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EXACT NULL CONTROLLABILITY OF SEMILINEAR INTEGRO-DIFFERENTIAL SYSTEMS

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Abstract. In this work, we establish a sufficient result for Exact null controllability of semilinear integro-differential system with non-autonomous functional evolution system. The results are obtained by using the Ascoli-Arzela theorem and Schauder fixed point theorem. An example is also provided to show an application of the obtained result.

Keywords: exact null controllability; resolvent operator; semilinear integro-differential systems; Schauder fixed point theorem.

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1. Introduction

Controllability of linear and non-linear systems represented by ordinary differential equations in finite dimensional space has been extensively studied. Several authors have extended the concept to infinite dimensional systems represented by the evolution equations with bounded operators in Banach spaces [1, 2, 3, 4]. The study of controllability results for such systems in infinite dimensional space is important. For the motivation of abstract systems and the null controllability of linear systems, one can refer to the book by Curtain and Pritchard [5] and by

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Curtain and Zwart [6]. For an earlier Survey on the controllability of nonlinear systems using fixed point theorems, including nonlinear delay systems.

We Consider the semilinear integrodifferential system with non-autonomous functional,

$$\begin{aligned} z'(s) &= Az(s)+Bv(s)+F_1(s,z_s)+\int_0^s F_2(\tau,z_\tau);\ s\in[0,T]=J \\ z_0(\xi) &= \phi(\xi);\ \xi\in[-\mu,0]=W.\ (1.1) \end{aligned}$$

where A is the infintesimal generator of a strongly continuous semigroup $S(s)$ in a Hilbert space Y, B is a linear bounded operator from a Hilbert space U into Y, $F_1 : J\times Y\to Y$, $F_2 : J\times C(W,Y)\to Y$, the control function $v(.)$ is given in $L_2(J,U)$. Here $C(W,Y)$ is the Banach space of all continuous functions $\phi : W\to Y$ endowed with the norm $\|\phi\| = sup\{\|\phi(\xi)\| : -r\le\xi\le 0\}$. Also for $z(.)\in C([-\mu,T]=L,Y)$ we have $z_s(.)\in C(W,Y)$ for $s\in J$, $z_s(\xi)=z(s+\xi)$ for $\xi\in W$.

This organization of this work is as follows. In section 2, we give some useful results on the integrated semigroups and we recall briefly some basic definitions and preliminary facts which will be used throughout this work. In section 3, we establish sufficient conditions for the exact null controllability of mild solutions of equation(1.1) by relying on a fixed point theorem due to Schauder. Last section is devoted to an application.

2. Preliminaries

Definition 2.1 : The linear control system $\dot{z} = A(s)z+B(s)v$ is said to be exact null controllable on the interval J, if for every ϕ and preassigned time b there exists a control $u(t)\in L_2([0,T],U)$ such that the corresponding solution $x(.)$ satisfies $x(b)=0$.

We prove the exact null controllability of mild solution of the above integro-differential equation (1.1). Before proving the exact null controllability, we define the mild solution of the integro-differential equation (1.1).

Let $T(s)$ be the C_0- semigroup generated by A and let z be a solution of (1.1) on J.

$$z(s) = S(s)\phi(0)+\int_0^s S(s-t)Bv(t)dt+\int_0^s S(s-t)F_1(t,z_t)dt+\int_0^s\int_0^t S(s-t)F_2(\tau,z_\tau)d\tau dt$$

$$z_0(\xi) \;=\; \phi(\xi),\ \xi \in W\ (2.1)$$

To prove our result on the exact null controllability of solutions we introduce the following assumptions:

The main assumptions created during this paper were that the semigroup $S(s), s > 0$ associated with the linear part of the functional equation is compact which the linear convolution operator $L_0^T v \;=\; \int_0^T S(s-t)Bv(t)dt$ includes a bounded inverse operator L_0^{-1} with values in $L^2(J,U)/ker(L_0^T$.

We begin with the subsequent assumptions,

$A_1)$ Y,U are Hilbert spaces

$A_2)$ $A : D(A) \subset Y \to Y$ generates a compact semigroup $S(s), s > 0$ on Y.

$A_3)$ The function $F_2 : J \times C(W,Y) \to Y$ is continuous and there exist functions $\lambda(.) \in L_1(J,R^+)$ and $g(.) \in L_1(C(W,Y),R^+)$ be such that

$$\|F_2(s,\phi)\| \;\leq\; \lambda(s)g(\phi)\ \forall\ (s,\phi) \in J \times C(W,Y)$$

Next, for convenience, allow us to introduce the following notation,

$K = Max\{\|S(s)\| : s \in J\}$, $\|B\| = M$, $\|H\| = \ell$,

$\|\lambda\| = \int_0^T \lambda(s)ds$, $P = Max(1, \ell, MK\sqrt{T})$, $a_1 = 4P\ell\sqrt{T}\|\lambda\|$,

$a_2 = 2KT\|\lambda\|$, $a_3 = 2P\ell\sqrt{T}\varepsilon^2$, $a_4 = -2KT\varepsilon$,

$C = Max(a_1,a_2,a_3,a_4)$, $d_1 = 2P\ell|\phi(0)|$,

$d_2 = 2K|\phi(0)|$, $d = Max(d_1,d_2)$.

$A_4)$ $lim\ Sup_{\mu\to+\infty}(\mu - C\ Sup\{g(\phi); |\phi| \leq \mu\}) \;=\; \infty$.

A_5) The linear system

$$\begin{aligned} X'(s) &= AX(s)+Bv(s)+f(s) \\ X(0) &= X_0 \ (2.2) \end{aligned}$$

is exact null controllable on J

A_6) Since for any $s \in J$, the function $F_1(s,.) : Y \to Y$ is continuous and for any given $z \in Y$, the function $F_1(.,z) : J \to Y$ is strongly measurable.

Moreover, for any $q > 0$, there is a function $f_q(s) \in L^2(J;R^+)$ such that

$$Sup_{z\in s_q}|F_1(s,z)| \leq f_q(s) \text{ for a.e } s \in J$$

and

$$lim \inf_{r\to+\infty} \|f_q(s)\|_{L^2} = \varepsilon < \infty$$

where $S_q = z \in C(J,Y), \|z(.)\| \leq q$

A_7) The function $g : C(J,Y) \to Y$ is continuous satisfying $|g(z)| \leq L(|z|)$ for a constant $L > 0$ and any $z \in C(J,Y)$, and there is a $\delta = \delta(\mu) \in (0,T)$ such that

$$g(v) = g(w) \text{ with } v(s) = w(s),\ s \in [\delta,T].$$

Define

$$\begin{aligned} L_0^T v &= \int_0^T S(T-s)Bv(t)dt \ L_2(J,U) \to Y \\ N_0^T(x,k) &= S(T)x + \int_0^T S(T-t)k(t)dt \ X \times L_2(J,U) \to Y. \end{aligned}$$

Then we have the following definition :

Definition 2.2 : The system (2.2) is said to be exactly null controllable on J if

$$ImL_0^T \supset ImN_0^T.$$

Remark 2.1 : It can be proved as in [9] that system (2.2) is exact null controllable if and only if there is a positive number γ such that

$$|(L_0^T)^* x| \geq \gamma |(N_0^T)^* x| \; \forall x \in Y$$

The following result plays crucial role in our discussion :

Lemma 2.1 : Suppose that linear system (2.2) is exact null controllable, then the linear operator $H := (L_0)^{-1}(N_0^T) : Y \times L^2(J;Y) \to L^2(J;U)$ is bounded and the control is of the form

$$\begin{aligned} v(s) : &= -(L_0)^{-1}(N_0^T(x,f)) = -H(x_0,f) \\ &= -(L_0)^{-1}\left[S(T)x_0 + \int_0^T S(T-t)\left(f_1(t) + \int_0^t f_2(\tau)\right)\right]; t \in J \; (2.3) \end{aligned}$$

Define the operator $\mathscr{F}$ on $C(L,X)$ as follows;

$$\mathscr{F}(z)(s) = \begin{cases} \phi(s); \; s \in W \\ S(s)\phi(0) + \int_0^s S(s-t)\Big[-BH(\phi(0),F_2) + F_1(t,z_t) \\ + \int_0^t F_2(\tau, z_\tau)d\tau\Big] dt; \; s \in J \; (2.4) \end{cases}$$

where

$$H(\phi(0),F) \; = \; (L_0)^{-1}\left[S(T)x_0 + \int_0^T S(T-t)\left(F_1(t,z_t) + \int_0^t F_2(\tau,z_\tau)d\tau\right)dt\right]$$

It will be shown that the operator $\mathscr{F}$ from $C(L,Y)$ into itself has a fixed point.

On Banach space $C(L,Y)$ introduce a set,

$$\begin{aligned} X_\mu = \{y(.) \in C(L,Y) \text{ such that } z(s) \; &= \; \phi(s), \; s \in W \\ &\text{and } |z(s)| \leq \mu \; \forall \; s \in L\} \end{aligned}$$

where μ is the positive constant.

3. Main results

In this section, we study the exact null controllability of mild solution of (1.1).

Theorem 3.1 : Assume $(A_1)-(A_7)$ satisfied. Then the system (2.1) is exact null controllable.

Proof. Let $\psi(\mu) = sup\{g(\phi) : |\phi| \leq \mu\}$

By the assumption (A_4), *there exist* $\mu > 0$ such that $d + c\psi(\mu) \leq \mu$. The proof are going to be given in several steps.

Step-1 :

$$
\begin{aligned}
\|v\| &= \left(\int_0^T \|H(\phi(0), F)(t)\|^2 dt \right)^{\frac{1}{2}} \\
&\leq \|H\| \left(\|\phi(0)\| + \left[\int_0^T \left(\|F_1(t, z_t) + \int_0^t F_2(\tau, z_\tau) d\tau\| \right)^2 dt \right]^{\frac{1}{2}} \right) \\
&\leq \ell \Bigg[\|\phi(0)\| + \Bigg(\int_0^T \|F_1(t, z_t)\|^2 + \int_0^T \left(\int_0^t \|F_2(\tau, z_\tau) d\tau\| \right)^2 \\
&\quad + 2 \int_0^T (\|F_1(t, z_t)\| \|F_2(\tau, z_\tau) d\tau\|)^2 \Bigg)^{\frac{1}{2}} dt \Bigg] \\
&\leq \ell \Bigg[\|\phi(0)\| + \Bigg(\int_0^T \varepsilon^2 + \int_0^T \left(\int_0^s \lambda(\tau) g(z_\tau) d\tau \right)^2 \\
&\quad + 2 \int_0^T \varepsilon^2 \left(\int_0^t \lambda(\tau) g(z_\tau) d\tau \right)^2 \Bigg)^{\frac{1}{2}} \Bigg] \\
&\leq \ell \left[\|\phi(0)\| + \sqrt{T} \left(\varepsilon^2 + \|\lambda\| \|\psi(\mu)\| + 2\varepsilon^2 \|\lambda\| \|\psi(\mu)\| \right) \right] \quad (3.1)
\end{aligned}
$$

Step-2 : There exist $\mu > 0$ such $\mathscr{F}$ sends X_μ into itself , $\mathscr{F} : X_\mu \to X_\mu$.

If $z(.) \in X_\mu$, from equation(2.4) and equation(3.1) for $s \in J$, we have,

$$
\begin{aligned}
\|(\mathscr{F}z)(s)\| \leq \|S(s)\phi(0)\| &+ \left\{ \| \int_0^T S(s-t)[-BH(\phi(0), F_2)(t)]\|^2 dt \right\}^{\frac{1}{2}} \\
&+ \int_0^s \|S(s-t)F_1(t, z_t)\| dt + \int_0^s \int_0^t \|S(s-t)F_2(\tau, z_\tau) d\tau\| dt
\end{aligned}
$$

$$
\begin{aligned}
&\le K\|\phi(0)\| + KM\sqrt{T}\Bigg[\ell\Bigg\{\|\phi(0)\| + \sqrt{T}\Bigg(\varepsilon^2 + \|\lambda\|\|\psi(\mu)\| \\
&+ 2\varepsilon^2\|\lambda\|\|\psi(\mu)\|\Bigg)\Bigg\}\Bigg] + KT\varepsilon + KT\|\lambda\|\psi(\mu) \\
&\le \frac{d}{2} + P\ell\Bigg[\|\phi(0)\| + \sqrt{T}\Bigg(\varepsilon^2 + \|\lambda\|\|\psi(\mu)\| + 2\varepsilon^2\|\lambda\|\|\psi(\mu)\|\Bigg)\Bigg] \\
&+ KT\varepsilon + KT\|\lambda\|\psi(r) \\
&\le \frac{d}{2} + P\ell\|\phi(0)\| + P\ell\sqrt{T}\varepsilon^2 + P\ell\sqrt{T}\|\lambda\|\|\psi(\mu)\| \\
&+ 2P\ell\sqrt{T}\varepsilon^2\|\lambda\|\|\psi(\mu)\| + KT\varepsilon + KT\|\lambda\|\|\psi(\mu)\| \\
&\le \frac{d}{2} + \frac{d}{2} + \frac{c}{2} + \frac{c}{4}\psi(\mu) + \frac{c}{4}\psi(\mu) - \frac{c}{2} + \frac{c}{2}\psi(\mu) \\
&\le \frac{1}{2}(d + c\psi(\mu)) + \frac{1}{2}(d + c\psi(\mu)) \\
&\le \frac{\mu}{2} + \frac{\mu}{2} \;=\; \mu.
\end{aligned}
$$

Hence $\mathscr{F}$ maps X_μ into itself.

Step-3 :

The operator $\mathscr{F}$ maps X_μ into equicontinuous set of $C(L,Y)$.

Let $0 < s_1 < s_2 \le T$, for each $z \in X_\mu$, let $\mathscr{F}(z)(s) = X(s)$. Then

$$
\begin{aligned}
X(s_1) - X(s_2) \;=\;& S(s_1)\phi(0) + \int_0^{s_1} S(s_1 - t)[-BH(\phi(0), F_2) + F_1(t, z_t) \\
&+ \int_0^t F_2(\tau, z_\tau)d\tau]dt - \Bigg[S(s_2)\phi(0) + \int_0^{s_2} S(s_2 - t) \\
&[-BH(\phi(0), F_2) + F_1(t, z_t) + \int_0^t F_2(\tau, z_\tau)d\tau]dt\Bigg] \\
=\;& [S(s_1)\phi(0) - S(s_2)\phi(0)] + \int_0^{s_1} S(s_1 - t)[-BH(\phi(0), F_2) \\
&+ F_1(t, z_t) + \int_0^t F_2(\tau, z_\tau)d\tau]dt - \int_0^{s_1} S(s_2 - t) \\
&[-BH(\phi(0), F_2) + F_1(t, z_t) + \int_0^t F_2(\tau, z_\tau)d\tau]dt \\
&- \int_{s_1}^{s_2} S(s_2 - t)[-BH(\phi(0), F_2) + F_1(t, z_t) \\
&+ \int_0^t F_2(\tau, z_\tau)d\tau]dt
\end{aligned}
$$

$$
\begin{aligned}
&= [S(s_1)-S(s_2)]\phi(0) - \int_{s_1}^{s_2} S(s_2-t)BH(\phi(0),F_2)(t)dt \\
&+ \int_0^{s_1} [S(s_1-t)-S(t_2-t)]BH(\phi(0),F_2)(t)dt \\
&- \int_{s_1}^{s2} S(s_2-t)F_1(t,z_t)dt + \int_0^{s_1} [S(s_1-t)-S(s_2-t)] \\
&F_1(t,z_t)dt - \int_{s_1}^{s_2} S(s_2-t)\int_0^t F_2(\tau,z_\tau)d\tau dt \\
&+ \int_0^{s_1} [S(s_1-t)-S(s_2-t)]\int_0^t F_2(\tau,z_\tau)d\tau dt \\
\Big\|X(s_1)-X(s_2)\Big\| &= \Big\|[S(s_1)-S(s_2)]\phi(0) - \int_{s_1}^{s_2} S(s_2-s)BH(\phi(0),F_2)(t)dt \\
&+ \int_0^{s_1} [S(s_1-t)-S(s_2-t)]BH(\phi(0),F_2)(t)dt \\
&- \int_{s_1}^{s_2} S(s_2-t)F_1(t,z_t)dt + \int_0^{s_1} [S(s_1-t)-S(s_2-t)] \\
&F_1(t,z_t)dt - \int_{s_1}^{s_2} S(s_2-t)\int_0^t F_2(\tau,z_\tau)d\tau dt \\
&+ \int_0^{s_1} [S(s_1-t)-S(s_2-t)]\int_0^t F_2(\tau,z_\tau)d\tau dt\Big\| \\
&\leq \|S(s_1)-S(s_2)\|\|\phi(0)\| + KM\int_{s_1}^{s_2} \|H(\phi(0),F_2)(t)\|dt \\
&+ M\int_0^{s_1} \|S(s_1-t)-S(s_2-t)\|\|H(\phi(0),F_2)(t)\|dt \\
&+ K\varepsilon(s_2-s_1) + \int_0^{s_1} \|S(s_1-t)-S(s_2-t)\|\varepsilon \\
&+ \int_0^{s_1} \|S(s_1-t)-S(s_2-t)\| \int_0^t \lambda(\tau)\phi(z_\tau)d\tau dt \\
&+ K\int_{s_1}^{s_2}\int_0^t \lambda(\tau)\phi(z_\tau)d\tau dt \\
&= J_1+J_2+J_3+J_4+J_5+J_6+J_7 \ (3.2)
\end{aligned}
$$

Since by equation (3.1) the control $'v'$ is bounded, the RHS of equation (3.2) doesn't depend on specific choices of $x(.)$. It is clear that, $J_2 \to 0$ and $J_7 \to 0$ as $(s_1-s_2) \to 0$.

Since the semigroup $S(.)$ is compact , $\|S(s_1-t)-S(s_2-t)\| \to 0$ as $(s_1-s_2) \to 0$ for arbitrary t,s such that $(s-t) > 0$. As $s_1 \to s_2$, J_4 and J_5 are tends to zero. Then $J_1 \to 0$ and by

Lebesgue dominated convergence theorem, $J_3 \to 0$ and $J_6 \to 0$ as $(s_1 - s_2) \to 0$. As $(s_1 - s_2) \to 0$, the RHS of equation (3.2) tends to zero.

The equicontinuity for the cases $s_1 < s_2 \leq 0$ and $s_1 \leq 0 \leq s_2$ follows from the uniform continuity of ϕ on W.

Step-4 :

For arbitrary $s \in [0,T]$ the set $V(s) = \{(\mathscr{F}z)(s) \ / \ z(.) \in X_\mu\}$ is relatively compact.

In fact, the case where $s = 0$ is trivial. Since $V(0) = \Phi(0)$.

So let s $(0 < s \leq T)$ be a fixed and let σ be a real number satisfying $0 < \sigma < s$.

For every $z(.) \in X_\mu$ define,

$$\begin{aligned}\mathscr{F}_\sigma(z)(s) \ &= \ S(s)\phi(0) + S(\sigma)\int_0^{s-\sigma} S(s-t-\sigma)\Big[-BH(\phi(0),F_2) + F_1(t,z_t) \\ &\qquad + \int_0^t F_2(\tau,z_\tau)d\tau\Big]dt\end{aligned}$$

Since $S(\sigma)$ is compact, the set

$$V_\sigma(s) \ = \ \{\mathscr{F}_\sigma(z)(s)/z(.) \in X_\mu\}$$

is relatively compact set in Y for every σ, $0 < \sigma < s$.

On the otherhand, for every $z(.) \in X_\mu$ by (1) we have,

$$\begin{aligned}\Big\|\mathscr{F}(z)(s) - \mathscr{F}_\sigma(z)(s)\Big\| &= \Big\|\int_{s-\sigma}^s S(s-t)\Big[BH(\phi(0),F_2) + F_1(t,z_t) \\ &\quad + \int_0^t F_2(\tau,z_\tau)d\tau\Big]dt\Big\| \\ &\leq \left(\int_{s-\sigma}^s \|S(s-t)\|^2\|B\|^2 dt\right)^{\frac{1}{2}}\left(\int_0^T \|H(\phi(0),F_2)(t)\|^2 dt\right)^{\frac{1}{2}} \\ &\quad + \int_{s-\sigma}^s \|S(s-t)\|\|F_1(t,z_t)\|dt \\ &\quad + \int_{s-\sigma}^s \|S(s-t)\|\int_0^t \|F_2(\tau,z_\tau)d\tau\|dt \\ &\leq KM\ell\Big(\|\phi(0)\| + \sqrt{T}\varepsilon^2 + \sqrt{T}\|\lambda\|\|\psi(\mu)\| \\ &\quad + 2\sqrt{T}\varepsilon^2\|\lambda\|\|\psi(\mu)\|\Big)\sqrt{\sigma} + K\varepsilon\sigma + K\|\lambda\|\psi(\mu)\sigma \\ &< \ \delta\end{aligned}$$

Therefore there are relatively compact sets arbitrarily close to the set $V(s)$. Hence for each $s \in [0,T]$, $V(s)$ is relatively compact in Y.

From steps [2-4] thanks to the Ascoli-Arzela theorm, one can conclude that $\mathscr{F}$ is compact. On the otherhand, it's determine that $\mathscr{F}$ is continuous on $C(L,Y)$.

From the Schauder fixed point theorm $\mathscr{F}$ has a fixed point. □

Corollary

If $F(.,.)$ is continuous and maps bounded sets in $[0,T] \times C([-r,0],X)$ into bounded sets in $C([-r,0],X)$ and

$$\lim_{\|\phi\| \to \infty} \frac{\|F(t,\phi)\|}{\|\phi\|} = 0$$

uniformly in t, then under the assumptions (A_2) and (A_5), the system (3.1) is exactly null controllable on $[0,T]$.

proof

Let $g(\phi) = sup\|F(t,\phi)\| : t \in [0,T]$. Then

$$\|F(t,\phi)\| \leq g(\phi) \ \forall \ (t,\phi) \in [0,T] \times C([-r,0],X).$$

It is suffitient to show that the assumption (A_4) holds. For a contradiction, suppose that it is not the case. Then the function

$$r - c \ sup\{g(\phi) : \|\phi\| \leq r\}$$

, is bounded from above with respect to r . From here it follows that

$$\liminf_{r \to \infty} \frac{sup[g(\phi) : \|\phi\| \leq r]}{r} \ \geq \ \frac{1}{c}.$$

Then, for some $\varepsilon > 0$, there exists a sequence r_n, $r_n \to \infty$ as $n \to \infty$ such that for all $n \geq 1$,

$$\frac{1}{r_n} sup[g(\phi) : \|\phi\| \leq r_n] > \varepsilon.$$

Hence, there further exists $\{\phi_n\}$, $\|\phi_n\| \leq r_n$, such that

$$\frac{g(\phi_n}{r_n} > \varepsilon for\ all\ n \geq 1$$

Next, we will show that $\{\phi_n\}$ is necessarily unbounded. Suppose that this is not true, that is, $\{\phi_n\}$ is bounded. Since F is continuous, we have

$$g(\phi_n) = \|F(t_n, \phi_n)\|\ for\ some\ t_n \in [0, T].$$

From boundedness of $F(.,.)$ it follows that $g(\phi_n)$ is bounded. This is a contradiction. We conclude that $\{x_n\}$ is unbounded. As a result, we can choose a subsequence $\{\phi_m\}$ of $\{\phi_n\}$ such that

$$\|\phi_m\| \to \infty$$

For this subsequence , we have

$$\lim_{m\to\infty} \frac{\|F(t_m, \phi_m)\|}{\|\phi_m\|} = \lim_{m\to\infty} \frac{g(\phi_m}{\|\phi_m\|} \geq \lim_{m\to\infty} \frac{g(\phi_m}{r_m} > \varepsilon > 0.$$

This however , contradicts the hypothesis of the corollary. Thus condition (A_4) holds and so,(3.1) is exactly null controllable on $[0, T]$.

4. APPLICATION

In this chapter, we consider the application of the main result.

Consider the integro-differential system of the form,

$$\frac{\partial}{\partial\tau} y(\tau, \eta) = \frac{\partial^2}{\partial\eta^2} y(\tau, \eta) + v(\tau, \eta) + h(\eta, y(\tau, \mu(\eta))) + \int_0^s F_2(s, y(s-k, \eta))ds$$

$$y_0(\tau, 0) = y_0(\tau, 1) = 0,\ t > 0$$

$$y(\tau, \eta) = \phi(\tau, \eta),\ -k \leq \tau \leq 0. \ (4.1)$$

where ϕ is continuous and $v \in L_2(0,T)$,$Y = L_2(0,1)$,$b \in Y$ and where $F_2 : R \times R \to R$ is continuous.

Let $A : Y \to Y$ be operator defined by $Ax = \frac{d^2x}{d\eta^2}$ with domain

$$D(A) = \left\{ x \in Y/x, \frac{dx}{d\eta} \text{ are absolutely continuous }, \frac{d^2x}{d\eta^2} \in Y, \ \frac{dx}{d\eta}(0) = \frac{dx}{d\eta}(1) = 0 \right\}$$

It is known that A is closed and A has the eigenvalue $\lambda_n = -n^2\pi^2, n \geq 0$ and the corresponding eigenvectors $e_n(\eta) = \sqrt{2}cos(n\pi\eta)$ for $n \geq 1$, $e_0 = 1$ for an orthonormal basis for $L_2(0,1)$.

Further , it is known that A generates a compact semigroup $S(s), \ s > 0$ in Y and is given by

$$\begin{aligned} S(\tau)x = & \ (x,1) + \sum_{n=1}^{\infty} e^{-n^2\pi^2 s}(x,e_n)e_n \\ = & \int_0^1 x(\alpha)d\alpha + \sum_{n=1}^{\infty} 2e^{-n^2\pi^2 t} cos(n\pi\eta) \int_0^1 cos(n\pi\alpha)x(\alpha)d\alpha, \ x \in Y \end{aligned}$$

and it is self adjoint.

To write system equation (4.1) in the form (1), we define $F_1 : J \times Y \to Y$, $g : C(J,Y) \to Y$ as follows $F_1(\eta,\tau) = h(\eta, y(\tau,\eta))$, $g(y(\tau,\eta)) = \sum_{i=1}^{p} c_i y(\tau,\eta_i)$

Then $g(.)$ clearly satisfies condition (A_6) because we may choose δ, τ_1. For function $h(.,.)$: $J \times R \to R$, we assume that

i) for any fixed $\tau \in R$, $h(.,\tau)$ is measurable on J.

ii) for any fixed $\eta \in J$, $h(\eta,.)$ is continuous and there is a constant $c \geq 0$ such that $|h(\eta,\tau)| \leq c|\tau| \ \forall \ \eta \in J$

under these assumptions, the function $F_1(\eta,\tau)$ verifies (A_6) as well.

If $v \in L_2(J,Y)$ then $B = I$ and $B^* = I$ and consequently by Remark (1) the condition for exact null controllability of the linear system with additive term $f \in L_2(J,Y)$,

$$\begin{aligned} \frac{\partial}{\partial\tau} y(\tau,\eta) = & \ \frac{\partial^2}{\partial\eta^2} y(\tau,\eta) + v(\tau,\eta) + F_1(\tau,\eta) + F_2(\tau,\eta) \\ \frac{\partial}{\partial\eta} y(\tau,0) = & \ \frac{\partial}{\partial\eta} y(\tau,1) = 0, \ \tau > 0 \\ y(\tau,\eta) \ = & \ \phi(\tau,\eta), \ -k \leq \tau \leq 0. \ (4.2) \end{aligned}$$

Due to Remark (2.1),the exact null controllability of equation(4.2) is corresponding to that there is a $\gamma > 0$, specified

$$\int_0^T \|B^*S^*(T-s)x\|^2 ds \geq \gamma\left(\|S^*(T)x\|^2 + \int_0^T \|S^*(T-s)x\|^2 ds\right)$$

or equivalently

$$\int_0^T \|S(T-s)x\|^2 ds \geq \gamma\left(\|S(T)x\|^2 + \int_0^T \|S(T-s)x\|^2 ds\right)$$

In [**?**], it is shown that the linear system equation(4.2) with $F_1 = 0$, $F_2 = 0$ is exact null controllable if

$$\int_0^T \|S(T-s)x\|^2 ds \geq T\|S(T)x\|^2$$

From here it follows that,

$$\int_0^T \|S(T-s)x\|^2 ds \geq \frac{T}{1+T}\left(\|S(T)x\|^2 + \int_0^T \|S(T-s)x\|^2 ds\right)$$

Thus by Remark (2.1) the linear system equation(4.2) is exact null controllable.

We assume that the nonlinear operator $F_2 : J \times Y \to Y$ is continuous and there is a constant $0 < \gamma < 1$ and a function $k \in L_2(J)$ such that

$$\|F_2(s,x)\| \ \leq \ k(s)\|x\|^\gamma$$

for all $(s,x) \in J \times Y$. So the conditions (A_3) and (A_4) are satisfied.

Thus all the conditions stated in main theorem are satisfied. Hence the system equation(4.1) is exact null controllable on J.

Conflict of Interests

The authors declare that there is no conflict of interests.

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