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FIXED POINT THEOREMS AND PARTIAL METRIC SPACES

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Abstract. In this paper, we give some fixed point theorems for generalized cyclic contraction and generalized $\varphi-$weak contraction in partial metric spaces, which improve the results of S. Romaguera in [1], M. Abbas in [2] and T. Abdeljiawad in [5].

Keywords: cyclic contraction mapping; generalized φ-weak contraction; *PMS*; fixed point.

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1. Introduction

In 1992, the notion of partial metric space was introduced by Matthews [7] as a part of the study of denotational semantics of dataflow networks. Henceforward, many authors made efforts to study various fixed point theorems and obtained a lot of perfect results. Ravi P Agarwal [10] proved some fixed point results for generalized cyclic contraction on partial metric spaces. Mădălina and Ioan [3] proved a Maia type fixed point theorem for cyclic φ-contraction, which extended the results of W.A. Kirk [9]. Z. Qingnian and S. Yisheng [12] proved fixed point results for single-valued hybrid generalized φ-weak contractions. Very recently, many new fixed point results were established by Yi Zhang, Jiang Zhu [8] and L.N. Mishra [11], etc.

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In this paper, we shall give some fixed point theorems for generalized cyclic contraction and generalized $\varphi-$weak contraction in partial metric spaces, which improve the results of S. Romaguera in [1], M. Abbas in [2] and T. Abdeljiawad in [5].

2. PRELIMINARIES

First, we shall introduce some essential definitions and lemmas.

Definition 2.1([1,2]) Suppose that the set X is nonempty. We call (X,p) a *PMS* (*the abbreviation of partial metric space*) if a function p maps $X\times X$ into nonnegative number and satisfies: (T_1) $x=y$ *iff* $p(x,x)=p(x,y)=p(y,y)$, (T_2) $p(x,x)\leq p(x,y)$, (T_3) $p(x,y)=p(y,x)$, and (T_4) $p(x,y)\leq p(x,z)+p(z,y)-p(z,z)$, $\forall x,y,z\in X$.

Now let (X,p) be a *PMS*, the mapping p^s maps $X\times X$ into nonnegative number, and

$$p^s(x,y)=2p(x,y)-p(x,x)-p(y,y).$$

We can verify the fact that (X,p^s) is a metric space. In addition, we can define the open ball on (X,p) denoted by $B(x,\varepsilon)=\{z\in X:p(x,z)<\varepsilon+p(x,x)\}$, where $x\in X$, $\varepsilon>0$. Meanwhile, we also use these open balls to form a base for T_0-topology τ.

Definition 2.2([1,2]) Suppose that (X,p) is a *PMS*, and $\{x_n\}\subset(X,p)$. Then

(1) $\{x_n\}$ converges to $x\in X$ iff $\lim\limits_{n\to\infty}p(x_n,x)=p(x,x)$.

(2) $\{x_n\}$ is called Cauchy sequence iff $\lim\limits_{n,m\to\infty}p(x_n,x_m)=\xi$ $(0\leq\xi<\infty)$.

(3) (X,p) is called complete if any Cauchy sequence $\{x_n\}$ converges w.r.t τ to $x\in X$ and $\lim\limits_{n,m\to\infty}p(x_n,x_m)=p(x,x)$.

(*Remark 1*([4]): $\lim\limits_{n\to\infty}p(x_n,x)=p(x,x)=0\ \Rightarrow\ \lim\limits_{n\to\infty}p(x_n,y)=p(x,y)$, $y\in X$.)

Lemma 2.3([1]) Let (X,p) be a *PMS*, and $\{x_n\}\subset(X,p)$. Then

(1) $\{x_n\}$ is a Cauchy sequence in (X,p) iff it is a Cauchy sequence in (X,p^s).

(2) A *PMS* (X,p) is complete iff (X,p^s) is complete. Moreover,

$$\lim_{n\to\infty} p^s(x,x_n)=0 \Longleftrightarrow p(x,x)=\lim_{n\to\infty} p(x,x_n)=\lim_{n,m\to\infty} p(x_n,x_m)$$

Definition 2.4([3]) Let X_i be nonempty set, $i=1,2,...,m$, and $X=\bigcup_{i=1}^{m} X_i$. X is called a cyclic representation of X w.r.t. f if there exists a self-map f on X satisfying $f(X_1)\subset X_2,...,f(X_{m-1})\subset X_m$, and $f(X_m)\subset X_1$.

Definition 2.5([6]) Assume that the set X is nonempty and $H,G:X\to X$. If $w=Hx=Gx$, $x,w\in X$, then x,w are called a coincidence point of H and G and a point of coincidence of H and G, respectively. Moreover, H, G are weakly compatible if $HGx=GHx$, whenever $Hx=Gx$.

Definition 2.6 To simplify notation, we shall introduce some abbreviations:

(1) we denote by Λ all upper semicontinuous from the right function (or u.s.r.f) if any $\phi\in\Lambda$ maps nonnegative real number into nonnegative real number with $\phi(t)<t$ for $t>0$.

(2) we denote by Δ all continuous function if any $\phi\in\Delta$ maps nonnegative real number into nonnegative real number with $\phi(t)<t$ for $t>0$.

(3) we denote by Θ all lower semi-continuous function if any $\varphi\in\Lambda$ maps positive real number into positive real number with $\varphi(0)=0$.

(4) we denote by by N and ω the set of all positive integer numbers and the set of all nonnegative integer numbers, respectively.

3. Main results

Part 1. Fixed point theorem for generalized cyclic contraction in partial metric spaces

To obtain fixed point theorem for generalized cyclic contraction in partial metric spaces, we shall firstly prove the following result by improving the Theorem 3 of S. Romaguera [1].

Theorem 3.1 If (X,p) is complete *PMS*, $\lambda\in[0,\frac{1}{2}]$, $\phi\in\Lambda$ and $f:X\to X$ satisfies:

$$p(fx,fy)\le\phi(\max\{p(x,y),p(fx,x),p(fy,y),\lambda p(x,fy)+(1-\lambda)p(fx,y)\}),\quad \forall x,y\in X,$$

then f has a unique fixed point $z \in X$. Moreover, $p(z,z) = 0$.

Proof To simplify notation, we define

$$M(x,y) = \max\{p(x,y), p(fx,x), p(fy,y), \lambda p(x,fy) + (1-\lambda)p(fx,y)\}.$$

First, let us construct $\{x_n\} \subset X$. Assume that $x_0 = x$ and $x_n = f^n x_0$, $\forall x \in X, n \in \omega$. If $f^n x = f^{n+1} x$ for some $n \in \omega$, then $f^n x$ is a fixed point of f, and $f^n x$'s uniqueness follows as in the last part.

So, suppose that $f^n x \neq f^{n+1} x$, $\forall n \in \omega$.

Next, we shall prove that $\lim\limits_{n\to\infty} p(x_n, x_{n+1}) = 0$.

We define

$$\begin{aligned} r^{Mn} &= M(x_n,x_{n+1}), \quad r^{Mnm} = M(x_{n_k},x_{m_k}), \quad r^n_{ij} = p(x_{n+i},x_{n+j}), \\ r^{nmk}_{ij} &= p(x_{n_k+i},x_{m_k+j}), \quad r^{nk}_{ij} = p(x_{n_k+i},x_{n_k+j}), \quad r^{mk}_{ij} = p(x_{m_k+i},x_{m_k+j}) \end{aligned}$$

where $i, j \in \{-1,0,1,2\}$.

For all $n \in \omega$, since

$$p(x_{n+1},x_{n+2}) = p(fx_n, fx_{n+1}) \leq \phi(M(x_n,x_{n+1})) < M(x_n,x_{n+1}) \tag{1}$$

where $M(x_n,x_{n+1}) = \max\{r^n_{01},\ r^n_{12},\ \lambda r^n_{02} + (1-\lambda) r^n_{11}\}$.

(a) If $r^{Mn} = r^n_{01}$, by (1) we have

$$r^n_{12} < r^{Mn} = r^n_{01}.$$

(b) If $r^{Mn} = r^n_{12}$, by (1) we have

$$r^n_{12} < r^{Mn} = r^n_{12}$$

which is contradictive.

(c) If $r^{Mn} = \lambda r_{02}^n + (1-\lambda) r_{11}^n$,

since

$$\begin{aligned} \lambda r_{02}^n + (1-\lambda) r_{11}^n &\leq \lambda (r_{01}^n + r_{12}^n - r_{11}^n) + (1-\lambda) r_{11}^n \\ &= \lambda (r_{01}^n + r_{12}^n) + (1-2\lambda) r_{11}^n \\ &\leq \lambda (r_{01}^n + r_{12}^n) + (1-2\lambda) r_{01}^n \\ &= \lambda r_{12}^n + (1-\lambda) r_{01}^n \qquad (2) \end{aligned}$$

by $(1)(2)$we obtain that

$$r_{12}^n < r^{Mn} = \lambda r_{02}^n + (1-\lambda) r_{11}^n \leq \lambda r_{12}^n + (1-\lambda) r_{01}^n$$

i.e.

$$r_{12}^n < \lambda r_{12}^n + (1-\lambda) r_{01}^n$$

i.e.

$$r_{12}^n < r_{01}^n.$$

Then by above $(a)(b)(c)$, we claim that $r_{12}^n < r_{01}^n, \forall n \in \omega$, and $\lim_{n\to\infty} p(x_n, x_{n+1}) = r_0$, where r_0 is a constant.

Now by $(1)(2)$ and taking limit of the inequality:

$$\begin{aligned} r_{12}^n \leq \phi(r^{Mn})) < r^{Mn} &= \max\{r_{01}^n,\ r_{12}^n,\ \lambda r_{02}^n + (1-\lambda) r_{11}^n\} \\ &\leq \max\{r_{01}^n,\ r_{12}^n,\ \lambda r_{12}^n + (1-\lambda) r_{01}^n\} \end{aligned}$$

so we have

$$r_0 = \lim_{n\to\infty} \phi(r^{Mn}) = \lim_{n\to\infty} r^{Mn}$$

Since $r^{Mn} \geq r_0$ for all $n \in \omega$ and ϕ is u.s.r.f., we obtain that

$$r_0 = \lim_{n\to\infty} \phi(r^{Mn}) = \limsup_{n\to\infty} \phi(r^{Mn}) \leq \phi(r_0).$$

So, $r_0 = 0$ and $\lim_{n\to\infty} p(x_n, x_{n+1}) = 0$

Consequently, for any subsequence $\{x_{n_k}\}$ of $\{x_n\}$, we have $\lim_{k\to\infty} r_{01}^{nk}=0$, and by (T_4), $\lim_{k\to\infty} r_{0q}^{nk}=0$ for any $q\in N$.

Next, let us prove that $\lim_{n,m\to\infty} p(x_n,x_m)=0$.

Now suppose the contradiction, then for sequence $(n_k)_{k\in N},(m_k)_{k\in N}\subset N$ with $m_k>n_k\geq k_0$, there exist $\varepsilon>0$, $k_0\in N$ such that $r_{00}^{nmk}\geq\varepsilon,\forall k\in N$.

Since $\lim_{n\to\infty} r_{01}^{n}=0$, w.l.g., we suppose that $r_{0-1}^{nmk}<\varepsilon$ and $r_{10}^{nmk}<\varepsilon$ for $m_k>n_k\geq k_0$ and $k\in N$, so for all $k\in N$, We have

$$\varepsilon\leq r_{00}^{nmk}\leq r_{0-1}^{nmk}+r_{-10}^{mk}<\varepsilon+r_{-10}^{mk}$$

hence $\lim_{k\to\infty} r_{00}^{nmk}=\varepsilon$.

Since

$$\begin{aligned}\lim_{k\to\infty}[\lambda r_{01}^{nmk}+(1-\lambda)r_{10}^{nmk}] &\leq \lim_{k\to\infty}[\lambda(r_{01}^{nk}+r_{10}^{nmk}+r_{01}^{mk})+(1-\lambda)r_{10}^{nmk}]\\ &= \lim_{k\to\infty}[\lambda(r_{01}^{nk}+r_{01}^{mk})+r_{10}^{nmk}]\\ &< \varepsilon \qquad (3)\end{aligned}$$

and by (3) we have

$$\begin{aligned}\varepsilon &= \lim_{k\to\infty} r_{00}^{nmk}\leq\lim_{k\to\infty} r^{Mnm}\\ &= \lim_{k\to\infty}\max\{r_{00}^{nmk},\ r_{10}^{nk},\ r_{10}^{mk},\ \lambda r_{01}^{nmk}+(1-\lambda)r_{10}^{nmk}\}\\ &< \max\{\varepsilon,0,0,\varepsilon\}=\varepsilon.\end{aligned}$$

Hence $\lim_{k\to\infty} r^{Mnm}=\varepsilon$.

Since $r^{Mnm}\geq\varepsilon$ for all $k\in N$, and ϕ is u.s.r.f., we have

$$\limsup_{k\to\infty}\phi(r^{Mnm})\leq\phi(\varepsilon).$$

at the same time, for all $k\in N$, we have

$$\begin{aligned}\varepsilon &\leq r_{00}^{nmk}\leq r_{01}^{nk}+r_{11}^{nmk}+r_{10}^{mk}\\ &\leq r_{01}^{nk}+\phi(r^{Mnm})+r_{10}^{mk}\end{aligned}$$

so

$$\varepsilon \leq \limsup_{k\to\infty} \phi(r^{Mnm}) \leq \phi(\varepsilon).$$

Hence $\lim_{n,m\to\infty} p(x_n,x_m) = 0$, and $\{x_n\}$ converges to a point $z \in X$ such that

$$\lim_{n,m\to\infty} p(x_n,x_m) = \lim_{n\to\infty} p(z,x_n) = p(z,z) = 0.$$

Now, for $\lambda \in [0,\frac{1}{2}]$ i.e. $1-\lambda \in [\frac{1}{2},1]$, we obtain that $\lim_{n\to\infty} M(z,x_n) = p(fz,z)$.

Since $M(z,x_n) \geq p(fz,z)$ for $n \in N$ and ϕ is u.s.r.f., we have

$$\limsup_{n\to\infty} \phi(M(z,x_n)) \leq \phi(p(fz,z)) \tag{4}$$

by (4) and the condition that $p(fz,z) \leq p(fz,x_n) + p(x_n,z)$, we obtain that

$$\begin{aligned} p(fz,z) &\leq \limsup_{n\to\infty}(p(fz,x_n)+p(x_n,z)) \\ &= \limsup_{n\to\infty} p(fz,fx_{n-1}) \\ &\leq \limsup_{n\to\infty} \phi(M(z,x_{n-1})) \leq \phi(p(fz,z)). \end{aligned}$$

Hence $p(fz,z) = 0$, i.e. $z = fz$. Thus z is a fixed point of f.

If there exists $z^* \in X$ such that $z^* = fz^*$, then we get

$$p(z,z^*) = p(fz,fz^*) \leq \phi(M(z,z^*)) = \phi(p(z,z^*)),$$

therefore $p(z,z^*) = 0$, i.e. $z = z^*$. Thus z is unique fixed point of f and $p(z,z) = 0$. □

Secondly, we shall improve the Theorem 2.3 of M.Abbas[2] and obtain fixed point theorem for generalized cyclic contraction mapping in partial metric spaces.

Theorem 3.2 Let (X,p) be complete *PMS*, $A_1,A_2,...,A_m$, m nonempty closed subsets of (X,p^s) and $Y = \bigcup_{i=1}^{m} A_i$ be a cyclic representation of Y w.r.t. f. If $\lambda \in [0,\frac{1}{2}]$, $\phi \in \Delta$, and

$$p(fx,fy) \leq \phi(M(x,y)), \qquad \forall x \in A_i, y \in A_{i+1}, i = 1,2,...,m$$

where $A_{m+1} = A_1$, and $M(x,y) = \max\{p(x,y), p(fx,x), p(fy,y), \lambda p(x,fy) + (1-\lambda)p(fx,y)\}$, then f has a unique fixed point $z \in \bigcap_{i=1}^{m} A_i \subset Y$.

Proof First, let us construct $\{x_n\} \subset Y$. Assume that $x_0 = x, \ \forall x \in Y$, thus we obtain that $x_0 \in A_{i_0}$ for some i_0. By definition 2.4, we claim that $fx_0 \in A_{i_0+1}$, and there exists $x_1 \in A_{i_0+1}$ such that $fx_0 = x_1$. Similarly, there exists $x_2 \in A_{i_0+2}$ such that $fx_1 = x_2$. By inductive method, for each $n \in \omega$, there exist $x_n \in A_{i_n}$ and $x_{n+1} \in A_{i_n+1}$, where $i_n \in \{1,2,...,m\}$, such that $fx_n = x_{n+1}$.

If $f^n x = f^{n+1}x$ for some $n \in \omega$, then $f^n x$ is a fixed point of f, and $f^n x$'s uniqueness follows as in the last part.

So, suppose that $f^n x \neq f^{n+1}x, \ \forall n \in \omega$.

First, by the proof of Theorem 3.1, we obtain that $\lim_{n\to\infty} p(x_n, x_{n+1}) = 0$.

Similarly, if $\{x_{n_k}\}$ is any subsequence of $\{x_n\}$, then we obtain that $\lim_{k\to\infty} p(x_{n_k}, x_{n_k+1}) = 0$, and $\lim_{k\to\infty} p(x_{n_k}, x_{n_k+i}) = 0, \ \forall i \in N.$ (5)

Next, we shall prove that $\lim_{n,m\to\infty} p(x_n, x_m) = 0$.

Now, we can decompose $\{x_n\}$ into m subsequence $\{x_n^{(i)}\}, i = 1,2,...,m$, where $x_n^{(i)} \in A_i, \ \forall n \in \omega$. In order to prove that $\{x_n\}$ is Cauchy sequence in (Y, p) or (Y, p^s) by Lemma 2.3, we only need to prove that any $\{x_n^{(i)}\}$ is Cauchy sequence in (Y, p), $i = 1,2,...,m$.

W.l.g., let $x_0 \in A_m$ and $x_m \in A_m$. Next, we shall show that $\lim_{j,l\to\infty} p(x_j^{(m)}, x_l^{(m)}) = 0$.

We define

$$\begin{aligned} r_{st}^{jlk} &= p(x_{j_k+s}^{(m)}, x_{l_k+t}^{(m)}), \quad r_{st}^{jk} = p(x_{j_k+s}^{(m)}, x_{j_k+t}^{(m)}), \quad r_{st}^{lk} = p(x_{l_k+s}^{(m)}, x_{l_k+t}^{(m)}), \\ r_{st}^{jlkh} &= p(x_{j_{k_h}+s}^{(m)}, x_{l_{k_h}+t}^{(m)}), \quad r_{st}^{jkh} = p(x_{j_{k_h}+s}^{(m)}, x_{j_{k_h}+t}^{(m)}), \quad r^M = M(x_{j_k}^{(m)}, x_{l_k-1}^{(m)}), \\ r^{Mh} &= M(x_{j_{k_h}}^{(m)}, x_{l_{k_h}-1}^{(m)}), \end{aligned}$$

where $s,t \in \{-m, \ -1, \ 0, \ 1\}$.

Now suppose the contradiction, thus for sequence $\{j_k\}, \{l_k\} \subset N$ with $l_k > j_k \geq k_0$, there exist $\varepsilon > 0$ such that

$$r_{00}^{jlk} \geq \varepsilon \ \ and \ \ r_{0-1}^{jlk} < \varepsilon, \quad \forall k \in N \tag{6}$$

Since (5) holds, then for any $k \in N$, by (6) we get

$$\varepsilon \leq r_{00}^{jlk} \leq r_{0-1}^{jlk} + r_{-10}^{lk} < \varepsilon + r_{-10}^{lk},$$

so

$$\lim_{k\to\infty} r_{00}^{jlk} = \varepsilon,$$

and

$$\lim_{k\to\infty} \phi(r_{00}^{jlk}) = \phi(\varepsilon) < \varepsilon.$$

Now, we claim that $\{r^M\}$ has a subsequence $\{r^{Mh}\}$, and $\lim_{k\to\infty} r^{Mh} = \alpha$, where $\alpha \in [\frac{\varepsilon}{2}, \varepsilon]$ is a constant.

Indeed, since

$$\varepsilon \le r_{00}^{jlk} \le r_{01}^{jk} + r_{1-1}^{jlk} + r_{-10}^{lk}$$

so

$$\varepsilon \le \lim_{k\to\infty} r_{1-1}^{jlk} \tag{7}$$

Besides, for any $\varepsilon^{'} \in (0,\varepsilon)$, there exists $k_{\varepsilon'} \in N$ such that for all $k \ge k_{\varepsilon'}$

$$r_{01}^{jk} < \varepsilon^{'}, \quad r_{-m-1}^{lk} < \varepsilon^{'} \quad and \quad r_{-10}^{lk} < \varepsilon^{'}.$$

Moreover, because $r_{0-1}^{jlk} < \varepsilon$, thus for $l_k > j_k \ge \min\{k_0, k_{\varepsilon'}\}$ we have

$$r_{00}^{jlk} \le r_{0-m}^{jlk} + r_{-m-1}^{lk} + r_{-10}^{lk} < \varepsilon + 2\varepsilon^{'}, \tag{8}$$

and

$$r_{1-1}^{jlk} \le r_{10}^{jk} + r_{0-m}^{jlk} + r_{-m-1}^{lk} < 2\varepsilon^{'} + \varepsilon, \tag{9}$$

Hence by (7)(8)(9), for each $l_k > j_k \ge \min\{k_0, k_{\varepsilon'}\}$, we get

$$\begin{aligned} \frac{\varepsilon}{2} &\le \lim_{k\to\infty} \frac{r_{1-1}^{jlk}}{2} \le \lim_{k\to\infty} (1-\lambda) r_{1-1}^{jlk} \le \lim_{k\to\infty} r^M \\ &= \lim_{k\to\infty} \max\{r_{0-1}^{jlk},\ r_{10}^{jk},\ r_{0-1}^{lk},\ \lambda r_{00}^{jlk} + (1-\lambda) r_{1-1}^{jlk}\} \\ &< \max\{\varepsilon,\ \varepsilon^{'},\ \varepsilon^{'},\ \varepsilon + 2\varepsilon^{'}\} \\ &= \varepsilon + 2\varepsilon^{'}. \end{aligned}$$

Then, $\lim_{k\to\infty} r^{Mh} = \alpha \in [\frac{\varepsilon}{2}, \varepsilon]$, so

$$\lim_{k\to\infty} \phi(r^{Mh}) = \phi(\alpha) < \alpha \leq \varepsilon.$$

Now, we choose $\beta \in (\phi(\alpha), \alpha)$. Hence there exists $h_\beta \in N$ such that $r_{01}^{jkh} < \beta - \phi(\alpha)$ for $h \geq h_\beta$, and for some $h \geq h_\beta$, we obtain that

$$r_{00}^{jlkh} \leq r_{01}^{jkh} + r_{10}^{jlkh} < \beta - \phi(\alpha) + \phi(r^{Mh}) < \alpha \leq \varepsilon,$$

i.e

$$r_{00}^{jlkh} < \varepsilon$$

which contradicts with $r_{00}^{jlk} \geq \varepsilon, \ \forall k \in N$.

By the proof above, we have that $\lim_{j,l\to\infty} p(x_j^{(m)}, x_l^{(m)}) = 0$, i.e. $\{x_n^{(m)}\}_{n\in\omega}$ is Cauchy sequence in (Y, p), and then $\{x_n^{(m)}\}_{n\in\omega}$ is Cauchy sequence in (Y, p^s). In the similar way, any $\{x_n^{(i)}\}_{n\in\omega}, i = 1, 2, ..., m-1$, is Cauchy sequence in (Y, p^s). Therefore $\{x_n\}_{n\in\omega}$ is Cauchy sequence in (Y, p) and (Y, p^s).

Because Y is a closed set of (Y, p^s), thus (Y, p^s) is a complete metric space. Then $\{x_n\}$ converges to a point $z \in Y$ in (Y, p^s), by Lemma 2.3, we have

$$0 = p(z,z) = \lim_{n\to\infty} p(z, x_n) = \lim_{n,m\to\infty} p(x_n, x_m)$$

Note that $\{x_n\}$ has an infinite number of terms in A_i, $i = 1, 2, ..., m$. Therefore, in $A_i, i = 1, 2, ..., m$, we can construct a subsequence of $\{x_n\}$ that converges to z. Since any $A_i, i = 1, 2, ..., m$, is a closed subset, it is easy to conclude that $z \in \bigcap_{i=1}^m A_i$ and $\bigcap_{i=1}^m A_i \neq \emptyset$.

Now, let $Z = \bigcap_{i=1}^m A_i$. Since each $A_i, i = 1, 2, ..., m$, is a closed subset, then Z is also a closed set, and (Z, p) is complete *PMS*. Since $f|_Z$ satisfies all conditions of Theorem 3.1 and so $f|_Z$ has a unique fixed point in Z, where $f|_Z$ is a restrictive self-map on Z.

If there exists $v \in Y$ such that $v = fv$, then we have

$$p(z,v) \ = \ p(fz, fv) \leq \phi(M(z,v)) = \phi(p(z,v))$$

which contradicts with $\phi(t) < t$ for $t > 0$. This concludes the proof. □

Remark 2 If we take $\lambda = \frac{1}{2}$ in Theorem 3.1 and Theorem 3.2, then we shall obtain Theorem 3 [1] and Theorem 2.3 [2], respectively.

Remark 3 If we take $\lambda = \frac{1}{2}$ and take $\phi(t) = kt$ for $k \in [0,1)$ in Theorem 3.2, then we shall obtain corollary 2.5 [2]

Part 2. Fixed point theorem for generalized $\varphi-$weak contraction in partial metric spaces

In this part, we shall extend the Theorem 5 of T.Abdeljiawad [5] and establish the common fixed point result of four self-maps which use generalized φ-weak contractions in partial metric spaces.

Theorem 3.3 If (X,p) is complete *PMS*, and $f,g,h,q : X \to X$ satisfy: $\lambda \in [0,\frac{1}{2}]$ $\varphi \in \Theta$, $fX \subseteq qX$, $gX \subseteq hX$ and for any $x,y \in X$,

$$p(fx,gy) \leq M(x,y) - \varphi(M(x,y)), \tag{10}$$

where $M(x,y) = \max\{p(hx,qy), p(fx,hx), p(gy,qy), \lambda p(hx,gy) + (1-\lambda)p(fx,qy)\}$.

If anyone of the subsets fX, gX, hX or qX is closed subset of (X,p), then $\{f,h\}$, $\{g,q\}$ have both a point of coincidence.

Besides, if $\{f,h\}$, $\{g,q\}$ are weakly compatible, then the maps f,g,h and q have a unique common fixed point in X.

Proof First, let us construct $\{y_n\}$, $\{x_n\} \subset X$. Assume that $\forall y_0 \in X$ and $x_0 = fy_0$. Because $fX \subseteq qX$, thus let $y_1 \in X$ such that $x_0 = fy_0 = qy_1$. In the similar way, we assume that $x_1 = gy_1$, and let $y_2 \in X$ such that $x_1 = gy_1 = hy_2$. By inductive method, we have that

$$x_{2n} = fy_{2n} = qy_{2n+1} \ \ and \ \ x_{2n+1} = gy_{2n+1} = hy_{2n+2}, \ \ \forall n \in N.$$

Next, we shall prove that $\lim_{n\to\infty} p(x_n, x_{n+1}) = 0$.

Indeed, by using (10) we get

$$p(x_{2n},x_{2n+1}) = p(fy_{2n},gy_{2n+1}) \leq M(y_{2n},y_{2n+1}) - \varphi(M(y_{2n},y_{2n+1})),\ \forall n \in N \quad (11)$$

where $M(y_{2n},y_{2n+1}) = \max\{r_{-10}^{2n},\ r_{10}^{2n},\ \lambda r_{-11}^{2n} + (1-\lambda)r_{00}^{2n}\}$, and $r_{ij}^{2n} = p(x_{2n+i},x_{2n+j})$, $i,j \in \{-1,0,1\}$.

(A) if $M(y_{2n},y_{2n+1}) = r_{-10}^{2n}$, then by (11) we have

$$r_{01}^{2n} \leq r_{-10}^{2n} - \varphi(r_{-10}^{2n}) \leq r_{-10}^{2n}.$$

(B) if $M(y_{2n},y_{2n+1}) = r_{10}^{2n}$, then by (11) we have

$$r_{01}^{2n} \leq r_{10}^{2n} - \varphi(r_{10}^{2n})$$

hence $\varphi(r_{10}^{2n}) \leq 0$, $r_{01}^{2n} = 0$ and $x_{2n} = x_{2n+1}$.

(C) if $M(y_{2n},y_{2n+1}) = \lambda r_{-11}^{2n} + (1-\lambda)r_{00}^{2n}$, then

$$\begin{aligned} r_{01}^{2n} &\leq M(y_{2n},y_{2n+1}) \\ &= \lambda(r_{-11}^{2n} + r_{00}^{2n}) + (1-2\lambda)r_{00}^{2n} \\ &\leq \lambda(r_{-10}^{2n} + r_{01}^{2n}) + (1-2\lambda)r_{01}^{2n} \end{aligned}$$

i.e. $r_{01}^{2n} \leq r_{-10}^{2n}$.

Then by above $(A)(B)(C)$, we obtain

$$p(x_{2n},x_{2n+1}) \leq p(x_{2n-1},x_{2n}),\ \ \forall n \in N.$$

Analogously,

$$p(x_{2n-1},x_{2n}) \leq p(x_{2n-2},x_{2n-1}),\ \ \forall n \in N.$$

Therefore we claim that $p(x_n,x_{n+1}) \leq p(x_{n-1},x_n)$, $\forall n \in N$, and $\lim_{n\to\infty} p(x_n,x_{n+1}) = r_0$, where $r_0 \geq 0$ is a constant.

Now, we assume that n is an even, and by using (11) and taking limit of the inequality:

$$\begin{aligned} & p(x_n,x_{n+1}) = p(fy_n,gy_{n+1}) \leq M(y_n,y_{n+1}) \\ = & \max\{r^n_{0-1},\ r^n_{01},\ \lambda r^n_{-11} + (1-\lambda) r^n_{00}\} \\ \leq & \max\{r^n_{0-1},\ r^n_{01},\ \lambda (r^n_{-10} + r^n_{01} - r^n_{00}) + (1-\lambda) r^n_{00}\} \\ = & \max\{r^n_{0-1},\ r^n_{01},\ \lambda (r^n_{-10} + r^n_{01}) + (1-2\lambda) r^n_{00}\} \\ \leq & \max\{r^n_{0-1},\ r^n_{01},\ \lambda (r^n_{-10} + r^n_{01}) + (1-2\lambda) r^n_{01}\} \end{aligned}$$

where $r^n_{ij} = p(x_{n+i},x_{n+j}),\ i,j \in \{-1,0,1\}$, so we get

$$\lim_{n\to\infty} p(x_n,x_{n+1}) = \lim_{n\to\infty} M(y_n,y_{n+1}) = r_0.$$

Because $\varphi \in \Theta$, so

$$\varphi(r_0) \leq \liminf_{n\to\infty} \varphi(M(y_n,y_{n+1}))$$

Now by taking upper limits as $n \to \infty$ for the following inequality:

$$p(x_n,x_{n+1}) \leq M(y_n,y_{n+1}) - \varphi(M(y_n,y_{n+1})),$$

so, we get

$$r_0 \leq r_0 - \liminf_{n\to\infty} \varphi(M(y_n,y_{n+1})) \leq r_0 - \varphi(r_0)$$

i.e. $\varphi(r_0) \leq 0$, then $r_0 = 0$ and $\lim_{n\to\infty} p(x_n,x_{n+1}) = 0.$ (12)

Now, let us prove that $\lim_{n,m\to\infty} p(x_n,x_m) = 0$.

Clearly, we need only to prove that $\lim_{n,m\to\infty} p(x_{2n},x_{2m}) = 0$. Now suppose the contradiction, thus there exist $\varepsilon > 0$, $k_0 \in N$ and two sequence $\{2n_k\},\{2m_k\}$ in N with $n_k > m_k \geq k_0$ $(k \in N)$ such that

$$p(x_{2n_k},x_{2m_k}) > \varepsilon \qquad (*)$$

where for $k \in N$, we shall denote $2n_k$ the smallest even integer outdoing $2m_k$ such that $(*)$ holds. Hence $p(x_{2n_k-2},x_{2m_k}) \leq \varepsilon$ and $\varphi(\varepsilon) > 0$ for $\varepsilon > 0$.

Since

$$\begin{aligned} \varepsilon &< p(x_{2n_k}, x_{2m_k}) \le p(x_{2m_k}, x_{2n_k-2}) + p(x_{2n_k-2}, x_{2n_k}) \\ &\le \varepsilon + p(x_{2n_k-2}, x_{2n_k}) \end{aligned}$$

therefore by (12) and definition 2.1, we obtain that $\lim_{k\to\infty} p(x_{2n_k}, x_{2m_k}) = \varepsilon$. (13)

Similarly, letting $r^0_{ij} = p(x_{2n_k+i}, x_{2m_k+j}),\ r^1_{ij} = p(x_{2n_k+i}, x_{2n_k+j}), r^2_{ij} = p(x_{2m_k+i}, x_{2m_k+j})$, $i, j \in \{0,1,2\}$, we get

$$\begin{aligned} |r^0_{10} - r^0_{00}| &\le r^1_{01}, \quad |r^0_{10} - r^0_{11}| \le r^2_{01} \\ |r^0_{11} - r^0_{21}| &\le r^1_{12}, \quad |r^0_{20} - r^0_{21}| \le r^2_{01} \end{aligned}$$

so by (12)(13), we get

$$\begin{aligned} \lim_{k\to\infty} p(x_{2m_k}, x_{2n_k+1}) &= \varepsilon, \ \lim_{k\to\infty} p(x_{2m_k+1}, x_{2n_k+1}) = \varepsilon \\ \lim_{k\to\infty} p(x_{2m_k+1}, x_{2n_k+2}) &= \varepsilon, \ \lim_{k\to\infty} p(x_{2m_k}, x_{2n_k+2}) = \varepsilon \end{aligned} \tag{14}$$

Then, by (14) we have that $\lim_{k\to\infty} M(y_{2n_k+2}, y_{2m_k+1}) = \varepsilon$. (15)

So by (10), we have

$$p(x_{2n_k+2}, x_{2m_k+1}) = p(fy_{2n_k+2}, gy_{2m_k+1}) \le M(y_{2n_k+2}, y_{2m_k+1}) - \varphi(M(y_{2n_k+2}, y_{2m_k+1}))$$

i.e.

$$\varphi(M(y_{2n_k+2}, y_{2m_k+1})) \le M(y_{2n_k+2}, y_{2m_k+1}) - p(x_{2n_k+2}, x_{2m_k+1}) \tag{16}$$

Hence, by (14)(15), the condition that $\varphi \in \Theta$, and letting $k \to \infty$ in (16) we get

$$\varphi(\varepsilon) \le \liminf_{k\to\infty} \varphi(M(y_{2n_k+2}, y_{2m_k+1})) \le \lim_{k\to\infty} [M(y_{2n_k+2}, y_{2m_k+1}) - p(x_{2n_k+2}, x_{2m_k+1})] = 0$$

which is contradictive with the assumption. Therefore, $\lim_{n,m\to\infty} p(x_n, x_m) = 0$.

Next, we assume that $h(X)$ is closed subset of complete PMS, thus $\{x_n\}$ converges to a point $z_0 \in h(X)$. Then we get

$$\lim_{n,m\to\infty} p(x_n, x_m) = \lim_{n\to\infty} p(x_n, z_0) = p(z_0, z_0) = 0$$

Now, let $z_1 \in X$ such that $h(z_1) = z_0$. Then we obtain that $\lim_{n\to\infty} M(z_1, y_{2n+1}) = p(fz_1, z_0)$.

Because $\varphi \in \Theta$, we have

$$\varphi(p(fz_1, z_0)) \leq \liminf_{n\to\infty} \varphi(M(z_1, y_{2n+1}))$$

moreover, by taking upper limits as $n \to \infty$ for the following inequality:

$$p(fz_1, x_{2n+1}) = p(fz_1), gy_{2n+1}) \leq M(z_1, y_{2n+1}) - \varphi(M(z_1, y_{2n+1}))$$

so, we obtain that

$$p(fz_1, z_0) \leq p(fz_1, z_0) - \liminf_{n\to\infty} \varphi(M(z_1, y_{2n+1})) \leq p(fz_1, z_0) - \varphi(p(fz_1, z_0))$$

hence $\varphi(p(fz_1, z_0)) \leq 0$, $p(fz_1, z_0) = 0$ and $z_0 = fz_1 = hz_1$.

Since $fX \subseteq qX$, then let $z_2 \in X$ such that $z_0 = fz_1 = qz_2$.

Because $M(z_1, z_2) = p(gz_2, z_0)$, and

$$p(z_0, gz_2) = p(fz_1, gz_2) \leq M(z_1, z_2) - \varphi(M(z_1, z_2)) = p(gz_2, z_0) - \varphi(p(gz_2, z_0))$$

hence $\varphi(p(gz_2, z_0)) \leq 0$, $p(gz_2, z_0) = 0$ and $z_0 = gz_2 = qz_2$.

By the proof above, we conclude that $fz_1 = hz_1 = z_0 = gz_2 = qz_2$.

Next, if $\{f, h\}$, $\{g, q\}$ are weakly compatible, thus

$$fz_0 = hz_0, \qquad gz_0 = qz_0$$

Since $M(z_0, z_2) = p(fz_0, z_0)$, then by (2.3.1), we get

$$p(f(z_0), z_0) \leq M(z_0, z_2) - \varphi(M(z_0, z_2)) = p(fz_0, z_0) - \varphi(p(fz_0, z_0))$$

hence $\varphi(p(fz_0, z_0)) \leq 0$, $z_0 = fz_0 = hz_0$. Similarly, we obtain that $z_0 = gz_0 = qz_0$. Thus z_0 is a common fixed point of f, g, h, q.

Finally, let us complete the proveness of the uniqueness. If $z = fz = gz = hz = qz$, $z \in X$, then we have

$$p(fz_0, gz) \leq M(z_0, z) - \varphi(M(z_0, z) = p(z_0, z) - \varphi(p(z_0, z)),$$

i.e $\varphi(p(z_0, z)) \leq 0$. Hence $p(z_0, z) = 0$ and $z_0 = z$. □

Remark 4 In Theorem 3.3, there are conditions: (i) h and v are both identity maps and (ii) $f = g$. If (i) or $(i)(ii)$ holds, we shall establish relevant fixed point theorems of two mappings or only one mapping, respectively.

Conflict of Interests

The authors declare that there is no conflict of interests.

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