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THE MONOCHROMATIC CONNECTIVITY OF 3-CHROMATIC GRAPHS

YIRONG YANG[1,*], HUAPING WANG[2]

[1]Department of Mathematics, Zhejiang Normal University, Jinhua 321004, P.R. China

[2]Department of Mathematics, Jiangxi Normal University, Nanchang 330022, P.R. China

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Abstract. In this paper, we solve completely the monochromatic connectivity of 3-chromatic graphs.

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1. Introduction

An edge-coloring of a connected graph is a *monochromatically connecting* coloring (MC-coloring, for short) if there is a monochromatic path joining any two vertices. The *monochromatic connection number* of a graph G, denoted by $mc(G)$, is defined to be the maximum number of colors used in an MC-coloring of a graph G. As proved in [3], an important property of an extremal MC-coloring(a coloring that use $mc(G)$ colors) is that each color forms a tree. For a color c, let T_c be the tree whose edges colored c. The color c is nontrivial if T_c has at least two edges. Otherwise c is trivial. A nontrivial color tree with m edges is said waste $m-1$ colors. For any two nontrivial colors b and c, the corresponding trees T_b and T_c intersect in at most one vertex [3]. Such an extremal coloring is called simple. Every connected graph has

*Corresponding author
E-mail address: 15558699651@163.com
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a simple extremal MC-coloring[3]. These concepts were introduced by Caro and Yuster in [3] and they gave some upper and lower bounds for $mc(G)$ characterized by other graph parameters. A straightforward lower bounds for $mc(G)$ is $m-n+2$(throughout this paper, n and m denote the number of vertices and edges respectively), which can be verified by coloring the edges of a spanning tree with one color, and coloring the remaining edges by new distinct colors.

Now we present some definitions and notations necessary. For a graph G, we use $V(G)$, $E(G)$, $|E(G)|$, $|V(G)|$ to denote the vertex set, edge set, number of vertices, number of edges of G, respectively. Given a graph G and $D \subseteq V(G)$, let $|D|$ be the number of vertices in D and $G[D]$ be the subgraph of G induced by D. If $\chi(G)=k$, then G is k-chromatic.

2. Preliminaries

Let $V_i, i=1,2,3$, be the vertex parts of the graph K_{n_1,n_2,n_3}. Let $E_{i,j}$ be the set of edges between V_i and $V_j, i,j \in \{1,2,3\}, i \neq j$. Let E_0 be a subset of $E(K_{n_1,n_2,n_3})$.

Lemma 2.1. [3] *If G is K_3-free, then $mc(G)=m-n+2$.*

Lemma 2.2. [3] *Any graph G satisfies $mc(G) \leq m-n+\chi(G)$.*

The join of two disjoint graphs G and H, denoted by $G+H$, is defined to be the graph $\overline{\overline{G}+\overline{H}}$.

Lemma 2.3. [4] *Let G be the join of two disconnected graphs G_1 and G_2. Then $mc(G)=|E(G)|-|V(G)|+2$.*

Moreover we have the following properties of the simple extremal MC-coloring.

Lemma 2.4. *If G is a connected spanning subgraph of some graph H, then $mc(G) \leq mc(H)-(|E(H)|-|E(G)|)$.*

Proof. It is clear that G has a simple extremal MC-coloring. Let f be an MC-coloring of G realizing $mc(G)$. Let the remaining $|E(H)|-|E(G)|$ edges of H receive trivial colors. Then we get an MC-coloring, denoted by f' of H. Clearly, f' is simple and it use $mc(G)+(|E(H)|-|E(G)|)$ colors. Then $mc(H) \geq mc(G)+(|E(H)|-|E(G)|)$, i.e., $mc(G) \leq mc(H)-(|E(H)|-|E(G)|)$, and we are done. The proof is completed.

Lemma 2.5. If G is a connected spanning subgraph of some graph H and let $mc(G)=m(G)-n(G)+k_1$, $mc(H)=m(H)-n(H)+k_2$, then $k_1 \leq k_2$.

Proof. By Lemma 2.4, it implies that $mc(G) \leq mc(H) - (|E(H)| - |E(G)|)$. Since G is a spanning subgraph of graph H, $n(G) = n(H)$. And we have $m(G) - n(G) + k_1 \leq m(H) - n(H) + k_2 - (|E(H)| - |E(G)|)$, i.e., $m(G) - n(G) + k_1 \leq m(G) - n(G) + k_2$. Hence we get that $k_1 \leq k_2$, and we are done. The proof is completed.

3. Main results

Lemma 3.1. *Let V_i, $i = 1,2,3$ be the vertex parts of the graph K_{n_1,n_2,n_3}. Let $G = K_{n_1,n_2,n_3} - \{uv, xy\}, u \in V_1, v, x \in V_2, y \in V_3$. Then $mc(G) = m - n + 2$*

Proof. The lower bound $mc(G) \geq m - n + 2$ is obvious and we only need to show $mc(G) \leq m - n + 2$.

Let f be a simple extremal MC-coloring of G. Suppose that f consists of k nontrivial color trees, denoted by $T_1, ..., T_k$, where $t_i = |V(T_i)|$. As T_i has $t_i - 1$ edges, it wastes $t_i - 2$ colors. Hence it suffices to prove that $\sum_{i=1}^{k} (t_i - 2) \geq n - 2$.

Case 1. Every vertex appears in at least two distinct nontrivial color trees.

In this case we have $\sum_{i=1}^{k} t_i \geq 2n$. So if $k \leq n/2 + 1$, we have $\sum_{i=1}^{k} (t_i - 2) \geq 2n - 2k \geq n - 2$, and we are done. So let $k > n/2 + 1$. Now we claim that we still have $\sum_{i=1}^{k} (t_i - 2) \geq n - 2$ when $k > n/2 + 1$. Since T_i can monochromatically connect at most $\binom{t_i - 1}{2}$ pairs of non-neighbors in G, we have $\sum_{i=1}^{k} \binom{t_i - 1}{2} \geq |E(G)| = \sum_{i=1}^{3} \binom{n_i}{2} + 2$.

Assume that $\sum_{i=1}^{k} (t_i - 2) < n - 2$. Since T_i is nontrivial, $t_i - 1 \geq 2$. By the straightforward convexity, the expression $\sum_{i=1}^{k} \binom{t_i - 1}{2}$, subject to $t_i - 1 \geq 2$, is maximized when $k - 1$ of the $t_i's$ equal 3 and one of the $t_i's$, say t_k, is as large as it can be, namely $t_k - 1$ is the largest integer smaller than $n - 2 + k - 2(k-1) = n - k$. Hence, $t_k - 1 = n - k - 1$. We have $\sum_{i=1}^{k} \binom{t_i - 1}{2} \leq k - 1 + \binom{n-k-1}{2}$.

Note that $g(k) = k - 1 + \binom{n-k-1}{2}$ is a decreasing function of k for $n/2 + 1 < k \leq n - 3$ and then $g(k) < g(n/2+1)$. Note that $\sum_{i=1}^{3} \binom{n_i}{2} + 2 - g(n/2+1) > 0$. This implies that $g(k) < g(n/2+1) < |E(G)|$, i.e., $\sum_{i=1}^{k} \binom{t_i - 1}{2} < |E(G)| = \sum_{i=1}^{3} \binom{n_i}{2} + 2$, a contradiction. Hence $\sum_{i=1}^{k} (t_i - 2) \geq n - 2$ and we are done.

Case 2. There are vertices that appear in unique nontrivial color trees.

Denote by S the vertices that appear in the unique nontrivial color trees. Note that u,v or x,y are monochromatically connected by a nontrivial color tree. So let T_u, T_x monochromatically connect u,v and x,y, respectively.

Subcase 2.1 $S\bigcap V_1, S\bigcap V_2, S\bigcap V_3 \neq \phi$.

Notice that vertices of the same part are not adjacent in G and any two of each part are monochromatically connected in a nontrivial color tree. So all the vertices of the same part must lie in a nontrivial color tree. So we can assume that $V_i \subseteq T_i, i=1,2,3$, and we have that $V_i \cap V(T_i) \cap S \neq \emptyset$.

Sub-subcase 2.1.1 $T_1 \neq T_2 \neq T_3 \neq T_1$.

Suppose that T_u , $T_x \notin \{T_1, T_2, T_3\}$. Since $V_i \subseteq V(T_i)$, we have that $t_i \geq n_i+1$, i.e., $t_i - 2 \geq n_i - 1$, $i=1,2,3$. That is to say that T_i waste at least $n_i - 1$ edges, $i=1,2,3$. Also, both T_u and T_x waste at least one edge. So the total waste of the coloring f is at least $n-2$ and we are done.

Suppose that $T_u \in \{T_1, T_2, T_3\}$ or $T_x \in \{T_1, T_2, T_3\}$. Without loss of generality, let $T_u = T_1$. Since $uv \notin E(G)$ and vertices of V_1 are not adjacent, then T_1 contains at least anther vertex besides v and vertices of V_1. It implies that $t_1 \geq n_1+2$, i.e., $t_1 - 2 \geq n_1$. Similarly, $V_i \subseteq V(T_i), i=2,3$, and we have that $t_i \geq n_i + 1$, i.e., $t_i - 2 \geq n_i - 1$ for $i=2,3$. So the total waste of T_1, T_2, T_3 is at least $n-2$ and we are done.

Sub-subcase 2.1.2 There are two trees in $\{T_1,\ T_2,\ T_3\}$ which are same.

Let $T_1 = T_2 \neq T_3$, now we have $V_1 \cup V_2 \subseteq V(T_1)$. Suppose that $y \in V(T_1)$. Then the waste of T_1 is at least $n_1 + n_2 - 1$. Clearly, $t_3 \geq n_3 + 1$, i.e., $t_3 - 2 \geq n_3 - 1$. Hence the total waste of the coloring f is at least $n-2$ and we are done. Suppose that $y \in V_3 - V(T_1)$. Then the waste of T_1 is at least $n_1 + n_2 - 2$ and $T_x \neq T_1$. This implies that $T_x = T_3$. Then $t_3 \geq n_3 + 2$, i.e., $t_3 - 2 \geq n_3$. Hence the total waste of the coloring f is at least $n-2$ and we are done. By the symmetry, if $T_2 = T_3 \neq T_1$, then the total waste of the coloring f is at least $n-2$ and we are done.

Let $T_1 = T_3 \neq T_2$, now we have $V_1 \cup V_3 \subseteq V(T_1)$. Suppose that $v \notin V(T_1)$ or $x \notin V(T_1)$. Without loss of generality, let $v \notin V(T_1)$, then $T_u = T_2$. It implies that $t_2 \geq n_2 + 2$, i.e., $t_2 - 2 \geq n_2$. Clearly, $t_1 \geq n_1 + n_3$, i.e., $t_1 - 2 \geq n_1 + n_3 - 2$. Hence the total waste of the coloring f is at least $n-2$ and we are done. Suppose that $v,x \in V(T_1)$. Since f is simple and $x,y \in V(T_2)$, we have that

$v = x$. Then $t_1 \geq n_1 + n_3 + 1$, i.e., $t_1 - 2 \geq n_1 + n_3 - 1$. Clearly, $t_2 \geq n_2 + 1$, i.e., $t_2 - 2 \geq n_2 - 1$. Hence the total waste of the coloring f is at least $n-2$ and we are done.

Sub-subcase 2.1.3 $T_1 = T_2 = T_3$.

Since $S \bigcap V_1 \bigcap V_2 \bigcap V_3 \neq \phi$, the tree T_1 is a spanning tree of G. So the waste of T_1 is $n-2$ and so we are done.

Subcase 2.2 The set S is exactly joint with two partite sets of G.

Here we only present the proof details of the case $S \cap V_1 \neq \emptyset$, $S \cap V_2 \neq \emptyset$. The other two cases can be proved similarly. Clearly, we can assume that $V_i \subseteq V(T_i), i = 1, 2$.

Assume that $T_1 = T_2$. Then we have that $V_1 \cup V_2 \subseteq V(T_1)$. Suppose that $y \in V(T_1)$. Since T_1 is not a spanning tree of G, there is a vertex $v_3 \in V_3 - V(T_1)$. Clearly, $v_3 y \notin E(G)$. Let T_{v_3} be the nontrivial color tree monochromatically connecting v_3, y. Since V_3 is an independent set in G, we have that $|V(T_{v_3}) \cap (V_1 \cup V_2)| \geq 1$. This implies that $|V(T_{v_3}) \cap V(T_1)| \geq 2$, a contradiction. Suppose that $y \notin V(T_1)$. Since $xy \notin E(G)$ and V_3 is an independent set in G, this means that $|V(T_x) \cap (V_1 \cup V_2)| \geq 2$, i.e., $|V(T_x) \cap V(T_1)| \geq 2$, a contradiction. So $T_1 \neq T_2$. Now we claim that $\sum\limits_{i=1}^{k}(t_i - 2) \geq n-2$. Since we have that $S \cap V_3 = \emptyset$, each vertex of V_3 appears in at least two nontrivial color trees. In order to monochromatically connect the $\binom{|V_3|}{2}$ distinct pairs of vertices of V_3, we need a set of nontrivial color trees, say $T_s, .., T_q$, and each $T_i, i = s, \cdots, q$ contains at least two vertices of V_3.

Suppose that $|V(T_1) \cap V_3| \geq 2$ and $|V(T_2) \cap V_3| \geq 2$, and let $w_1, w_2 \in V(T_1) \cap V_3$, $z_1, z_2 \in V(T_2) \cap V_3$. Notice that $|V(T_1) \cap V(T_2)| \leq 1$. Let $w_1 \in V(T_1) \cap V_3 - V(T_2)$ and $z_1 \in V(T_2) \cap V_3 - V(T_1)$. Since $w_1 z_1 \notin E(G)$, we have w_1, z_1 lie in a nontrivial color tree and let T_s be such nontrivial color tree in f. Since V_3 is an independent set, we have that $|V(T_s) \cap (V_1 \cup V_2)| \geq 1$. This implies that $V(T_s) \cap V_1 \neq \emptyset$ or $V(T_s) \cap V_2 \neq \emptyset$. Along with $w_1 \in V(T_s) \cap V(T_1)$ and $z_1 \in V(T_s) \cap V(T_2)$, we have that $|V(T_1) \cap V(T_s)| \geq 2$ or $|V(T_2) \cap V(T_s)| \geq 2$, a contradiction.

Suppose that $|V(T_1) \cap V_3| < 2$ and $|V(T_2) \cap V_3| < 2$, then $T_1, T_2 \notin \{T_s, \ldots, T_q\}$. It is clear that $t_i \geq n_i + 1$, i,e., $t_i - 2 \geq n_i - 1$ for $i = 1, 2$. Notice that $t_i \geq 3$, i.e., $t_i - 2 \geq 1$, for $i = s, \ldots, q$. If $q - s + 1 \geq n_3$, then we have $\sum\limits_{i=s}^{q}(t_i - 2) \geq q - s + 1 \geq n_3$. Hence we get that $\sum\limits_{i=1}^{k}(t_i - 2) = (\sum\limits_{i=s}^{q}(t_i - 2)) + n_1 + n_2 - 2 \geq n - 2$ and we are done.

So let $q-s+1<n_3$. Since $V_3\subset\cup_{i=s}^{q}V(T_i)$ and each vertex of V_3 appears in at least two distinct nontrivial color trees, every vertex of V_3 is covered by at least two edges of $T_s,..,T_q$ and each such edge in G exactly covers one vertex of V_3. So, the total number of edges of $T_s,..,T_q$ is at least $2n_3$ and we have $\sum_{i=s}^{q}(t_i-1)\ge 2n_3$, i.e., $\sum_{i=s}^{q}(t_i-2)=\sum_{i=s}^{q}(t_i-1)-(q-s+1)>n_3$. Hence $\sum_{i=1}^{k}(t_i-2)=(\sum_{i=s}^{q}(t_i-2))+n_1+n_2-2>n-2$ and we are done.

Suppose that $|V(T_1)\cap V_3|<2$ or $|V(T_2)\cap V_3|<2$. Without loss of generality, let $|V(T_1)\cap V_3|\ge 2$ and $|V(T_2)\cap V_3|<2$. Then we have that $T_1\in\{T_s,..,T_q\}$ and $T_2\notin\{T_s,..,T_q\}$. It is clear that $t_1\ge n_1+2$, i.e., $t_1-2\ge n_1$ and that $t_2\ge n_2+1$, i,e., $t_2-2\ge n_2-1$. Notice that $t_i\ge 3$, i.e., $t_i-2\ge 1$, for $i=s,\dots,q$ and $t_1-2\ge n_1$. If $q-s+1\ge n_3$, then we have $\sum_{i=s}^{q}(t_i-2)\ge n_1+q-s\ge n_1+n_3-1$. Hence $\sum_{i=1}^{k}(t_i-2)\ge n-2$ and we are done.

So let $q-s+1<n_3$. Notice that each $\{T_s,..,T_q\}\backslash\{T_1\}$ contains at least a vertex out of V_3. So the sum of the orders of $\{T_s,..,T_q\}$ is at least $2n_3+n_1+q-s$. This implies that $\sum_{i=s}^{q}(t_i-1)\ge 2n_3+n_1-1$, i.e., $\sum_{i=s}^{q}(t_i-2)=\sum_{i=s}^{q}(t_i-1)-(q-s+1)>n_3+n_1-1$. Hence $\sum_{i=1}^{k}(t_i-2)=(\sum_{i=s}^{q}(t_i-2))+n_2-1>n-2$ and we are done.

Sub-case 2.3 The set S is exactly joint with one partite set of G.

Without loss of generality, let $S\cap V_1\neq\emptyset, S\cap V_2=\emptyset, S\cap V_3=\emptyset$, then $V_1\subseteq V(T_1)$ and each vertex of $V_2\cup V_3$ appears in at least two distinct nontrivial color trees. Let $T_2,\dots,T_k$ be the nontrivial color trees which monochromatically connect all vertices of $V_2\cup V_3$. Then each T_i contains at least two vertices of $V_2\cup V_3$ for $2\le i\le k$.

Suppose that $|V(T_1)\cap(V_2\cup V_3)|<2$. Then $T_1\notin\{T_2,...,T_k\}$. It is clearly that every vertex of $V_2\cup V_3$ appears in at least two distinct nontrivial color trees. By the same way as case 1, we we can deduce that $\sum_{i=2}^{k}(t_i-2)\ge n_2+n_3-1$. Since $V_1\subseteq V(T_1)$, we have that $t_1\ge n_1+1$, i.e., $t_1-2\ge n_1-1$. Hence $\sum_{i=1}^{k}(t_i-2)\ge n-2$ and we are done.

Suppose that $|V(T_1)\cap(V_2\cup V_3)|\ge 2$. Then $T_1\in\{T_2,...,T_k\}$. Now we still claim that $\sum_{i=1}^{k}(t_i-2)\ge n-2$. Recall that we have $\sum_{i=2}^{k}(t_i-2)\ge n_2+n_3-1$ for $T_1\notin\{T_2,...,T_k\}$. But now $T_1\in\{T_2,...,T_k\}$ and $V_1\subset V(T_1)$, then T_1 will have other n_1-1 edges of $E(G)$ such that all vertices of V_1 are monochromatically connected. That is to say that $\sum_{i=1}^{k}(t_i-2) \geq n-2 = \sum_{i=2}^{k}(t_i-2) \geq n_2+n_3-1+n_1-1 = n-2$ for this case, and we are done.

The proof is completed.

Theorem 3.2. *Let G be a connected 3-chromatic spanning subgraph of K_{n_1,n_2,n_3} with partite sets $V_i, |V_i| = n_i, i = 1,2,3$. If $G = K_{n_1,n_2,n_3} - E_0$ with $E_0 \cap E_{i,j} \neq \emptyset$ and $E_0 \cap E_{j,k} \neq \emptyset$, $\{i,j,k\} = \{1,2,3\}$, then $mc(G) = m-n+2$.*

Proof. The lower bound $mc(G) \geq m-n+2$ is obvious and we only need to show $mc(G) \leq m-n+2$. It is clearly that G is a connected spanning subgraph of $K_{n_1,n_2,n_3} - \{uv, xy\}$ for some $uv \in E_{i,j}$ and $xy \in E_{j,k}$. By Lemmas 2.5-3.1, we have that $mc(G) \leq m-n+2$ and we are done.

The proof is completed.

Theorem 3.3. *Let G be a connected 3-chromatic spanning subgraph of K_{n_1,n_2,n_3} with partite sets $V_i, |V_i| = n_i, i = 1,2,3$. If $G = K_{n_1,n_2,n_3} - E_0, E_0 \subset E_{i,j}, \{i,j\} \subset \{1,2,3\}$ such that $G[V_i,V_j]$ is disconnected, then $mc(G) = m-n+2$.*

Proof. Without loss of generality, we assume that $i=1, j=2$. Then $E_0 \subset E_{1,2}$ and $G[V_1,V_2]$ is disconnected. Let $G_1 = G[V_1,V_2]$ and $G_2 = G[V_3]$. So $G = G_1 + G_2$. Notice that both G_1 and G_2 are disconnected. Hence, from Lemma 2.3 we have that $mc(G) = m(G) - n(G) + 2$, and we are done.

The proof is completed.

Theorem 3.4. *Let G be a connected 3-chromatic spanning subgraph of K_{n_1,n_2,n_3} with partite sets $V_i, |V_i| = n_i, i = 1,2,3$. Let $G = K_{n_1,n_2,n_3} - E_0$. Then $mc(G) = m-n+3$ if and only if $E_0 \subseteq E_{i,j}$ and $G[V_i,V_j]$ is still connected for some $i,j \in [3]$.*

Proof. Now we show the necessity of this proof. Let $mc(G) = m-n+3$. We show that $E_0 \subseteq E_{i,j}$ and $G[V_i,V_j]$ is still connected for some $i,j \in [3]$. Suppose that E_0 is not a subset of $E_{i,j}$ for any $i,j \in [3]$. This implies that $E_0 \cap E_{i,j} \neq \emptyset, E_0 \cap E_{j,k} \neq \emptyset, \{i,j,k\} = \{1,2,3\}$. Then it follows from Theorem 3.2 that $mc(G) = m-n+2$, a contradiction. So $E_0 \subseteq E_{i,j}$for some $i,j \in [3]$. Suppose that $G[V_i,V_j]$ is disconnected. Then it follows from Theorem 3.3 that $mc(G) = m-n+2$, a contradiction and we are done.

The sufficiency of this proof can be proved by coloring the spanning tree of $G[V_i,V_j]$ with a color c_1 and One vertex from $V_i \cup V_j$ is adjacent to all vertices of V_k by a color c_2, where $k \neq i, j$ and $k \in [3]$. The remaining edges of G receive trivial colors. Then we get an simple extremal MC-coloring, say f of G. Clearly, f contains $m(G)-n(G)+3$ colors and we are done.

The proof is completed.

Conflict of Interests

The authors declare that there is no conflict of interests.

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