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CONCIRCULAR CURVATURE TENSOR OF KENMOTSU MANIFOLDS ADMITTING GENERALIZED TANAKA-WEBSTER CONNECTION

D. L. KIRAN KUMAR*, H. G. NAGARAJA, DIPANSHA KUMARI

Department of Mathematics, Bangalore University, Jnana Bharathi Campus, Bengaluru - 560 056, INDIA

Abstract. The objective of the present paper is to study concircular curvature tensor of Kenmotsu manifold with respect to generalized Tanaka-Webster connection, whose concircular curvature tensor satisifies certain conditions and it is shown that if the curvature tensor of a Kenmotsu manifold admitting generalized Tanaka-Webster connection ∇^* vanishes, then the Kenmotsu manifold is locally isometric to the hyperbolic space $H^{2n+1}(-1)$. Further we have studied ξ-concircularly flat, ϕ-concircularly flat, pseudo-concircularly flat, $C^*.\phi = 0$, $C^*.S^* = 0$ and we have shown that $R^*.C^* = R^*.R^*$. Finally, an example of a 5-dimensional Kenmotsu manifold with respect to the generalized Tanaka-Webster connection is given to verify our result.

Keywords: Kenmotsu manifolds; generalized Tanaka-Webster connection; concircular curvature tensor; ξ-concircularly flat; ϕ-concircularly flat; pseudo-concircularly flat.

2010 AMS Subject Classification: 53D10, 53D15.

1. Introduction

The Tanaka-Webster connection is canonical affine connection defined on a non-degenerate pseudo-Hermition CR-manifold [18, 21]. The generalized Tanaka-Webster connection for contact metric manifolds by the canonical connection was first studied by Tanno [19]. This connection coincides with the Tanaka-Webster connection if the associated CR-structure is integrable.

*Corresponding author
E-mail address: kirankumar250791@gmail.com
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For a real hypersurface in a Kahler manifold with almost contact structure (ϕ,ξ,η,g), Cho [4, 5] adapted Tanno's generalized Tanaka-Webster connection for a non-zero real number k. Using the generalized Tanaka-Webster connection, some geometers have studied some characterizations of real hypersurfaces in complex space forms [17]. Kenmotsu manifolds introduced by Kenmotsu in 1971[10]. Kenmotsu manifolds have been studied by various others such as Ozgur [14], yildz et al [25], Hui et al [8, 9], Nagaraja et al [11, 12, 13] and many others [2, 22]. Recently many authors[15, 7, 16] have been studied generalized Tanaka-Webster connection in Kenmotsu manifolds.

The present paper is organized as follows: After a brief review of Kenmotsu manifolds in section 2, we study concircular curvature tensor of Kenmotsu manifold with generalized Tanaka-Webster connection and prove that if the curvature tensor of a Kenmotsu manifold admitting generalized Tanaka-Webster connection ∇^* vanishes, then the Kenmotsu manifold is locally isometric to the hyperbolic space $H^{2n+1}(-1)$. Next, we study ξ-concircularly flat, ϕ-concircularly flat, pseudo-concircularly flat, $C^*.\phi=0$ and $C^*.S^*=0$ with respect to generalized Tanaka-Webster connection. Then we have proved $R^*.C^*=R^*.R^*$. Finally, in the last section we give an example of a 5-dimensional Kenmotsu manifold admitting generalized Tanaka-Webster connection to verify our results.

2. PRELIMINARIES

A $(2n+1)$-dimensional smooth manifold M is said to be an almost contact metric manifold if it admits an almost contact metric structure (ϕ,ξ,η,g) consisting of a tensor field ϕ of type $(1,1)$, a vector field ξ, a 1-form η and a Riemannian metric g compatible with (ϕ,ξ,η) satisfying

$$\phi^2X=-X+\eta(X)\xi,\ \phi\xi=0,\ g(X,\xi)=\eta(X),\ \eta(\xi)=1,\eta\circ\phi=0 \tag{1}$$

and

$$g(\phi X,\phi Y)=g(X,Y)-\eta(X)\eta(Y). \tag{2}$$

An almost contact metric manifold is said to be a Kenmotsu manifold [3] if

$$(\nabla_X\phi)Y=-g(X,\phi Y)\xi-\eta(Y)\phi X, \tag{3}$$

where ∇ denotes the Riemannian connection of g.

In a Kenmotsu manifold the following relations hold [6].

$$\nabla_X\xi = X - \eta(X)\xi, \tag{4}$$

$$(\nabla_X\eta)Y = g(\nabla_X\xi, Y), \tag{5}$$

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X, \tag{6}$$

$$\eta(R(X,Y)Z) = g(X,Z)\eta(Y) - g(Y,Z)\eta(X), \tag{7}$$

$$S(X,\xi) = -2n\eta(X), \tag{8}$$

$$Q\xi = -2n\xi, \tag{9}$$

$$S(\phi X,\phi Y) = S(X,Y) + 2n\eta(X)\eta(Y), \tag{10}$$

for any vector fields X,Y,Z on M, where R denote the curvature tensor of type $(1,3)$ on M.

3. Main results

Througout this paper we associate $*$ with the quantities with respect to generalized Tanaka-Webster connection. The generalized Tanaka-Webster connection ∇^* associated to the Levi-Civita connection ∇ is given by [20, 7]

$$\nabla^*_X Y = \nabla_X Y - \eta(Y)\nabla_X\xi + (\nabla_X\eta)(Y)\xi - \eta(X)\phi Y, \tag{11}$$

for any vector fields X, Y on M.

Using (4) and (5), the above equation yields,

$$\nabla^*_X Y = \nabla_X Y + g(X,Y)\xi - \eta(Y)X - \eta(X)\phi Y. \tag{12}$$

By taking $Y = \xi$ in (12) and using (4) we obtain

$$\nabla^*_X\xi = 0. \tag{13}$$

We now calculate the Riemann curvature tensor R^* using (12) as follows:

$$R^*(X,Y)Z = R(X,Y)Z + g(Y,Z)X - g(X,Z)Y. \tag{14}$$

Using (6) and taking $Z = \xi$ in (14) we get

$$R^*(X,Y)\xi = 0. \tag{15}$$

On contracting (14), we obtain the Ricci tensor S^* of a Kenmotsu manifold with respect to the generalized Tanaka-Webster connection ∇^* as

$$S^*(Y,Z) = S(Y,Z) + 2ng(Y,Z). \tag{16}$$

This gives

$$Q^*Y = QY + 2nY. \tag{17}$$

Contracting with respect to Y and Z in (16), we get

$$r^* = r + 2n(2n+1), \tag{18}$$

where r^* and r are the scalar curvatures with respect to the generalized Tanaka-Webster connection ∇^* and the Levi-Civita connection ∇ respectively.

Definition 3.1. A Kenmotsu manifold with respect to the Levi-Civita connection is of constant curvature if its curvature tensor R is of the form

$$g(R(X,Y)Z,U) = k\{g(Y,Z)g(X,U) - g(X,Z)g(Y,U)\},$$

where k is a constant.

If $R^* = 0$, then the equation (14) becomes

$$R(X,Y,Z,U) = -\{g(Y,Z)g(X,U) - g(X,Z)g(Y,U)\}. \tag{19}$$

From which it follows that the Kenmotsu manifold with respect to the Levi-Civita connection is of constant curvature -1.

This leads to the following :

Theorem 3.1. If curvature tensor of a Kenmotsu manifold with respect to generalized Tanaka-Webster connection ∇^* is vanishes, then the Kenmotsu manifold is locally isometric to the hyperbolic space $H^{2n+1}(-1)$.

Definition 3.2. [1] For each plane p in the tangent space $T_x(M)$, the sectional curvature $K(p)$ is defined by $K(p) = \frac{R(X,Y,X,Y)}{g(X,X)g(Y,Y)-g(X,Y)^2}$, where $\{X,Y\}$ is orthonormal basis for p. Clearly $K(p)$ is the independent of the choice of the orthonormal basis $\{X,Y\}$.

Taking $Z = X$, $U = Y$ in (19), we get

$$R(X,Y,X,Y) = \{g(X,X)g(Y,Y) - g(X,Y)g(X,Y)\}. \tag{20}$$

Then from the above equation we conclude that

$$K(p) = \frac{R(X,Y,X,Y)}{g(X,X)g(Y,Y)-g(X,Y)^2} = -1. \tag{21}$$

Thus we can state the following theorem :

Theorem 3.2. If in a Kenmotsu manifold, the curvature tensor of a generalized Tanaka-Webster connection ∇^* vanishes, then the sectional curvature of the plane determined by two vectors $X,Y \in \xi^\perp$ is -1.

Now, an interesting invariant of a concircular transformation is the concircular curvature tensor. The concircular curvature tensor [23] C^* with respect to the generalized Tanaka-Webster connection ∇^* is defined by

$$C^*(X,Y)Z = R^*(X,Y)Z - \frac{r^*}{2n(2n+1)}\{g(Y,Z)X - g(X,Z)Y\}, \tag{22}$$

for all vector fields X, Y, Z on M.

By interchanging X and Y in (22), we have

$$C^*(Y,X)Z = R^*(Y,X)Z - \frac{r^*}{2n(2n+1)}\{g(X,Z)Y - g(Y,Z)X\}. \tag{23}$$

On adding (22) and (23) and using the fact that $R(X,Y)Z + R(Y,X)Z = 0$, we get

$$C^*(X,Y)Z + C^*(Y,X)Z = 0. \tag{24}$$

From (14), (22) and first Bianchi identity $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$ with respect to ∇, we obtain

$$C^*(X,Y)Z + C^*(Y,Z)X + C^*(Z,X)Y = 0. \tag{25}$$

Hence, from (24) and (25), shows that concircular curvature tensor with respect to generalized Tanaka-Webster connection in a Kenmotsu manifold is skew-symmetric and cyclic.

Next, we assume that the manifold M with respect to the generalized Tanaka-Webster connection is concircularly flat, that is, $C^*(X,Y)Z = 0$. Then from (22), it follows that

$$R^*(X,Y)Z = \frac{r^*}{2n(2n+1)}\{g(Y,Z)X - g(X,Z)Y\}. \tag{26}$$

Taking inner product of the above equation with ξ, we have

$$g(R^*(X,Y)Z,\xi) = \frac{r^*}{2n(2n+1)}\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}. \tag{27}$$

Using (1), (7), (14) and (18) in (27), we get

$$\frac{r+2n(2n+1)}{2n(2n+1)}\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\} = 0. \tag{28}$$

Replacing X by QX in (28), we obtain

$$\frac{r+2n(2n+1)}{2n(2n+1)}\{g(Y,Z)\eta(QX) - g(QX,Z)\eta(Y)\} = 0. \tag{29}$$

Using (8) in (29), we get

$$\frac{r+2n(2n+1)}{2n(2n+1)}\{-2ng(Y,Z)\eta(X) - S(X,Z)\eta(Y)\} = 0. \tag{30}$$

Taking $Y = \xi$ in (30), yields

$$\frac{r+2n(2n+1)}{2n(2n+1)}\{-2n\eta(X)\eta(Z) - S(X,Z)\} = 0. \tag{31}$$

This implies either the scalar curvature of M is $-2n(2n+1)$ or

$$S(X,Z) = -2n\eta(X)\eta(Z). \tag{32}$$

Hence we can state the following theorem:

Theorem 3.3. For a concircularly flat Kenmotsu manifold with respect to the generalized Tanaka-Webster connection, either the scalar curvature is $-2n(2n+1)$ or the manifold is a special type of η-Einstein manifold.

Definition 3.3. A Kenmotsu manifold with respect to the generalized Tanaka-Webster connection ∇^* is said to be ξ- concircularly flat if $C^*(X,Y)\xi = 0$.

In view of (14) and (18) in (22), we get

$$C^*(X,Y)Z = R(X,Y)Z + g(Y,Z)X - g(X,Z)Y - \frac{r+2n(2n+1)}{2n(2n+1)}\{g(Y,Z)X - g(X,Z)Y\}. \tag{33}$$

By taking $Z = \xi$ in (33) and then using (1) and (6), we find

$$C^*(X,Y)\xi = \frac{r+2n(2n+1)}{2n(2n+1)}R(X,Y)\xi. \tag{34}$$

Thus from (14), (18), (33) and (34), we have the following theorem:

Theorem 3.4. Let M be a Kenmotsu manifold with generalized Tanaka-Webster connection. In M, the following three conditions are equivalent:

i) M is ξ- concircularly flat.

ii) $r = -2n(2n+1)$.

iii) $r^* = 0$.

Now, we assume that the manifold M with respect to the generalized Tanaka-Webster connection is ξ-concircularly flat, that is, $C^*(X,Y)\xi = 0$. Then from (22), it follows that

$$R^*(X,Y)\xi = \frac{r^*}{2n(2n+1)}\{\eta(Y)X - \eta(X)Y\}. \tag{35}$$

In view of (15) and (18), we have

$$\frac{r+2n(2n+1)}{2n(2n+1)}\{\eta(Y)X - \eta(X)Y\} = 0. \tag{36}$$

Taking $Y = \xi$ in (36) and using (1), we get

$$\frac{r+2n(2n+1)}{2n(2n+1)}\{X - \eta(X)\xi\} = 0. \tag{37}$$

Taking inner product of the above equation with U, we have

$$\frac{r+2n(2n+1)}{2n(2n+1)}\{g(X,U) - \eta(X)\eta(U)\} = 0. \tag{38}$$

Now, replacing X by QX in (38), we obtain

$$\frac{r+2n(2n+1)}{2n(2n+1)}\{g(QX,U) - \eta(QX)\eta(U)\} = 0. \tag{39}$$

Using (9) in (39), we get

$$\frac{r+2n(2n+1)}{2n(2n+1)}\{S(X,U)+2n\eta(X)\eta(U)\}=0. \tag{40}$$

This implies either the scalar curvature of M is $-2n(2n+1)$ or

$$S(X,U)=-2n\eta(X)\eta(U). \tag{41}$$

Hence we can state the following theorem:

Theorem 3.5. For a ξ-concircularly flat Kenmotsu manifold with respect to the generalized Tanaka-Webster connection, either the scalar curvature is $-2n(2n+1)$ or the manifold is a special type of η-Einstein manifold.

Definition 3.4. A Kenmotsu manifold is said to be ϕ-concircularly flat with respect to the generalized Tanaka-Webster connection ∇^* if

$$g(C^*(\phi X,\phi Y)\phi Z,\phi W)=0, \tag{42}$$

for any vector fields X,Y,Z on M.

Using (22) in (42), we have

$$g(R^*(\phi X,\phi Y)\phi Z,\phi W)=\frac{r^*}{2n(2n+1)}\{g(\phi Y,\phi Z)g(\phi X,\phi W)-g(\phi X,\phi Z)g(\phi Y,\phi W)\}. \tag{43}$$

Let $\{e_1,e_2,e_3,.......e_{2n+1}\}$ be a local orthonormal basis of vector fields in M. Then $\{\phi e_1,\phi e_2,\phi e_3,.......\phi e_{2n+1}\}$ is also a local orthonormal basis. If we put $X=W=e_i$ in (43) and summing up with respect to $i, 1\le i\le 2n+1$, we obtain

$$\begin{aligned}\sum_{i=1}^{2n} g(R^*(\phi e_i,\phi Y)\phi Z,\phi e_i)=\frac{r^*}{2n(2n+1)}\sum_{i=1}^{2n}\{&g(\phi Y,\phi Z)g(\phi e_i,\phi e_i)\\ &-g(\phi e_i,\phi Z)g(\phi Y,\phi e_i)\}.\end{aligned} \tag{44}$$

From (44), it follows that

$$S^*(\phi Y,\phi Z)=\frac{r^*(2n-1)}{2n(2n+1)}g(\phi Y,\phi Z). \tag{45}$$

Using (1), (16) and (18) in (45), we get

$$S(\phi Y,\phi Z)+2ng(\phi Y,\phi Z)=\frac{(r+2n(2n+1))(2n-1)}{2n(2n+1)}g(\phi Y,\phi Z). \tag{46}$$

By using (2) and (10) in (46), we obtain

$$S(Y,Z)+2n\eta(Y)\eta(Z)+\{2n-\frac{(r+2n(2n+1))(2n-1)}{2n(2n+1)}\}g(\phi Y,\phi Z)=0. \tag{47}$$

Hence contracting (47), we get

$$r=-2n. \tag{48}$$

By substituting equation (48) in (22), we get

$$C^*(X,Y)Z=R(X,Y)Z+\frac{1}{2n+1}\{g(Y,Z)X-g(X,Z)Y\}. \tag{49}$$

This leads to the following:

Theorem 3.6. Let the Kenmotsu manifold M with generalized Tanaka-Webster connection be ϕ-concircularly flat. Then M is of constant sectional curvature $-\frac{1}{2n+1}$ if and only if concircular curvature tensor C^* vanishes.

Definition 3.5. A Kenmotsu manifold is said to be pseudo-concircularly flat with respect to the generalized Tanaka-Webster connection ∇^* if it satisfies

$$g(C^*(\phi X,Y)Z,\phi W)=0, \tag{50}$$

for any vector fields X,Y,Z on M.

In view of (22) and (50), we have

$$g(R^*(\phi X,Y)Z-\frac{r^*}{2n(2n+1)}\{g(Y,Z)\phi X-g(\phi X,Z)Y\},\phi W)=0. \tag{51}$$

Making use of (14) and (18) in (51), we get

$$g(R(\phi X,Y)Z,\phi W)-\frac{r}{2n(2n+1)}\{g(Y,Z)g(\phi X,\phi W)-g(\phi X,Z)g(Y,\phi W)\}=0. \tag{52}$$

Let $\{e_1,e_2,e_3,.......e_{2n+1}\}$ be a local orthonormal basis of vector fields in M. Then by putting $Y=Z=e_i$ in (52) and summing up with respect to $i, 1\leq i\leq 2n+1$, we obtain

$$S(\phi X,\phi W)=\frac{r}{2n+1}g(\phi X,\phi W). \tag{53}$$

On using (1) and (10) in (53), we get

$$S(X,W)=\frac{r}{2n+1}g(X,W)-\{2n+\frac{r}{2n+1}\}\eta(X)\eta(W). \tag{54}$$

Again taking $X = W = e_i$ in (54) and summing up with respect to $i, 1 \leq i \leq 2n+1$, we obtain

$$r = -2n(2n+1). \tag{55}$$

By virtue of (54) and (55), we get

$$S(X,W) = -2ng(X,W). \tag{56}$$

Thus M is an Einstein manifold.

Again by substituting (55) in (33), we obtain

$$C^*(X,Y)Z = R(X,Y)Z + \{g(Y,Z)X - g(X,Z)Y\}. \tag{57}$$

Thus, from the above discussions we state the following:

Theorem 3.7. Let the Kenmotsu manifold M with generalized Tanaka-Webster connection be pseudo-concircularly flat if and only if $S(Y,Z) = -2ng(Y,Z)$.

Further if $C^* = 0$, then M is isomorphic to the hyperbolic space $H^{2n+1}(-1)$.

Definition 3.6. A Kenmotsu manifold is said to be ϕ-concircularly semisymmetric with respect to generalized Tanaka-Webster connection ∇^* if $C^*(X,Y).\phi = 0$ holds on M.

Now, we consider ϕ-concircularly semisymmetric Kenmotsu manifold with respect to generalized Tanaka-Webster connection. Then

$$(C^*(X,Y).\phi)Z = C^*(X,Y)\phi Z - \phi C^*(X,Y)Z = 0. \tag{58}$$

for all X,Y,Z on M.

Taking $Z = \xi$ in (58), we get

$$\phi(C^*(X,Y)\xi) = 0. \tag{59}$$

Using (34) and (6) in (59), we get

$$\frac{r+2n(2n+1)}{2n(2n+1)}\{\eta(X)\phi Y - \eta(Y)\phi X\} = 0. \tag{60}$$

Replace Y by ξ and X by ϕX in (60) and using (1), we get

$$\frac{r+2n(2n+1)}{2n(2n+1)}\{X - \eta(X)\xi\} = 0. \tag{61}$$

Taking inner product of the above equation with U, we have

$$\frac{r+2n(2n+1)}{2n(2n+1)}\{g(X,U)-\eta(X)\eta(U)\}=0. \tag{62}$$

Now, replacing X by QX in (62), we obtain

$$\frac{r+2n(2n+1)}{2n(2n+1)}\big\{g(QX,U)-\eta(QX)\eta(U)\big\}=0. \tag{63}$$

Using (9) in (63), we get

$$\frac{r+2n(2n+1)}{2n(2n+1)}\big\{S(X,U)+2n\eta(X)\eta(U)\big\}=0. \tag{64}$$

This implies either the scalar curvature of M is $-2n(2n+1)$ or

$$S(X,U)=-2n\eta(X)\eta(U). \tag{65}$$

Hence we can state the following:

Theorem 3.8. For a ϕ-concircularly semisymmetric Kenmotsu manifold with respect to the generalized Tanaka-Webster connection, either the scalar curvature is $-2n(2n+1)$ or the manifold is a special type of η-Einstein manifold.

Now, we consider

$$C^*.S^*=S^*(C^*(X,Y)Z,U)+S^*(Z,C^*(X,Y)U). \tag{66}$$

By making use of (22) and (16) in (66), we obtain

$$\begin{aligned}C^*.S^* &= S(R(X,Y)Z-\frac{r}{2n(2n+1)}\{g(Y,Z)X-g(X,Z)Y\},U)\\ &\quad +S(Z,R(X,Y)U-\frac{r}{2n(2n+1)}\{g(Y,U)X-g(X,U)Y\}).\end{aligned} \tag{67}$$

Suppose $C^*.S^*=0$. Then we have

$$S^*(C^*(X,Y)Z,U)+S^*(Z,C^*(X,Y)U)=0. \tag{68}$$

Taking $U=\xi$ in (68) and using (16), it follows that

$$S^*(Z,C^*(X,Y)\xi)=0. \tag{69}$$

Making use of (1), (6) and (33) in (69), we get

$$\frac{r+2n(2n+1)}{2n(2n+1)}S^*(Z,\eta(X)Y-\eta(Y)X)=0. \tag{70}$$

Replacing X by ξ in (70) and using (1) and (16), we get

$$\frac{r+2n(2n+1)}{2n(2n+1)}\{S(Z,Y)+2ng(Z,Y)\}=0. \tag{71}$$

Contracting (71) with respect to Y and Z, we get

$$r=-2n(2n+1). \tag{72}$$

From (67) and (72), we obtain

$$S(Y,Z)=-2ng(Y,Z). \tag{73}$$

Thus M is an Einstein manifold.

Again by substituting (72) in (33), we obtain

$$C^*(X,Y)Z=R(X,Y)Z+\{g(Y,Z)X-g(X,Z)Y\}. \tag{74}$$

Thus, from the above discussions we state the following:

Theorem 3.9. Let M be a Kenmotsu manifold with generalized Tanaka-Webster connection, then $C^*.S^*=0$ if and only if $S(Y,Z)=-2ng(Y,Z)$.

Further if $C^*=0$ then M is isomorphic to the hyperbolic space $H^{2n+1}(-1)$.

Further, we have

$$\begin{aligned}(R^*(X,Y).C^*)(U,V,W)&=R^*(X,Y)C^*(U,V)W-C^*(R^*(X,Y)U,V)W\\&-C^*(U,R^*(X,Y)V)W-C^*(U,V)R^*(X,Y)W.\end{aligned} \tag{75}$$

With the use of (22), (75) becomes

(76)

$$\begin{aligned}(R^*(X,Y).C^*)(U,V,W)&=R^*(X,Y)R^*(U,V)W-R^*(R^*(X,Y)U,V)W-R^*(U,R^*(X,Y)V)W\\&-R^*(U,V)R^*(X,Y)W+\frac{r^*}{2n(2n+1)}\{g(R^*(X,Y)V,W)U+g(V,R^*(X,Y)W)U\\&-g(R^*(X,Y)U,W)V-g(U,R^*(X,Y)W)V\}.\end{aligned}$$

By the symmetric properties of the curvature tensor R^* [7, 16], we get

$$\begin{aligned}(R^*(X,Y).C^*)(U,V,W)&=R^*(X,Y)R^*(U,V)W-R^*(R^*(X,Y)U,V)W\\&-R^*(U,R^*(X,Y)V)W-R^*(U,V)R^*(X,Y)W.\end{aligned} \tag{77}$$

Finally, we get

$$(R^*(X,Y).C^*)(U,V,W) = (R^*(X,Y).R^*)(U,V,W). \tag{78}$$

Thus we state the following:

Theorem 3.10. Let M be a Kenmotsu manifold with generalized Tanaka-Webster connection. Then $R^*.C^* = R^*.R^*$.

4. Example of a 5-dimensional Kenmotsu manifold with respect to the generalized Tanaka-Webster connection

We consider the five-dimensional manifolod $M = \{(x,y,z,u,v) \in R^5\}$, where (x,y,z,u,v) are the standard coordinates in R^5. The vector fields

$$E_1 = e^{-v}\frac{\partial}{\partial x},\ E_2 = e^{-v}\frac{\partial}{\partial y},\ E_3 = e^{-v}\frac{\partial}{\partial z},\ E_4 = e^{-v}\frac{\partial}{\partial u},\ E_5 = \frac{\partial}{\partial v}$$

are linearly independent at each point of M. Let g be the Riemannian metric defined by

$$g_{ij} = \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{for } i \neq j. \end{cases}$$

Let η be the 1-form defined by $\eta(Z) = g(Z,E_3)$ for any $Z \in \chi(M)$. Let ϕ be the $(1,1)$ tensor field defined by $\phi E_1 = E_3, \phi E_2 = E_4, \phi E_3 = -E_1, \phi E_4 = -E_2, \phi E_5 = 0$. Then using the linearity of ϕ and g we have

$$\eta(E_5) = 1,\ \phi^2(Z) = -Z + \eta(Z)E_5,\ g(\phi Z, \phi U) = g(Z,U) - \eta(Z)\eta(U),$$

for any $Z,U \in \chi(M)$. Thus for $E_5 = \xi, (\phi,\xi,\eta,g)$ defines an almost contact metric structure on M.

Let ∇ be the Levi-Civita connection with respect to the metric g. Then we have

$$[E_1,E_2] = [E_1,E_3] = [E_1,E_4] = [E_2,E_3] = 0,\ \ [E_1,E_5] = E_1,$$

$$[E_4,E_5] = E_4,\ \ [E_2,E_4] = [E_3,E_4] = 0,\ \ [E_2,E_5] = E_2,\ \ [E_3,E_5] = E_3.$$

The Riemannian connection ∇ of the metric g is given by the Koszul's formula

$$2g(\nabla_X Y,Z) = X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) - g(X,[Y,Z]) - g(Y,[X,Z]) + g(Z,[X,Y]).$$

By Koszul's formula, we get

$$\nabla_{E_1}E_1 = -E_5,\ \nabla_{E_1}E_2 = 0,\ \nabla_{E_1}E_3 = 0,\ \nabla_{E_1}E_4 = 0,\ \nabla_{E_1}E_5 = E_1,$$
$$\nabla_{E_2}E_1 = 0,\ \nabla_{E_2}E_2 = -E_5,\ \nabla_{E_2}E_3 = 0,\ \nabla_{E_2}E_4 = 0,\ \nabla_{E_2}E_5 = E_2,$$
$$\nabla_{E_3}E_1 = 0,\ \nabla_{E_3}E_2 = 0,\ \nabla_{E_3}E_3 = -E_5,\ \nabla_{E_3}E_4 = 0,\ \nabla_{E_3}E_5 = E_3,$$
$$\nabla_{E_4}E_1 = 0,\ \nabla_{E_4}E_2 = 0,\ \nabla_{E_4}E_3 = 0,\ \nabla_{E_4}E_4 = -E_5,\ \nabla_{E_4}E_5 = E_4,$$
$$\nabla_{E_5}E_1 = 0,\ \nabla_{E_5}E_2 = 0,\ \nabla_{E_5}E_3 = 0,\ \nabla_{E_5}E_4 = 0,\ \nabla_{E_5}E_5 = 0.$$

Further we obtain the following:

$$\nabla^*_{E_i}E_j = 0, \quad i,\ \ j = 1,2,3,4,5$$

and hence

$$(\nabla^*_{E_i}\phi)E_j = 0, \quad i,\ \ j = 1,2,3,4,5.$$

From the above expressions it follows that the manifold satisfies (2), (3) and (4) for $\xi = E_5$. Hence the manifold is a Kenmotsu manifold. With the help of the above results we can verify the following results.

$$R(E_1,E_2)E_2 = R(E_1,E_3)E_3 = R(E_1,E_4)E_4 = R(E_1,E_5)E_5 = -E_1,$$
$$R(E_1,E_2)E_1 = E_2,\ R(E_1,E_3)E_1 = R(E_5,E_3)E_5 = R(E_2,E_3)E_5 = E_3,$$
$$R(E_2,E_3)E_3 = R(E_2,E_4)E_4 = R(E_2,E_5)E_5 = -E_2,\ R(E_3,E_4)E_4 = -E_3,$$
$$R(E_2,E_5)E_2 = R(E_1,E_5)E_1 = R(E_4,E_5)E_4 = R(E_3,E_5)E_3 = E_5,$$
$$R(E_1,E_4)E_1 = R(E_2,E_4)E_2 = R(E_3,E_4)E_3 = R(E_5,E_4)E_5 = E_4$$

and

$$R^*(E_i,E_j)E_k = 0,\ \ i,\ \ j,\ k = 1,2,3,4,5.$$

From the above expressions of the curvature tensor of the Kenmotsu manifold it can be easily seen that the manifold has a constant sectional curvature -1.

Making use of the above results we obtain the Ricci tensors as follows:

$$S(E_1,E_1) = g(R(E_1,E_2)E_2,E_1) + g(R(E_1,E_3)E_3,E_1) + g(R(E_1,E_4)E_4,E_1)$$
$$+ g(R(E_1,E_5)E_5,E_1) = -4.$$

Similarly, we have

$$S(E_2,E_2) = S(E_3,E_3) = S(E_3,E_3) = S(E_4,E_4) = S(E_5,E_5) = -4$$

and

$$S^*(E_1,E_1) = S^*(E_2,E_2) = S^*(E_3,E_3) = S^*(E_4,E_4) = S^*(E_5,E_5) = 0.$$

Therefore, it can be easily verified that the manifold is an Einstein manifold with respect to Levi-Civita connection.

Conflict of Interests

The authors declare that there is no conflict of interests.

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