^2\theta}{\sin\theta\cos\theta}\psi(\theta) + n\varphi(\theta) - \psi'(\theta) \\ a_1(\theta) &= \frac{n}{(p-q)\sin\theta\cos\theta} - \frac{\cos^2\theta - \sin^2\theta}{\sin\theta\cos\theta}\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned} \tag{2.8}$$

Here we can state our main result

Theorem 1. *If*

$$[\frac{2nq}{p-q}+c\cos^2\theta]P(\cos\theta,\sin\theta)+[\frac{-2np}{p-q}+c\sin^2\theta]Q(\cos\theta,\sin\theta)$$

$$+\sin\theta\cos\theta[qP'(\cos\theta,\sin\theta)-pQ'(\cos\theta,\sin\theta)]\ \neq 0$$

in the domain of definition $[0,2\pi]$*, then the system 1.1 has at most one limit cycle surrounding the origin.*

Proof. We have from 2.8

$$\begin{aligned}
a_2(\theta) &= \frac{-2n\psi}{(p-q)\sin\theta\cos\theta}+\frac{\cos^2\theta-\sin^2\theta}{\sin\theta\cos\theta}\psi(\theta)+n\varphi(\theta)-\psi'(\theta)\\
&= \frac{-2n+(p-q)(\cos^2\theta-\sin^2\theta)}{(p-q)\sin\theta\cos\theta}\psi(\theta)+n\varphi(\theta)-\psi'(\theta)\\
&= \frac{-2n+(p-q)(\cos^2\theta-\sin^2\theta)}{(p-q)}[-qP(\cos\theta,\sin\theta)+pQ(\cos\theta,\sin\theta)]\\
&\quad +n[\cos^2\theta P(\cos\theta,\sin\theta)+\sin^2\theta Q'(\cos\theta,\sin\theta)]-\psi'(\theta)
\end{aligned}$$

From 2.3, we get

$$\begin{aligned}
\psi'(\theta) &= -q(\cos^2\theta-\sin^2\theta)P(\cos\theta,\sin\theta)-q\sin\theta\cos\theta P'(\cos\theta,\sin\theta)\\
&\quad +p(\cos^2\theta-\sin^2\theta)Q(\cos\theta,\sin\theta)+p\sin\theta\cos\theta Q'(\cos\theta,\sin\theta)
\end{aligned}$$

Therefore, after simplification we obtain

$$\begin{aligned}
a_2(\theta) &= [\frac{2nq}{p-q}+c\cos^2\theta]P(\cos\theta,\sin\theta)+[\frac{-2np}{p-q}+c\sin^2\theta]Q(\cos\theta,\sin\theta)\\
&\quad +\sin\theta\cos\theta[qP'(\cos\theta,\sin\theta)-pQ'(\cos\theta,\sin\theta)]
\end{aligned}$$

Now we apply the following fact proved by Gasull and Llibre [7] which say, " *If* $a_2(\theta)\neq 0$, *in* $[0,2\pi]$*, then the system 1.2 has at most one isolated periodic solution".*

This proves the Theorem. ∎

Conclusion 1. *The validity of the condition in the Theorem 1 is equivalent to the existence of the solution of the first order differential equation in* $Q(\cos\theta, \sin\theta)$

$$\begin{aligned}&[\frac{2nq}{p-q} + c\cos^2\theta]P(\cos\theta, \sin\theta) + [\frac{-2np}{p-nq} + c\sin^2\theta]Q(\cos\theta, \sin\theta)\\&+\sin\theta\cos\theta[q\acute{P}(\cos\theta, \sin\theta) - p\acute{Q}(\cos\theta, \sin\theta)] = 0\end{aligned}$$

regarding that $P(\cos\theta, \sin\theta)$ *given.*

Secondly, The most classical hypothesis on which many results depend is the definite sign hypothesis for the coefficients of 2.7, see for instance, [7, 8, 11, 14]. Generalized Abel equations with some coefficients of definite signs are also investigated in several papers, see [3,6,12, 13]. Recently, [4] requires fixed sign hypotheses for some linear combinations of the coefficients. The new results of this paper, on the top of Theorem 1, are the approach and introducing a new generalized transformation, and the polynomial is quasi-homogeneous.

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Conflict of Interests

The authors declare that there is no conflict of interests.

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