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COMPATIBLE MAPPINGS OF TYPE (A-1) AND TYPE (A-2) IN CONE METRIC SPACE

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Abstract: In this paper we introduce the concept of compatible mapping of type (A-1) and type (A-2) and prove two common fixed point theorems for two pairs of self mappings in a cone metric space without assuming its normality. Our results extend and modify the results of [11, 12, 16].

Keywords: Weakly compatible mappings, Compatible mappings, Compatible mappings type (A-1), Compatible mappings type (A-2), Common fixed point, Cone metric space.

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1. Introduction

Huang and Zhang [6] introduced the concept of cone metric space as generalisation of metric space, replacing the set of real numbers in metric space by an ordered Banach space. They also obtained some fixed point theorems in this space for mappings satisfying different contractive conditions. Subsequently, Abbas and Jungck [1], Abbas and Rhoades [3] studied common fixed point theorems in cone metric spaces. Vetro [17] proved some fixed point theorem for two self mappings satisfying a contractive condition through weak compatibility in a normal cone metric space. Amit Singh, R.C. Dimri and Sandeep Bhatt [2] proved a unique common fixed point theorem for four weakly compatible self mappings in complete cone metric spaces without using the notion of continuity. Rezapour and Hamlbarani [15]

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omit the assumption of normality in cone metric space making another dimension in developing fixed point theory in cone metric space. Non normal cone metric space is found to be used in [4, 10]. Jain *et.al.* [11, 12] also proved some fixed point theorems without using the normality of a cone metric space. In their first paper they introduced the concept of compatibility of mappings and in the second, they used two pairs of weakly compatible mappings.

The first important result in the theory of fixed point of compatible mappings was obtained by Gerald Jungck in 1986 [8] as a generalization of commuting mappings. In 1993 Jungck, Murthy and Cho [9] introduced the concept of compatible mappings of type (A) by generalizing the definition of weakly uniformly contraction maps. Pathak and Khan [13] introduced the concept of A-compatible and S-compatible by splitting the definition of compatible mappings of type (A). Pathak *et. al.* [14] renamed A-compatible and S-compatible as compatible mappings of type (A-1) and compatible mappings of type (A-2) respectively and introduced it in fuzzy metric space.

The aim of this paper is to introduce the concept of compatible mappings of type(A-1) and type(A-2) in cone metric space.

2. Preliminaries

Definition 2.1[6] Let E be a real Banach Space and P a subset of E. The set P is called a cone if and only if

(i) P is closed, non-empty and $P \neq 0$;

(ii) $a, b \in R$, $a, b \geq 0$, $x, y \in P \Rightarrow ax + by \in P$;

(iii) $P \cap (-P) = 0$.

For a given cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x << y$ will stand for $y - x \in P^0$, where P^0 denotes the interior of the set P.

Definition 2.2[6] Let X be a non-empty set. Suppose that the mapping d: $X \times X \rightarrow E$ satisfies:

(a) $0 < d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

(b) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(c) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

Example: [6] Let $E=R^2$, $P=\{(x, y) \in E : x, y \geq 0\}$, $X = R$ and d: $X \times X \rightarrow E$ defined by $d(x, y) = (|x-y|, a|x-y|)$, where $a \geq 0$ is a constant. Then (X, d) is a cone metric space.

Definition 2.3 [6] Let (X, d) be a cone metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$.

(i) $\{x_n\}$ converges to x if for every $c \in E$ with $0 << c$, there is an integer N_c such that for all $n \geq N_c$, $d(x_n, x) << c$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$, $(n \to \infty)$.

(ii) If for any $c \in E$ with $0 << c$, there is an integer N_c such that for all $n, m \geq N_c$, $d(x_n, x_m) << c$, then $\{x_n\}$ is called a Cauchy sequence in X.

(iii) If every Cauchy sequence in X is convergent in X, then X is called a complete cone metric space.

Proposition 2.4 [5] Let P be a cone in a Banach space E. If $a \in P$ and $a \leq ka$, for some $k \in [0,1)$ then, $a = 0$.

Proposition 2.5[5] Let P be a cone in a Banach space E. If for $a \in E$ and $a << c$, for all $c \in P^0$, then $a = 0$.

Proposition 2.6[11] Let (X, d) be a cone metric space and P be a cone real Banach space E. If $u \leq v$, $v << w$ then $u << w$.

Remark 2.7 [15] $\lambda P^0 \subseteq P^0$, for $\lambda > 0$ and $P^0 + P^0 \subseteq P^0$.

Lemma 2.8 [11] Let (X, d) be a cone metric space and P be a cone in a real Banach space E and $k_1, k_2, k_3, k_4, k > 0$.If $x_n \to x$, $y_n \to y$, $z_n \to z$ and $p_n \to p$ in X and

$$ka \leq k_1 d(x_n, x) + k_2 d(y_n, y) + k_3 d(z_n, z) + k_4 d(p_n, p),$$

then $a = 0$.

Definition 2.9 [1] Let A and S be self maps of a set X. If $w = Ax = Sx$, for some $x \in X$, then w is called a coincidence point of A and S.

Definition 2.10 [15] Let X be any set. A pair of self maps (A, S) in X is said to be weakly compatible if $u \in X$, $Au = Su$ imply $SAu = ASu$.

Definition 2.11[11] Let (X, d) be a cone metric space. A pair of self maps (A, S) in X is said to be compatible if for every $c \in P^0$, there is a positive integer N_c such that $d(ASx_n, SAx_n) << c$, for all $n > N_c$ whenever $\{x_n\}$ is a sequence in X such that $Ax_n \to u$ and $Sx_n \to u$ for some $u \in X$.

Proposition 2.12[11] In a cone metric space every commuting pair of self maps is compatible.

Proposition 2.13[11] In a cone metric space every compatible pair of self maps is weakly compatible.

Definition 2.14 Let(X, d) be a cone metric space. A pair of self maps (A, S) in X is said to be compatible of type (A) if for every $c \in P^0$, there is a positive integer N_c such that $d(ASx_n, SSx_n) << c$ and $d(SAx_n, AAx_n) << c$, for all $n > N_c$ whenever $\{x_n\}$ is a sequence in X such that $Ax_n \to u$ and $Sx_n \to u$ for some $u \in X$.

Definition 2.15 Let(X, d) be a cone metric space. A pair of self maps (A, S) in X is said to be compatible of type (A-1) if for every $c \in P^0$, there is a positive integer N_c such that $d(SAx_n, AAx_n) << c$, for all $n > N_c$ whenever $\{x_n\}$ is a sequence in X such that $Ax_n \to u$ and $Sx_n \to u$ for some $u \in X$.

Definition 2.16 Let(X, d) be a cone metric space. A pair of self maps (A, S) in X is said to be compatible of type (A-2) if for every $c \in P^0$, there is a positive integer N_c such that $d(ASx_n, SSx_n) << c$, for all $n > N_c$ whenever $\{x_n\}$ is a sequence in X such that $Ax_n \to u$ and $Sx_n \to u$ for some $u \in X$.

Proposition 2.17 Let S and A be self maps of a cone metric space (X, d). If the pair (S, A) is compatible type (A-1) and $Su=Au$ for some u in X then $SAu=AAu$.

Proof: Let $\{x_n\}$ be a sequence in X defined by $x_n=u$ for $n=1, 2, ...$ and let $Su=Au$. Then we have $Ax_n \to u$ and $Sx_n \to u$. Since the pair (A, S) is compatible of type (A-1) we have for every $c \in P^0$, there is a positive integer N_c such that $d(SAu, AAu) = d(SAx_n, AAx_n) << c$, for all $n > N_c$. Hence $SAu=AAu$.

Proposition 2.18 Let S and A be self maps of a cone metric space (X, d). If the pair (S, A) is compatible type (A-2) and $Su=Au$ for some u in X then $ASu=SSu$.

Proof: Let $\{x_n\}$ be a sequence in X defined by $x_n=u$ for $n=1, 2, ...$ and let $Su=Au$. Then we have $Ax_n \to u$ and $Sx_n \to u$. Since the pair (A, S) is compatible of type (A-2) we have for every $c \in P^0$, there is a positive integer N_c such that $d(ASu, SSu) = d(ASx_n, SSx_n) << c$, for all $n > N_c$. Hence $ASu=SSu$.

Proposition 2.19 Let S and A be self maps of a cone metric space (X, d). If the pair (S, A) is compatible type (A-1) and $Ax_n \to u$ and $Sx_n \to u$ for some u in X and a sequence $\{x_n\}$ in X then $AAx_n \to Su$ if S is continuous at u.

Proof: We have $d(Au, SSx_n) \le d(Au, ASx_n)+d(ASx_n, SSx_n)$

Since S is continuous at u we have $ASx_n \to Au$. Since the pair (S, A) is compatible of type (A-1) we have for every $c \in P^0$, there is a positive integer N_c such that

$$\frac{c}{2} - d(ASx_n, SSx_n), \ \frac{c}{2} - d(ASx_n, Au) \in P^0 \text{ for all } n > N_c.$$

Therefore, $c - d(ASx_n, SSx_n) - d(ASx_n, Au) \in P^0$ for all $n > N_c$.

So $d(ASx_n, SSx_n)+d(ASx_n, Au) - d(Au, SSx_n) \in P^0$ for all $n > N_c$.

$c - d(Au, SSx_n) \in P^0$ for all $n > N_c$

$d(Au, SSx_n) << c$ for all $n > N_c$.

Hence $SSx_n \to Au$.

Proposition 2.20 Let S and A be self maps of a cone metric space (X, d). If the pair (S, A) is compatible type (A-2) and $Ax_n \to u$ and $Sx_n \to u$ for some u in X and a sequence $\{x_n\}$ in X then $SSx_n \to Au$ if A is continuous at u.

Proof: We have $d(Su, AAx_n) \leq d(Su, SAx_n)+d(SAx_n, AAx_n)$

Since A is continuous at u we have $SAx_n \to Su$. Since the pair (S, A) is compatible of type (A-2) we have for every $c \in P^0$, there is a positive integer N_c such that

$$\frac{c}{2} - d(SAx_n, AAx_n),\ \ \frac{c}{2} - d(SAx_n, Su) \in P^0 \text{ for all } n > N_c.$$

Therefore, $c - d(SAx_n, AAx_n) - d(SAx_n, Su) \in P^0$ for all $n > N_c$.

So $d(SAx_n, AAx_n)+d(SAx_n, Su) - d(Su, AAx_n) \in P^0$ for all $n > N_c$.

$c - d(Su, AAx_n) \in P^0$ for all $n > N_c$

$d(Su, AAx_n) << c$ for all $n > N_c$.

Hence $AAx_n \to Su$.

3. Main results

We prove the following theorem.

Theorem 3.1: Let (X, d) be complete cone metric space with respect to a cone P contained in a real Banach space E. Let A, B, S and T be self mappings on X satisfying

(i) $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$,

(ii) pairs $\{A, S\}$ is compatible mappings of type (A-1) and $\{B, T\}$ is weakly compatible;

(iii) one of A or S is continuous,

(iv) for some $a_1, a_2, a_3, a_4 \in [0, 1)$ with $a_1 + a_2 + a_3 + 2a_4 < 1$ such that for all $x, y \in X$

$d(Ax, By) \leq a_1 d(Ax, Sx)+ a_2 d(By, Ty)+ a_3 d(Sx, Ty)+ a_4[d(Ax, Ty)+d(Sx, By)]$

then A, B, S and T have a unique common fixed point in X.

Proof: Let $x_0 \in X$ be any point in X. Define sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$Ax_{2n} = Tx_{2n+1} = y_{2n},\ Bx_{2n+1} = Sx_{2n+2} = y_{2n+1}, \text{ for all } n. \tag{2.1}$$

Proceeding as in Jain *et. al.*[12] it can be proved that $\{y_n\}$ is a Cauchy sequence in X and hence X is complete. So $\{y_n\} \to u \in X$. Hence its subsequences

$$\{Ax_{2n}\} \to u \text{ and } \{Bx_{2n+1}\} \to u \tag{2.2}$$

$$\{Sx_{2n}\} \to u \text{ and } \{Tx_{2n+1}\} \to u \tag{2.3}$$

As S is continuous, we have $SSx_{2n} \to Su$, $SAx_{2n} \to Su$

Since the pair $\{A, S\}$ is compatible of type (A-1) by proposition (1.19), we have

$$AAx_{2n} \to Su$$

Now,

$$d(Su, u) \le d(Su, AAx_{2n}) + d(AAx_{2n}, Bx_{2n+1}) + d(Bx_{2n+1}, u)$$
$$= d(Su, AAx_{2n}) + d(y_{2n+1}, u) + d(AAx_{2n}, Bx_{2n+1})$$

Again using (iv) we have

$$d(Su, u) \le d(Su, AAx_{2n}) + d(y_{2n+1}, u) + a_1 d(AAx_{2n}, SAx_{2n}) + a_2 d(Bx_{2n+1}, Tx_{2n+1})$$
$$+ a_3 d(SAx_{2n}, Tx_{2n+1}) + a_4[d(AAx_{2n}, Tx_{2n+1}) + d(SAx_{2n}, Bx_{2n+1})]$$
$$= d(Su, AAx_{2n}) + d(y_{2n+1}, u) + a_1 d(AAx_{2n}, SAx_{2n}) + a_2 d(y_{2n+1}, y_{2n})$$
$$+ a_3 d(SAx_{2n}, y_{2n}) + a_4[d(AAx_{2n}, y_{2n}) + d(SAx_{2n}, y_{2n+1})]$$
$$\le d(Su, AAx_{2n}) + d(y_{2n+1}, u) + a_1[d(AAx_{2n}, Su) + d(Su, SAx_{2n})]$$
$$+ a_2[d(y_{2n+1}, u) + d(u, y_{2n})] + a_3[d(SAx_{2n}, Su) + d(Su, u) + d(u, y_{2n})]$$
$$+ a_4[d(AAx_{2n}, Su) + d(Su, u) + d(u, y_{2n}) + d(SAx_{2n}, Su) + d(Su, u) + d(u, y_{2n+1})]$$

As $AAx_{2n} \to Su$, $SSx_{2n} \to Su$, $\{y_{2n}\} \to u$ and $\{y_{2n+1}\} \to u$, using Lemma 1.8, we have $d(Su, u) = 0$ which implies $Su = u$.

Now,

$$d(Au, Su) \le d(Au, Bx_{2n+1}) + d(Bx_{2n+1}, Su)$$
$$= d(y_{2n+1}, Su) + d(Au, Bx_{2n+1})$$

Using (iv) with $x = u$ and $y = x_{2n+1}$ we have

$$d(Au, Su) \le d(y_{2n+1}, Su) + a_1 d(Au, Su) + a_2 d(Bx_{2n+1}, Tx_{2n+1}) + a_3 d(Su, Tx_{2n+1})$$
$$+ a_4[d(Au, Tx_{2n+1}) + d(Su, Bx_{2n+1})]$$
$$= d(y_{2n+1}, Su) + a_1 d(Au, Su) + a_2 d(y_{2n+1}, y_{2n}) + a_3 d(Su, y_{2n})$$
$$+ a_4[d(Au, y_{2n}) + d(Su, y_{2n+1})]$$
$$\le d(y_{2n+1}, Su) + a_1 d(Au, Su) + a_2[d(y_{2n+1}, Su) + d(Su, y_{2n})] + a_3 d(Su, y_{2n})$$
$$+ a_4[d(Au, Su) + d(Su, y_{2n}) + d(Su, y_{2n+1})]$$

So $\quad [1 - a_1 - a_4] d(Au, Su) \le [1 + a_2 + a_4] d(y_{2n+1}, Su) + [a_2 + a_3 + a_4] d(Su, y_{2n})$.

Using $Su = u$ we have

$$[1 - a_1 - a_4] d(Au, u) \le [1 + a_2 + a_4] d(y_{2n+1}, u) + [a_2 + a_3 + a_4] d(u, y_{2n}).$$

Which implies $d(Su, u) = 0$, and we get $Au = Su = u$. Thus u is a common fixed point of the pair of maps (A, S).

As $A(X) \subseteq T(X)$, there exists $v \in X$ such that $u = Au = Tv$. So

$$u = Au = Su = Tv. \tag{2.4}$$

Taking $x = u$ and $y = v$ in (iv) we have

$d(Au, Bv) \le a_1d(Au, Su)+ a_2d(Bv, Tv)]+ a_3d(Su, Tv)+ a_4[d(Au, Tv)+d(Su, Bv)]$

$d(u, Bv) \le [a_2+ a_4]d(u, Bv)$

which implies $d(Bv, u) = 0$ and we get $Bv = u$.

Thus $Bv = Tv = u$. As the pair (B, T) is weak compatible we get $Bu = Tu$. Taking $x = u$ and $y = u$ in (iv) and using $Au = Su$, $Bu = Tu$ we get

$d(Au, Bu) \le [a_3 + 2a_4]d(Au, Bu)$

Hence $Au = Bu$, thus we have $u = Au = Su = Bu = Tu$.

Thus u is a common fixed point of four self maps A, B, S and T.

If A is continuous we can prove the same result.

In order to prove the uniqueness of fixed point, if possible let $w =Aw = Bw = Sw =Tw$. be another common fixed point of the four self maps. Taking $x = w$ and $y =u$ in (iv) we get

$d(Aw, Bu) \le a_1d(Aw, Sw)+ a_2d(Bu, Tu)]+ a_3d(Sw, Tu)+ a_4[d(Aw, Tu)+d(Sw, Bu)]$

$= a_1d(w, w)+ a_2d(u, u)]+ a_3d(w, u)+ a_4[d(w, u) +d(w, u)]$

Implies $d(w, u) \le [a_3 + 2a_4]d(w, u)$.

Hence $u = w$. Thus the four self maps A, B, S and T have a unique common fixed point. This completes the proof.

Theorem 3.2: Let (X, d) be complete cone metric space with respect to a cone P contained in a real Banach space E. Let A, B, S and T be self mappings on X satisfying

(i) $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$,

(ii) pairs $\{A, S\}$ is compatible mappings of type (A-2) and $\{B, T\}$ is weakly compatible;

(iii) one of A or S is continuous,

(iv) for some $a_1, a_2, a_3, a_4 \in [0, 1)$ with $a_1+a_2 + a_3+2a_4 < 1$ such that for all $x, y \in X$

$d(Ax, By) \le a_1d(Ax, Sx)+ a_2d(By, Ty)+ a_3d(Sx, Ty)+ a_4[d(Ax, Ty)+d(Sx, By)]$

then A, B, S and T have a unique common fixed point in X.

Proof: Let $x_0 \in X$ be any point in X. Define sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$Ax_{2n} = Tx_{2n+1} = y_{2n},\ Bx_{2n+1}= Sx_{2n+2} = y_{2n+1}, \text{ for all } n \tag{2.5}$$

Proceeding as in Jain *et. al*.[12] it can be proved that $\{y_n\}$ is a Cauchy sequence in X and hence X is complete. So $\{y_n\} \to u \in X$. Hence its subsequences

$$\{Ax_{2n}\} \to u \text{ and } \{Bx_{2n+1}\} \to u \tag{2.6}$$

$$\{Sx_{2n}\} \to u \text{ and } \{Tx_{2n+1}\} \to u \tag{2.7}$$

As A is continuous, we have $AAx_{2n} \to Au$, $ASx_{2n} \to Au$

As the pair $\{A, S\}$ is compatible of type (A-2) by proposition (1.20), we have $SSx_{2n} \to Au$

Now, $d(Au, u) \le d(Au, ASx_{2n}) + d(ASx_{2n}, Bx_{2n+1}) + d(Bx_{2n+1}, u)$

$$= d(Au, ASx_{2n}) + d(y_{2n+1}, u) + d(ASx_{2n}, Bx_{2n+1})$$

Again using (iv) we have

$$\begin{aligned} d(Au, u) &\le d(Au, ASx_{2n}) + d(y_{2n+1}, u) + a_1 d(ASx_{2n}, SSx_{2n}) + a_2 d(Bx_{2n+1}, Tx_{2n+1}) \\ &\quad + a_3 d(SSx_{2n}, Tx_{2n+1}) + a_4[d(ASx_{2n}, Tx_{2n+1}) + d(SSx_{2n}, Bx_{2n+1})] \\ &= d(Au, ASx_{2n}) + d(y_{2n+1}, u) + a_1 d(ASx_{2n}, SSx_{2n}) + a_2 d(y_{2n+1}, y_{2n}) \\ &\quad + a_3 d(SSx_{2n}, y_{2n}) + a_4[d(ASx_{2n}, y_{2n}) + d(SSx_{2n}, y_{2n+1})] \\ &\le d(Au, ASx_{2n}) + d(y_{2n+1}, u) + a_1[d(ASx_{2n}, Au) + d(Au, SSx_{2n})] \\ &\quad + a_2 [d(y_{2n+1}, u) + d(u, y_{2n})] + a_3[d(SSx_{2n}, Au) + d(Au, u) + d(u, y_{2n})] \\ &\quad + a_4[d(ASx_{2n}, Au) + d(Au, u) + d(u, y_{2n}) + d(SSx_{2n}, Au) + d(Au, u) + d(u, y_{2n+1})] \end{aligned}$$

As $SSx_{2n} \to Au$, $AAx_{2n} \to Au$, $\{y_{2n}\} \to u$ and $\{y_{2n+1}\} \to u$, using Lemma 1.8, we have $d(Au, u) = 0$, and we get $Au = u$.

Now,

$$\begin{aligned} d(Au, Su) &\le d(Au, Bx_{2n+1}) + d(Bx_{2n+1}, Su) \\ &= d(y_{2n+1}, Su) + d(Au, Bx_{2n+1}) \end{aligned}$$

Using (iv) we have

$$\begin{aligned} d(Au, Su) &\le d(y_{2n+1}, Su) + a_1 d(Au, Su) + a_2 d(Bx_{2n+1}, Tx_{2n+1}) + a_3 d(Su, Tx_{2n+1}) \\ &\quad + a_4[d(Au, Tx_{2n+1}) + d(Su, Bx_{2n+1})] \\ &= d(y_{2n+1}, Su) + a_1 d(Au, Su) + a_2 d(y_{2n+1}, y_{2n}) + a_3 d(Su, y_{2n}) \\ &\quad + a_4[d(Au, y_{2n}) + d(Su, y_{2n+1})] \\ &\le d(y_{2n+1}, Su) + a_1 d(Au, Su) + a_2[d(y_{2n+1}, Su) + d(Su, y_{2n})] + a_3 d(Su, y_{2n}) \\ &\quad + a_4[d(Au, Su) + d(Su, y_{2n}) + d(Su, y_{2n+1})] \end{aligned}$$

So $[1 - a_1 - a_4] d(Au, Su) \le [1 + a_2 + a_4] d(y_{2n+1}, Su) + [a_2 + a_3 + a_4] d(Su, y_{2n})$.

Using $Au = u$ we have

$$[1 - a_1 - a_4] d(u, Su) \le [1 + a_2 + a_4] d(y_{2n+1}, u) + [a_2 + a_3 + a_4] d(u, y_{2n}).$$

So $d(Su, u) = 0$, and we get $Au = Su = u$. Thus u is a common fixed point of the pair of maps (A, S).

As $A(X) \subseteq T(X)$, there exists $v \in X$ such that $u = Au = Tv$. So

$$u = Au = Su = Tv. \tag{2.8}$$

Taking $x = u$ and $y = v$ in (iv) we have

$$d(Au, Bv) \le a_1 d(Au, Su) + a_2 d(Bv, Tv)] + a_3 d(Su, Tv) + a_4[d(Au, Tv) + d(Su, Bv)]$$

$$d(u, Bv) \le [a_2 + a_4] d(u, Bv)$$

So $d(Bv, u) = 0$ and we get $Bv = u$.

Thus $Bv = Tv = u$. As the pair (B, T) is weak compatible we get $Bu = Tu$. Taking $x = u$ and $y=u$ in (iv) and using $Au = Su$, $Bu = Tu$ we get

$$d(Au, Bu) \leq [a_3 + 2a_4]d(Au, Bu)$$

So $Au = Bu$ and hence we have $u = Au = Su = Bu = Tu$.

Thus u is a common fixed point four self maps A, B, S and T in this case.

If S is continuous we can prove the same result.

In order to prove the uniqueness of fixed point, if possible let $w =Aw = Bw = Sw =Tw$ be another common fixed point of the four self maps. Taking $x = w$ and $y =u$ in (iv) we get

$$d(Aw, Bu) \leq a_1d(Aw, Sw)+ a_2d(Bu, Tu)]+ a_3d(Sw, Tu)+ a_4[d(Aw, Tu)+d(Sw, Bu)]$$
$$= a_1d(w, w)+ a_2d(u, u)]+ a_3d(w, u)+ a_4[d(w, u) +d(w, u)]$$

implies

$$d(w, u) \leq [a_3 + 2a_4]d(w, u).$$

Hence $u = w$. Thus the four self maps A, B, S and T have a unique common fixed point. This completes the proof.

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