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ON CONSTRUCTION OF BARNES-WALL LATTICES

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Abstract: This paper presents the construction for decomposable Barnes-Wall principal sublattices (BWPSL) especially in rotated version, based on augmenting the product lattices involving lower dimensional component lattices from the same family. Particularly, a product lattice is constructed for which no additional augmenting is required. Certain properties of parameters including minimum distance and coding gain of augmented lattice are derived with a few examples.

Keywords: Barnes-Wall principal sublattice (BWPSL), product lattice (PL), Reed-Muller code, augmented product construction (AP-construction), expanded generator matix, lattice holes.

2000 AMS Subject Classification: 68P30

I. Introduction

In the 1950'^s and 1960'^s mathematicians developed some constructive dense sphere packings based on lattices in spaces of moderate to high dimensional namely, 2^n-dimensional Barnes-Wall lattices [1] and ultra dense 24-dimensional Leech lattice [2]. Forney [8], [12] exposed the outstanding squaring construction of Barnes-Wall lattices, their principal sublattices and their rotated versions with the code formulas in real and complex forms. Salomon and Amrani [11] found some product lattices which are easy to encode decode, yet the dimensions of these product lattices are typically smaller than the good binary lattices of similar minimum distance. Goldberg [5], Rao and Reddy [6], Peng and Farrell [7] discussed a technique called

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augmentation of original product code with several of its cosets or (AP construction) to solve this problem such that the minimum distance is not affected or marginally reduced. Salomon and Amrani [9], extended this AP construction to lattices and described it using a particular matrix representation of component lattices. They also have shown that BWPSL can be represented as a union of cosets of a product of two lower dimensional lattices from the same family.

The principal purpose of this paper is to describe that the product of rotated versions of two distinct lattices from Barnes-Wall family is a sublattice of higher dimensional lattice of the same family. Significantly, the higher dimensional lattice can be represented as a union of cosets of the original product lattice. More precisely, it is shown that the real and complex form of aforementioned product gives a sublattice of rotated version and of rotated square version of higher dimensional Barnes-Wall principal sublattice. Particularly, a product lattice involving rotated versions of two distinct (m^{th} and n^{th}) members of Barnes-Wall family in complex form is again a higher dimensional member of the same family which is a promising construction in this paper. The description comprises an expanded generator matrix representation of lattices and their products. Expanded generator matrices are built using code formulas [11] however, we have presented a slightly modified form of code formulas of rotated versions given by Forney [12] to properly account for the coefficient of Reed-Muller codes. Indeed, it is proved that minimum distance remains invariant for each product lattice and coding gain of augmented lattice, as compared to the PL, is improved. For completeness, we give some illustrations in tabular form embracing a few known lattices, their products along with the parameters.

II. Preliminaries

The Barnes-Wall lattices are a family of 2^n-dimensional binary lattices. The n^{th} member of this family $\Lambda(0,n)$ may be regarded as a 2^{n+1}-dimensional real lattice or 2^n-dimensional complex lattice. The Barnes-Wall lattices are decomposable with code formulas that involve the code family of Reed-Muller codes which when considered of a given length are nested in the sense that RM(n,n)/RM(n,n-1)/...../RM(0,n) is a code partition chain [8]. The principal sublattices of Barnes-Wall lattices are a family of lattices $\Lambda(r,n),\ 0 \le n,\ 0 \le r \le n,$ which may be defined as decomposable 2^n-dimensional complex lattice and 2^{n+1}-dimensional real lattice with the following code formulas [12].

Complex formula

$$\Lambda_c(r,n) = \phi^{(n-r)} G^N + \sum_{r \le r' \le n} RM(r',\ n)\ \phi^{(r'-r)} \tag{1}$$

Real formulas

For $n-r$ even

$$\Lambda_r(r,n) = 2^{(n-r)/2} Z^{2N} + \sum_{r+1 \le r' \le n,\ n-r'\ even} RM(r',\ n+1)\ 2^{(n-r')/2} \tag{2}$$

For $n-r$ odd

$$\Lambda_r(r,n) = 2^{(n-r+1)/2} Z^{2N} + \sum_{r+1 \le r' \le n,\ n-r'\ even} RM(r',\ n+1)\ 2^{(n-r')/2} \tag{3}$$

Further, the rotated version of Barnes-Wall principal sublattices $R\Lambda(r,n)$ are easily accomplished by operating the 2N-dimensional rotation operator *R*, defined in [8] on the principal sublattices $\Lambda(r,n)$. We provide a slightly modified form of the code formulas given by Forney in [12] of these rotated versions as follows:

Complex formula

$$R\Lambda_c(r,n) = \phi^{(n-r+1)} G^N + \sum_{r \le r' \le n} RM(r',\ n)\ \phi^{(r'-r+1)} \tag{4}$$

Real formulas

For $n-r$ even

$$R\Lambda_r(r,n) = 2^{(n-r+2)/2} Z^{2N} + \sum_{r \le r' \le n,\ n-r'\ even} RM(r',\ n+1)\ 2^{(r'-r)/2} \tag{5}$$

For $n-r$ odd

$$R\Lambda_r(r,n) = 2^{(n-r+1)/2} Z^{2N} + \sum_{r \le r' \le n,\ n-r'\ odd} RM(r',\ n+1)\ 2^{(r'-r)/2} \tag{6}$$

where $N=2^n$, $\phi = 1+i$ and *RM(r, n)* represents Reed-Muller code. Thus $\Lambda(n,n)$ are the Gaussian integer lattice G^N in complex form and the integer lattice Z^{2N} in real form, $\Lambda(0,n)$ is the Barnes-Wall lattice as mentioned above and $\Lambda(n-1,n) : n \ge 1$ is the checkerboard lattice D_{2N}. Also note that according to real form formulas (2) and (3), $\Lambda(-1,n)$ is a rotated version of the Barnes-Wall lattice i.e., $\Lambda_r(-1,n) = R\Lambda_r(0,n)$ where *R* is 2N-dimensional rotation operator.

The minimum squared distance and the fundamental volume of $R\Lambda(r,n)$ are given as 2^{n--r+1} and $V(R\Lambda(r,n)) = 2^{N(n-r+1)} \Big/ 2^{\sum_{r\le r'<n} k(r',n)}$ respectively where $k(r,n)$ is the number of information bits associated with RM((r ,n) (the dimension of $G(r, n)$) and $\sum_{r\le r'<n} k(r',n)$ represents the total number of coded bits. For these values of minimum squared distance $d^2_{\min}$ and the fundamental volume $V(R\Lambda(r,n))$, the fundamental coding gain is [8]

$$\gamma(R\Lambda(r,n)) = \frac{d^2_{\min}(R\Lambda(r,n))}{V(R\Lambda(r,n))^{2/2N}} = 2^{\frac{1}{N}\sum_{r\le r'<n} k(r',n)} \tag{7}$$

Barnes-Wall Product lattice: Lattices are often analyzed in terms of their generator matrices [10]. Thus, let us denote $M(r_1,n_1)$ *and* $M(r_2,n_2)$ as the generator matrices of lattices $\Lambda(r_1,n_1)$ and $\Lambda(r_2,n_2)$ respectively. The product of these two lattices yields a product lattice whose generator matrix is $M(r_1,n_1)\otimes M(r_2,n_2)$ [11].

Salomon and Amrani presented the augmented product construction of Reed-Muller codes in [9] and proved an important Lemma stated as:

Lemma I: For any integer l and set of Reed-Muller codes $\{RM(r_i,n_i)\}_{i=1}^{l}$, with $0 \le r_i \le n_i$

$$RM(r_1,n_1)\otimes RM(r_2,n_2)\otimes .....RM(r_l,n_l) \subseteq RM(\sum_{i=1}^{l} r_i, \sum_{i=1}^{l} n_i) .$$

III. Augmented product construction of rotated version of Barnes-Wall principal sublattices in real form

Let us consider two real form rotated version of Barnes-Wall principal sublattices $R\Lambda(r_1,n_1)$ and $R\Lambda(r_2,n_2)$. Salomon and Amrani provided the representation of lattice in terms of expanded generator matrix [11] which is a proper generator matrix and span the lattice i.e., every lattice point can be represented as an integral linear combination of its rows and any such combination is necessarily a lattice point. Using (5) and (6), we construct below the expanded generator matrices $R\tilde{M}_r(r_i,n_i)$ for each lattice; in each case, the first matrix is for (n_i - r_i) even while the second matrix is for (n_i - r_i) odd:

$$\tilde{RM}_r(r_1,n_1)= \begin{cases} \begin{pmatrix} 2^{(n_1-r_1+2)/2}G_{RM}(n_1+1,n_1+1) \\ 2^{(n_1-r_1)/2}\quad G_{RM}(n_1,n_1+1) \\ 2^{(n_1-r_1-2)/2}G_{RM}(n_1-2,n_1+1) \\ \cdots \\ \cdots\cdot \\ 2G_{RM}(r_1+2,n_1+1) \\ G_{RM}(r_1,n_1+1) \end{pmatrix} & ;\quad n_1-r_1 \ \ even \\ \begin{pmatrix} 2^{(n_1-r_1+1)/2}G_{RM}(n_1+1,n_1+1) \\ 2^{(n_1-r_1-1)/2}G_{RM}(n_1-1,n_1+1) \\ 2^{(n_1-r_1-3)/2}G_{RM}(n_1-3,n_1+1) \\ \cdots \\ \cdots\cdot \\ 2G_{RM}(r_1+2,n_1+1) \\ G_{RM}(r_1,n_1+1) \end{pmatrix} & ;\quad n_1-r_1 \ \ odd \end{cases} \quad ; \quad \tilde{RM}_r(r_2,n_2)= \begin{cases} \begin{pmatrix} 2^{(n_2-r_2+2)/2}G_{RM}(n_2+1,n_2+1) \\ 2^{(n_2-r_2)/2}\quad G_{RM}(n_2,n_2+1) \\ 2^{(n_2-r_2-2)/2}G_{RM}(n_2-2,n_2+1) \\ \cdots \\ \cdots\cdot \\ 2G_{RM}(r_2+2,n_2+1) \\ G_{RM}(r_2,n_2+1) \end{pmatrix} & ;\quad n_2-r_2 \ \ even \\ \begin{pmatrix} 2^{(n_2-r_2+1)/2}G_{RM}(n_2+1,n_2+1) \\ 2^{(n_2-r_2-1)/2}G_{RM}(n_2-1,n_2+1) \\ 2^{(n_2-r_2-3)/2}G_{RM}(n_2-3,n_2+1) \\ \cdots \\ \cdots\cdot \\ 2G_{RM}(r_2+2,n_2+1) \\ G_{RM}(r_2,n_2+1) \end{pmatrix} & ;\quad n_2-r_2 \ \ odd \end{cases}$$

$$Consequently,\ \tilde{RM}_r(r_1+r_2,n_1+n_2+1)= \begin{cases} \begin{pmatrix} 2^{(n_1+n_2-r_1-r_2+3)/2}G_{RM}(n_1+n_2+2,n_1+n_2+2) \\ 2^{(n_1+n_2-r_1-r_2+1)/2}G_{RM}(n_1+n_2+1,n_1+n_2+2) \\ 2^{(n_1+n_2-r_1-r_2-1)/2}G_{RM}(n_1+n_2-1,n_1+n_2+2) \\ 2^{(n_1+n_2-r_1-r_2-3)/2}G_{RM}(n_1+n_2-3,n_1+n_2+2) \\ \cdots \\ \cdots\cdot \\ 2G_{RM}(r_1+r_2+2,n_1+n_2+2) \\ G_{RM}(r_1+r_2,n_1+n_2+2) \end{pmatrix} & ;\ n_1+n_2+1-r_1-r_2 \ \ even \\ \begin{pmatrix} 2^{(n_1+n_2-r_1-r_2+4)/2}G_{RM}(n_1+n_2+2,n_1+n_2+2) \\ 2^{(n_1+n_2-r_1-r_2+2)/2}G_{RM}(n_1+n_2+2,n_1+n_2+2) \\ 2^{(n_1+n_2-r_1-r_2)/2}\quad G_{RM}(n_1+n_2,n_1+n_2+2) \\ 2^{(n_1+n_2-r_1-r_2-2)/2}G_{RM}(n_1+n_2-2,n_1+n_2+2) \\ 2^{(n_1+n_2-r_1-r_2-4)/2}G_{RM}(n_1+n_2-4,n_1+n_2+2) \\ \cdots \\ \cdots\cdot \\ 2G_{RM}(r_1+r_2+2,n_1+n_2+2) \\ G_{RM}(r_1+r_2,n_1+n_2+2) \end{pmatrix} & ;\ n_1+n_2+1-r_1-r_2 \ \ odd \end{cases}$$

where $G_{RM}(r,n)$ are generator matrices of $RM(r,n)$. Next we prove that all the rows of the expanded matrix $R\tilde{M}_r(r_1,n_1)\otimes R\tilde{M}_r(r_2,n_2)$ are contained in the matrix $R\tilde{M}_r(r_1+r_2,n_1+n_2+1)$ in the following Lemma.

Lemma II: For any $0\le r_1\le n_1,\ 0\le r_2\le n_2$

$$R\tilde{M}_r(r_1,n_1)\otimes R\tilde{M}_r(r_2,n_2)\subseteq R\tilde{M}_r(r_1+r_2,n_1+n_2+1)$$

Proof: In accordance with the parity of (n_1-r_1) and (n_2-r_2) appear in the expression $R\tilde{M}_r(r_1,n_1)\otimes R\tilde{M}_r(r_2,n_2)$ we will have four distinct cases:

Case I: n_1-r_1 odd and n_2-r_2 odd imply $n_1+n_2-r_1-r_2+1$ odd

$R\tilde{M}_r(r_1,n_1)\otimes R\tilde{M}_r(r_2,n_2)=$

$$\begin{pmatrix}
2^{(n_1+n_2-r_1-r_2+2)/2}\,G_{RM}(n_1+1,n_1+1)\otimes G_{RM}(n_2+1,n_2+1)\\
2^{(n_1+n_2-r_1-r_2)/2}\begin{cases}G_{RM}(n_1+1,n_1+1)\otimes G_{RM}(n_2-1,n_2+1)\\ G_{RM}(n_1-1,n_1+1)\otimes G_{RM}(n_2+1,n_2+1)\end{cases}\\
2^{(n_1+n_2-r_1-r_2-2)/2}\begin{cases}G_{RM}(n_1+1,n_1+1)\otimes G_{RM}(n_2-3,n_2+1)\\ G_{RM}(n_1-3,n_1+1)\otimes G_{RM}(n_2+1,n_2+1)\\ G_{RM}(n_1-1,n_1+1)\otimes G_{RM}(n_2-1,n_2+1)\end{cases}\\
2^{(n_1+n_2-r_1-r_2-4)/2}\begin{cases}G_{RM}(n_1+1,n_1+1)\otimes G_{RM}(n_2-5,n_2+1)\\ G_{RM}(n_1-5,n_1+1)\otimes G_{RM}(n_2+1,n_2+1)\\ G_{RM}(n_1-1,n_1+1)\otimes G_{RM}(n_2-3,n_2+1)\\ G_{RM}(n_1-3,n_1+1)\otimes G_{RM}(n_2-1,n_2+1)\end{cases}\\
....\\
.....\\
2\begin{cases}G_{RM}(r_1+2,n_1+1)\otimes G_{RM}(r_2,n_2+1)\\ G_{RM}(r_1,n_1+1)\otimes G_{RM}(r_2+2,n_2+1)\end{cases}\\
G_{RM}(r_1,n_1+1)\otimes G_{RM}(r_2,n_2+1)
\end{pmatrix}
\subseteq
\begin{pmatrix}
2^{(n_1+n_2-r_1-r_2+2)/2}\,G_{RM}(n_1+n_2+2,n_1+n_2+2)\\
2^{(n_1+n_2-r_1-r_2)/2}\,G_{RM}(n_1+n_2,n_1+n_2+2)\\
2^{(n_1+n_2-r_1-r_2-2)/2}\,G_{RM}(n_1+n_2-2,n_1+n_2+2)\\
2^{(n_1+n_2-r_1-r_2-4)/2}\,G_{RM}(n_1+n_2-4,n_1+n_2+2)\\
....\\
.....\\
2\,G_{RM}(r_1+r_2+2,n_1+n_2+2)\\
G_{RM}(r_1+r_2,n_1+n_2+2)
\end{pmatrix}$$

Using Lemma I and the facts that $G_{RM}(r,n)\subset G_{RM}(r+1,n)$ and that changing the order of the rows of an (expanded) generator matrix yields the same lattice, all the the rows of $R\tilde{M}_r(r_1,n_1)\otimes R\tilde{M}_r(r_2,n_2)$ in Case I are explicitly contained in $R\tilde{M}_r(r_1+r_2,n_1+n_2+1)$. The other three Cases constitute the same characteristics and the expressions are given below

Case II: n_1-r_1 odd and n_2-r_2 even imply $n_1+n_2-r_1-r_2+1$ even

$R\tilde{M}_r(r_1,n_1)\otimes R\tilde{M}_r(r_2,n_2)=$

$$\begin{pmatrix} 2^{(n_1+n_2-r_1-r_2+3)/2}\, G_{RM}(n_1+1, n_1+1) \otimes G_{RM}(n_2+1, n_2+1) \\ 2^{(n_1+n_2-r_1-r_2+1)/2} \begin{cases} G_{RM}(n_1+1, n_1+1) \otimes G_{RM}(n_2, n_2+1) \\ G_{RM}(n_1-1, n_1+1) \otimes G_{RM}(n_2+1, n_2+1) \end{cases} \\ 2^{(n_1+n_2-r_1-r_2-1)/2} \begin{cases} G_{RM}(n_1+1, n_1+1) \otimes G_{RM}(n_2-2, n_2+1) \\ G_{RM}(n_1-1, n_1+1) \otimes G_{RM}(n_2, n_2+1) \\ G_{RM}(n_1-3, n_1+1) \otimes G_{RM}(n_2+1, n_2+1) \end{cases} \\ 2^{(n_1+n_2-r_1-r_2-3)/2} \begin{cases} G_{RM}(n_1+1, n_1+1) \otimes G_{RM}(n_2-4, n_2+1) \\ G_{RM}(n_1-1, n_1+1) \otimes G_{RM}(n_2-2, n_2+1) \\ G_{RM}(n_1-3, n_1+1) \otimes G_{RM}(n_2, n_2+1) \\ G_{RM}(n_1-5, n_1+1) \otimes G_{RM}(n_2+2, n_2+1) \end{cases} \\ \cdots \\ \cdots\cdot \\ 2 \begin{cases} G_{RM}(r_1+2, n_1+1) \otimes G_{RM}(r_2, n_2+1) \\ G_{RM}(r_1, n_1+1) \otimes G_{RM}(r_2+2, n_2+1) \end{cases} \\ G_{RM}(r_1, n_1+1) \otimes G_{RM}(r_2, n_2+1) \end{pmatrix} \subseteq \begin{pmatrix} 2^{(n_1+n_2-r_1-r_2+3)/2}\, G_{RM}(n_1+n_2+2, n_1+n_2+2) \\ 2^{(n_1+n_2-r_1-r_2+1_1)/2}\, G_{RM}(n_1+n_2+1, n_1+n_2+2) \\ 2^{(n_1+n_2-r_1-r_2-1)/2}\, G_{RM}(n_1+n_2-1, n_1+n_2+2) \\ 2^{(n_1+n_2-r_1-r_2-3)/2}\, G_{RM}(n_1+n_2-3, n_1+n_2+2) \\ \cdots \\ \cdots\cdot \\ 2\, G_{RM}(r_1+r_2+2, n_1+n_2+2) \\ G_{RM}(r_1+r_2, n_1+n_2+2) \end{pmatrix}$$

$$= R\tilde{M}_r(r_1+r_2, n_1+n_2+1)$$

Case III: $n_1 - r_1$ even and $n_2 - r_2$ odd imply $n_1+n_2-r_1-r_2+1$ even

$$R\tilde{M}_r(r_1, n_1) \otimes R\tilde{M}_r(r_2, n_2) =$$

$$\begin{pmatrix} 2^{(n_1+n_2-r_1-r_2+3)/2}\, G_{RM}(n_1+1, n_1+1) \otimes G_{RM}(n_2+1, n_2+1) \\ 2^{(n_1+n_2-r_1-r_2+1)/2} \begin{cases} G_{RM}(n_1, n_1+1) \otimes G_{RM}(n_2+1, n_2+1) \\ G_{RM}(n_1+1, n_1+1) \otimes G_{RM}(n_2-1, n_2+1) \end{cases} \\ 2^{(n_1+n_2-r_1-r_2-1)/2} \begin{cases} G_{RM}(n_1-2, n_1+1) \otimes G_{RM}(n_2+1, n_2+1) \\ G_{RM}(n_1, n_1+1) \otimes G_{RM}(n_2-1, n_2+1) \\ G_{RM}(n_1+1, n_1+1) \otimes G_{RM}(n_2-3, n_2+1) \end{cases} \\ 2^{(n_1+n_2-r_1-r_2-3)/2} \begin{cases} G_{RM}(n_1-4, n_1+1) \otimes G_{RM}(n_2+1, n_2+1) \\ G_{RM}(n_1-2, n_1+1) \otimes G_{RM}(n_2-1, n_2+1) \\ G_{RM}(n_1, n_1+1) \otimes G_{RM}(n_2-3, n_2+1) \\ G_{RM}(n_1+2, n_1+1) \otimes G_{RM}(n_2-5, n_2+1) \end{cases} \\ \cdots \\ \cdots\cdot \\ 2 \begin{cases} G_{RM}(r_1, n_1+1) \otimes G_{RM}(r_2+2, n_2+1) \\ G_{RM}(r_1+2, n_1+1) \otimes G_{RM}(r_2, n_2+1) \end{cases} \\ G_{RM}(r_1, n_1+1) \otimes G_{RM}(r_2, n_2+1) \end{pmatrix} \subseteq \begin{pmatrix} 2^{(n_1+n_2-r_1-r_2+3)/2}\, G_{RM}(n_1+n_2+2, n_1+n_2+2) \\ 2^{(n_1+n_2-r_1-r_2+1_1)/2}\, G_{RM}(n_1+n_2+1, n_1+n_2+2) \\ 2^{(n_1+n_2-r_1-r_2-1)/2}\, G_{RM}(n_1+n_2-1, n_1+n_2+2) \\ 2^{(n_1+n_2-r_1-r_2-3)/2}\, G_{RM}(n_1+n_2-3, n_1+n_2+2) \\ \cdots \\ \cdots\cdot \\ 2\, G_{RM}(r_1+r_2+2, n_1+n_2+2) \\ G_{RM}(r_1+r_2, n_1+n_2+2) \end{pmatrix}$$

$$= R\tilde{M}_r(r_1+r_2, n_1+n_2+1)$$

Case IV: $n_1 - r_1$ even and $n_2 - r_2$ even then $n_1+n_2-r_1-r_2+1$ odd

$$R\tilde{M}_r(r_1, n_1) \otimes R\tilde{M}_r(r_2, n_2) =$$

$$\begin{pmatrix} 2^{(n_1+n_2-r_1-r_2+4)/2}\, G_{RM}(n_1+1,n_1+1)\otimes G_{RM}(n_2+1,n_2+1) \\ 2^{(n_1+n_2-r_1-r_2+2)/2}\begin{cases} G_{RM}(n_1+1,n_1+1)\otimes G_{RM}(n_2,n_2+1) \\ G_{RM}(n_1,n_1+1)\otimes G_{RM}(n_2+1,n_2+1) \end{cases} \\ 2^{(n_1+n_2-r_1-r_2)/2}\begin{cases} G_{RM}(n_1+1,n_1+1)\otimes G_{RM}(n_2-2,n_2+1) \\ G_{RM}(n_1,n_1+1)\otimes G_{RM}(n_2,n_2+1) \\ G_{RM}(n_1-2,n_1+1)\otimes G_{RM}(n_2+1,n_2+1) \end{cases} \\ 2^{(n_1+n_2-r_1-r_2-2)/2}\begin{cases} G_{RM}(n_1+1,n_1+1)\otimes G_{RM}(n_2-4,n_2+1) \\ G_{RM}(n_1,n_1+1)\otimes G_{RM}(n_2-2,n_2+1) \\ G_{RM}(n_1-2,n_1+1)\otimes G_{RM}(n_2,n_2+1) \\ G_{RM}(n_1-4,n_1+1)\otimes G_{RM}(n_2+1,n_2+1) \end{cases} \\ \cdots\cdot \\ \cdots\cdot\cdot \\ 2\begin{cases} G_{RM}(r_1+2,n_1+1)\otimes G_{RM}(r_2,n_2+1) \\ G_{RM}(r_1,n_1+1)\otimes G_{RM}(r_2+2,n_2+1) \end{cases} \\ G_{RM}(r_1,n_1+1)\otimes G_{RM}(r_2,n_2+1) \end{pmatrix} \subseteq \begin{pmatrix} 2^{(n_1+n_2-r_1-r_2+4)/2}\, G_{RM}(n_1+n_2+2,n_1+n_2+2) \\ 2^{(n_1+n_2-r_1-r_2+2)/2}\, G_{RM}(n_1+n_2+1,n_1+n_2+2) \\ 2^{(n_1+n_2-r_1-r_2)/2}\, G_{RM}(n_1+n_2,n_1+n_2+2) \\ 2^{(n_1+n_2-r_1-r_2-2)/2}\, G_{RM}(n_1+n_2-2,n_1+n_2+2) \\ \cdots\cdot \\ \cdots\cdot\cdot \\ 2\,G_{RM}(r_1+r_2+2,n_1+n_2+2) \\ G_{RM}(r_1+r_2,n_1+n_2+2) \end{pmatrix}$$

$$= R\tilde{M}_r(r_1+r_2,n_1+n_2+1)$$

We shall now show the main result of this Section in the following theorem.

Theorem 1: For any $0 \le r_1 \le n_1,\ 0 \le r_2 \le n_2$

$$R\Lambda_r(r_1,n_1)\otimes R\Lambda_r(r_2,n_2)\subseteq R\Lambda_r(r_1+r_2,n_1+n_2+1)$$

Proof: The kronecker product of any two lattices is given by kronecker product of their corresponding expanded generator matrices. Thus, the matrix $R\tilde{M}_r(r_1,n_1)\otimes R\tilde{M}_r(r_2,n_2)$ is the expanded generator matrix of $R\Lambda_r(r_1,n_1)\otimes R\Lambda_r(r_2,n_2)$. Lemma II reveals that in each case, expanded generator matrix of $R\Lambda_r(r_1+r_2,n_1+n_2+1)$ contains all the rows of expanded generator matrix of $R\Lambda_r(r_1,n_1)\otimes R\Lambda_r(r_2,n_2)$.

Corollary 1.1: The minimum distance of real PL involving rotated versions of the two Barnes-Wall principal sublattices as its component lattices is the product of the minimum distances of the two component lattices despite of dimension.

Proof: It is known that [11]

$$d^2_{\min}(R\Lambda_r(r_1,n_1)\otimes R\Lambda_r(r_2,n_2))\le d^2_{\min}(R\Lambda_r(r_1,n_1))\,d^2_{\min}(R\Lambda_r(r_2,n_2)) \tag{8}$$

Also Theorem 1 tells that

$$
\begin{aligned}
d^2_{\min}(R\Lambda_r(r_1,n_1)\otimes R\Lambda_r(r_2,n_2)) &\geq d^2_{\min}(R\Lambda_r(r_1+r_2,n_1+n_2+1))\\
&= 22^{n_1+n_2+1-r_1-r_2}\\
&= 2^{n_1-r_1+1}\,2^{n_2-r_2+1}\\
&= d^2_{\min}(R\Lambda_r(r_1,n_1))\,d^2_{\min}(R\Lambda_r(r_2,n_2)) \qquad (9)
\end{aligned}
$$

Equations (8) and (9) simultaneously give

$$d^2_{\min}(R\Lambda_r(r_1,n_1)\otimes R\Lambda_r(r_2,n_2)) = d^2_{\min}(R\Lambda_r(r_1,n_1))\,d^2_{\min}(R\Lambda_r(r_2,n_2)) \qquad (10)$$

The equalities in (10) prove the corollary and also make known that the minimum distance remains same by augmenting cosets of the product lattice $R\Lambda_r(r_1,n_1)\otimes R\Lambda_r(r_2,n_2)$ until the complete rotated version of Barnes-Wall principal sublattice is obtained which in turn suggests the following result for fundamental coding gain:

Corollary 1.2: The fundamental coding gain of augmented lattice is improved as compared to the PL.

In general from [11],

$$\gamma(R\Lambda_r(r_1,n_1)\otimes R\Lambda_r(r_2,n_2)) = \gamma(R\Lambda_r(r_1,n_1))\gamma(R\Lambda_r(r_2,n_2)) \qquad (11)$$

Using (7) and the fact that fundamental coding gain remains same under scaled orthogonal transformation, the equality (11) becomes

$$
\begin{aligned}
\gamma(R\Lambda_r(r_1,n_1)\otimes R\Lambda_r(r_2,n_2)) &= \gamma(\Lambda_r(r_1,n_1))\gamma(\Lambda_r(r_2,n_2))\\
&= 2^{\frac{1}{2^{n_1}}\sum_{r_1\leq r'\leq n_1} k(r',n_1)+\frac{1}{2^{n_2}}\sum_{r_2\leq r'\leq n_2} k(r',n_2)}
\end{aligned}
$$

Substituting for the dimension of RM code, $k(r,n)=\sum_{0\leq m\leq n}\binom{n}{m}$

$$
\begin{aligned}
&= 2^{\frac{1}{2^{n_1}}\sum_{r_1\leq r'\leq n_1}\sum_{0\leq m\leq r'}\binom{n_1}{m}+\frac{1}{2^{n_2}}\sum_{r_2\leq r'\leq n_2}\sum_{0\leq m\leq r'}\binom{n_2}{m}}\\
&\leq 2^{\frac{1}{2^{n_1+n_2+1}}\sum_{r_1+r_2\leq r'\leq n_1+n_2+1}\sum_{0\leq m\leq r'}\binom{n_1+n_2+1}{m}}\\
&= \gamma((R\Lambda_r(r_1+r_2,n_1+n_2+1))
\end{aligned}
$$

Corollary 1.3: For any integer *m* and a set of lattices $\{R\Lambda_r(r_i,n_i)\}_{i=1}^{m}$, with $0\leq r_i\leq n_i$

$$R\Lambda_r(r_1,n_1)\otimes R\Lambda_r(r_2,n_2)\otimes\ldots\ldots\otimes R\Lambda_r(r_m,n_m)\subseteq R\Lambda_r\left(\sum_{i=1}^{m}r_i,\, m-1+\sum_{i=1}^{m}n_i\right)$$

The proof of above corollary is a simple extension of Theorem 1.

Example 1.4: The product of rotated versions of two "checkerboard" lattices is a sublattice of rotated version of a lattice from Barnes-Wall family of different type i.e.

$$\begin{aligned} RD_{2^{n_1+1}} \otimes \quad RD_{2^{n_2+1}} &= R\Lambda_r(n_1-1,n_1) \otimes R\Lambda_r(n_2-1,n_2) \\ &\subseteq R\Lambda_r(n_1+n_2-2, n_1+n_2+1) \end{aligned}$$

Here RD_{2N}^r represents the real form "checkerboard" lattice of dimension 2*N*. In particular if we take *N=2*, $RD_4 \otimes RD_4 = R\Lambda_r(0,1) \otimes R\Lambda_r(0,1) \subseteq R\Lambda_r(0,3) = R\Lambda_{16}$.

IV. Augmented product construction of rotated version of Barnes-Wall principal sublattices in Complex form

Corresponding to an N_1N_2- dimensional complex form PL $\Lambda_c(r_1,n_1) \otimes \Lambda_c(r_1,n_1)$, the dimension of PL in real form is $2N_1N_2$ where $N_1,\ N_2$ denote the dimensions of (complex) lattices $\Lambda_c(r_1,n_1)$ and $\Lambda_c(r_2,n_2)$ respectively. Thus, $\Lambda_c(r_1,n_1) \otimes \Lambda_c(r_1,n_1)$ does not correspond to $4N_1N_2$ dimensional real form product lattice $\Lambda_r(r_1,n_1) \otimes \Lambda_r(r_1,n_1)$ and it needs a separate treatment. Neverthless, AP construction is provided involving the rotated versions of two different principal sublattices.

Consider $R\Lambda_c(r_1,n_1)$ and $R\Lambda_c(r_2,n_2)$ be two complex form Barnes-Wall principal sublattice in rotated version. Corresponding expanded generator matrices are deemed as $R\tilde{M}_c(r_1,n_1)$ and $R\tilde{M}_c(r_1,n_1)$ respectively. Using (1), (4) and the equality $R^{\mu}\Lambda_c(r_i,n_i) = \phi^{\mu}\Lambda_c(r_i,n_i)$ [8] we can construct the complex expanded generator matrices of $R\Lambda_c(r_1,n_1)$, $R\Lambda_c(r_2,n_2)$ and $R^2\Lambda_c(r_1+r_2,n_1+n_2)$ as :

$$R\tilde{M}_c(r_1,n_1) = \begin{pmatrix} \phi^{n_1-r_1+1}G_{RM}(n_1,n_1) \\ \phi^{n_1-r_1}\ G_{RM}(n_1-1,n_1) \\ \phi^{n_1-r_1-1}G_{RM}(n_1-2,n_1) \\ \ldots \\ \ldots \\ \phi^2 G_{RM}(r_1+1,n_1) \\ \phi\ G_{RM}(r_1,n_1) \end{pmatrix}; \quad R\tilde{M}_c(r_2,n_2) = \begin{pmatrix} \phi^{n_2-r_2+1}G_{RM}(n_2,n_2) \\ \phi^{n_2-r_2}\ G_{RM}(n_2-1,n_2) \\ \phi^{n_2-r_2-1}G_{RM}(n_2-2,n_2) \\ \ldots \\ \ldots \\ \phi^2 G_{RM}(r_2+1,n_2) \\ \phi\ G_{RM}(r_2,n_2) \end{pmatrix}; \quad R^2\tilde{M}_c(r_1+r_2,n_1+n_2) = \begin{pmatrix} \phi^{n_1+n_2-r_1+r_2+2}G_{RM}(n_1+n_2,n_1+n_2) \\ \phi^{n_1+n_2-r_1+r_2+1}G_{RM}(n_1+n_2-1,n_1+n_2) \\ \phi^{n_1+n_2-r_1+r_2}\ G_{RM}(n_1+n_2-2,n_1+n_2) \\ \ldots \\ \ldots \\ \phi^3 G_{RM}(r_1+r_2+1,n_1+n_2) \\ \phi^2 G_{RM}(r_1+r_2,n_1+n_2) \end{pmatrix}$$

We shall now prove that the kronecker product of rotated versions of two distinct BWPSL is a sublattice of square rotated version of higher dimensional lattice from the same family. For this, we need to prove the following Lemma.

Lemma III: For any $0 \le r_1 \le n_1,\ 0 \le r_2 \le n_2$

(i) $R\tilde{M}_c(r_1, n_1) \otimes R\tilde{M}_c(r_2, n_2) \subseteq R^2 \tilde{M}_c(r_1 + r_2, n_1 + n_2)$

(ii) $R\tilde{M}_c(0, n_1) \otimes R\tilde{M}_c(0, n_2) = R^2 \tilde{M}_c(0, n_1 + n_2)$

Proof: Using the (complex) expanded generator matrices mentioned before, Lemma I along with the facts that $G_{RM}(r, n) \subset G_{RM}(r+1, n)$ and that changing the order of the rows of an (expanded) generator matrix yields the same lattice, prove (i) of Lemma and it can be shown as:

$$R\tilde{M}_c(r_1, n_1) \otimes R\tilde{M}_c(r_2, n_2) =$$

$$\begin{pmatrix} \phi^{(n_1+n_2-r_1-r_2+2)} & G_{RM}(n_1, n_1) \otimes G_{RM}(n_2, n_2) \\ \phi^{(n_1+n_2-r_1-r_2+1)} & \begin{cases} G_{RM}(n_1-1, n_1) \otimes G_{RM}(n_2, n_2) \\ G_{RM}(n_1, n_1) \otimes G_{RM}(n_2-1, n_2) \end{cases} \\ \phi^{(n_1+n_2-r_1-r_2)} & \begin{cases} G_{RM}(n_1, n_1) \otimes G_{RM}(n_2-2, n_2) \\ G_{RM}(n_1-1, n_1) \otimes G_{RM}(n_2-1, n_2) \\ G_{RM}(n_1-2, n_1) \otimes G_{RM}(n_2, n_2) \end{cases} \\ \quad .... & \\ \quad ..... & \\ \phi^3 \begin{cases} G_{RM}(r_1+1, n_1) \otimes G_{RM}(r_2, n_2) \\ G_{RM}(r_1, n_1) \otimes G_{RM}(r_2+1, n_2) \end{cases} & \\ \phi^2 G_{RM}(r_1, n_1) \otimes G_{RM}(r_2, n_2) & \end{pmatrix} \subseteq \begin{pmatrix} \phi^{(n_1+n_2-r_1-r_2+2)} G_{RM}(n_1+n_2, n_1+n_2) \\ \phi^{(n_1+n_2-r_1-r_2+1)} G_{RM}(n_1+n_2-1, n_1+n_2) \\ \phi^{(n_1+n_2-r_1-r_2)} \quad G_{RM}(n_1+n_2-2, n_1+n_2) \\ ..... \\ \phi^3 G_{RM}(r_1+r_2+1, n_1+n_2) \\ \phi^2 G_{RM}(r_1+r_2, n_1+n_2) \end{pmatrix} = R^2 \tilde{M}_r(r_1+r_2, n_1+n_2)$$

Using the following result for Reed-Muller codes [9] stated as

For any $0 \le r \le n_1 + n_2;\ \bigcup_{r_1+r_2=r,\, 0\le r_1\le n_1,\, 0\le r_2\le n_2} G_{RM}(r_1, n_1) \otimes G_{RM}(r_2, n_2) = G_{RM}(r, n_1+n_2)$

(where $A \cup B$ denotes a matrix that includes all the rows of matrices *A* and *B* such that each row appears exactly once) and on substituting $r_1 = r_2 = 0$ in (i), the subset notation $\subseteq$ is turned into an equality which proves (ii) of Lemma.

Theorem 2: For any $0 \le r_1 \le n_1,\ 0 \le r_2 \le n_2$

(i) $R\Lambda_c(r_1, n_1) \otimes R\Lambda_c(r_2, n_2) \subseteq R^2 \Lambda_c(r_1 + r_2, n_1 + n_2)$

(ii) $R\Lambda_c(0, n_1) \otimes R\Lambda_c(0, n_2) = R^2 \Lambda_c(0, n_1 + n_2)$

Proof: Case (i) of Theorem follows immediately from (i) of Lemma III. Namely, lattice $R^2\Lambda_c(r_1 + r_2, n_1 + n_2)$ consists of all the lattice points of the product lattices $R\Lambda_c(r_1, n_1) \otimes R\Lambda_c(r_2, n_2)$ while (ii) of Lemma III imply a promising constructions of the lattices $R^2\Lambda_c(0, n_1 + n_2)$. In contrast to the real form case herein in (ii) of Theorem, augmenting is not required on top of the kronecker product which exhibits promising constructions in this correspondence.

Corollary 2.1: The minimum distance of complex PL in Theorem-2 remains invariant i.e., it is the product of the minimum distances of the two component lattices, regardless of dimension.

Proof: In general, following inequality holds [11]:

$$d^2_{\min}(\Lambda_c(r_1,n_1)\otimes\Lambda_c(r_2,n_2)) \le d^2_{\min}(\Lambda_c(r_1,n_1))\, d^2_{\min}(\Lambda_c(r_2,n_2)) \tag{12}$$

which allow us to write respectively for the kronecker product $R\Lambda_c(r_1,n_1)\otimes R\Lambda_c(r_2,n_2)$

$$d^2_{\min}(R\Lambda_c(r_1,n_1)\otimes R\Lambda_c(r_2,n_2)) \le d^2_{\min}(R\Lambda_c(r_1,n_1))\, d^2_{\min}(R\Lambda_c(r_2,n_2)) \tag{13}$$

Also it follows from Theorem-2(i)

$$\begin{aligned} d^2_{\min}(R\Lambda_r(r_1,n_1)\otimes R\Lambda_r(r_2,n_2)) &\ge d^2_{\min}(R^2\Lambda_c(r_1+r_2,n_1+n_2) \\ &= 2^2 2^{n_1+n_2-r_1-r_2} \\ &= 2^{n_1-r_1+1}\, 2^{n_2-r_2+1} \\ &= d^2_{\min}(R\Lambda_c(r_1,n_1))\, d^2_{\min}(R\Lambda_c(r_2,n_2)) \end{aligned} \tag{14}$$

Combining inequalities (13) with (14) yields

$$d_{\min}(R\Lambda_c(r_1,n_1)\otimes R\Lambda_c(r_2,n_2)) = d_{\min}(R\Lambda_c(r_1,n_1))\, d_{\min}(R\Lambda_c(r_2,n_2)) \tag{15}$$

The equality in (15) proves the corollary and by augmenting cosets of the product lattice $R\Lambda_c(r_1,n_1)\otimes R\Lambda_c(r_2,n_2)$, we can fill its “holes” such that the minimum distance remains same.

Corollary 2.2: For any integer *m* and a set of lattices $\{R\Lambda_c(r_i,n_i)\}_{i=1}^m$, with $0 \le r_i \le n_i$

(i) $R\Lambda_c(r_1,n_1)\otimes R\Lambda_c(r_2,n_2)\otimes......\otimes R\Lambda_c(r_m,n_m) \subseteq R^m\Lambda_c\left(\sum_{i=1}^m r_i, \sum_{i=1}^m n_i\right)$

(ii) $R\Lambda_c(0,n_1)\otimes R\Lambda_c(0,n_2)\otimes......\otimes R\Lambda_c(0,n_m) = R^m\Lambda_c\left(0, \sum_{i=1}^m n_i\right)$

The proof of above corollary is an extension of Theorem 2.

Example 2.3: The real and complex form representation of RD_4 and RE_8 are given by (4), (5) and (6)

$$RD_4 = R\Lambda_r(0,1) = 2Z^4 + (4,1,4)$$

$$RE_8 = R\Lambda_r(0,2) = 4Z^8 + 2(8,7,2) + (8,1,8)$$

$$RD_4 = R\Lambda_c(0,1) = \phi^2 G^2 + \phi RM(0,1)$$

$$RE_8 = R\Lambda_c(0,2) = \phi^3 G^4 + \phi^2\,(4,3,2) + \phi(4,1,4)$$

The generator matrices of RD_4 and RE_8 in complex form can be written as

$$M_{RD_4} = \begin{pmatrix} \phi^2 & 0 \\ \phi & \phi \end{pmatrix}; M_{RE_8} = \begin{pmatrix} \phi^3 & 0 & 0 & 0 \\ \phi^2 & \phi^2 & \phi^2 & \phi^2 \\ \phi^2 & 0 & \phi^2 & 0 \\ \phi & \phi & \phi & \phi \end{pmatrix}$$; which gives immediately

$$M_{RD_4} \otimes M_{RD_4} = \begin{pmatrix} \phi^4 & 0 & 0 & 0 \\ \phi^3 & \phi^3 & 0 & 0 \\ \phi^3 & 0 & \phi^3 & 0 \\ \phi^2 & \phi^2 & \phi^2 & \phi^2 \end{pmatrix}; M_{RD_4} \otimes M_{RE_8} = \begin{pmatrix} \phi^5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \phi^4 & \phi^4 & 0 & 0 & 0 & 0 & 0 & 0 \\ \phi^4 & 0 & \phi^4 & 0 & 0 & 0 & 0 & 0 \\ \phi^4 & \phi^4 & \phi^4 & \phi^4 & 0 & 0 & 0 & 0 \\ \phi^4 & 0 & 0 & 0 & \phi^4 & 0 & 0 & 0 \\ \phi^3 & \phi^3 & 0 & 0 & \phi^3 & \phi^3 & 0 & 0 \\ \phi^3 & 0 & \phi^3 & 0 & \phi^3 & 0 & \phi^3 & 0 \\ \phi^2 & \phi^2 & \phi^2 & \phi^2 & \phi^2 & \phi^2 & \phi^2 & \phi^2 \end{pmatrix}$$

and these are also the complex form generator matrices of R^2E_8 and $R^2\Lambda_{16}$.

V. Conclusion

In summary, we extended the AP-construction to lattices using the architecture provided by Salomon and Amrani [9] and thus constructed BWPSL of rotated version in real as well as in complex form. For each construction, expanded generator matrices of product lattices are given. Of particular interest is a product lattice involving rotated versions of two distinct (m^{th} and n^{th}) members of Barnes-Wall family in complex form which is again a higher dimensional member of the same family. Given below is a table which explicitly shows some illustrations embracing a few known lattices viz. *Schläfli lattice* D_4 , Gosset lattice E_8, checkerboard lattice D_N etc., their products along with the parameters.

Λ_1	Λ_2	$\Lambda_1^r \otimes \Lambda_2^r \subseteq \Lambda_3^r$	$R\Lambda_1^r \otimes R\Lambda_2^r \subseteq \Lambda_4^r$	$\Lambda_1^c \otimes \Lambda_2^c \subseteq \Lambda_3^c$	$R\Lambda_1^c \otimes R\Lambda_2^c \subseteq \Lambda_4^c$	$\mu(\Lambda_3^r)$	$\mu(\Lambda_4^r)$	$\mu(\Lambda_3^c)$	$\mu(\Lambda_4^c)$
Z^2	Z^2	$Z^2 \otimes Z^2 \subseteq Z^4$	$RZ^2 \otimes RZ^2 \subseteq RD_4^r$	$G \otimes G = G$	$RG \otimes RG = R^2 G$	0	1	0	0
Z^2	Z^4	$Z^2 \otimes Z^4 \subseteq Z^8$	$RZ^2 \otimes RZ^4 \subseteq RD_8^r$	$G \otimes G^2 \subseteq G^2$	$RG \otimes RG^2 \subseteq R^2 G^2$	0	1	0	0
Z^2	D_4	$Z^2 \otimes D_4^r \subseteq D_8^r$	$RZ^2 \otimes RD_4^r \subseteq RE_8^r$	$G \otimes D_4^c = D_4^c$	$RG \otimes RD_4^c = R^2 D_4^c$	1	2	1	1
Z^4	D_4	$Z^4 \otimes D_4^r \subseteq D_{16}^r$	$RZ^4 \otimes RD_4^r \subseteq RH_{16}^r$	$G^2 \otimes D_4^c \subseteq D_8^c$	$RG^2 \otimes RD_4^c \subseteq R^2 D_8^c$	1	2	1	1
D_4	D_4	$D_4^r \otimes D_4^r \subseteq H_{16}^r$	$RD_4^r \otimes RD_4^r \subseteq R\Lambda_{16}^r$	$D_4^c \otimes D_4^c = E_8^c$	$RD_4^c \otimes RD_4^c = R^2 E_8^c$	2	3	2	2
D_4	D_8	$D_4^r \otimes D_8^r \subseteq X_{32}^r$	$RD_4^r \otimes RD_8^r \subseteq RH_{32}^r$	$D_4^c \otimes D_8^c \subseteq H_{16}^c$	$RD_4^c \otimes RD_8^c \subseteq R^2 H_{16}^c$	2	3	2	2
D_4	E_8	$D_4^r \otimes E_8^r \subseteq H_{32}^r$	$RD_4^r \otimes RE_8^r \subseteq R\Lambda_{32}^r$	$D_4^c \otimes E_8^c = \Lambda_{16}^c$	$RD_4^c \otimes RE_8^c = R^2 \Lambda_{16}^c$	3	4	3	3

Λ_i^r, Λ_i^c *are real and complex form lattices with* $R\Lambda_i^r$, $R\Lambda_i^c$ *as their rotated versions respectively. The 3rd and 5th columns are explicable by* [9] *and that of 4th and 6th by Theorem 1 and Theorem 2 of this paper.* $R^2\Lambda_i^c = \phi^2\Lambda_i^c$ *and* $\mu(\Lambda)$ *denotes the depth of lattice* Λ [8].

$d^2_{\min}(\Lambda_1^r \otimes \Lambda_2^r)$	$d^2_{\min}(R\Lambda_1^r \otimes R\Lambda_2^r)$	$d^2_{\min}(\Lambda_1^c \otimes \Lambda_2^c)$	$d^2_{\min}(R\Lambda_1^c \otimes R\Lambda_2^c)$	$\gamma(\Lambda_1^r \otimes \Lambda_2^r) = \gamma(R\Lambda_1^r \otimes R\Lambda_2^r) = \gamma(\Lambda_1^c \otimes \Lambda_2^c) = \gamma(R\Lambda_1^c \otimes R\Lambda_2^c)$	$\gamma(\Lambda_3^r)$	$\gamma(\Lambda_4^r)$	$\gamma(\Lambda_3^c)$	$\gamma(\Lambda_4^c)$
1	4	1	4	1	1	$2^{1/2}$	1	1
1	4	1	4	1	1	$2^{3/4}$	1	1
2	8	2	8	$2^{1/2}$	$2^{3/4}$	2	$2^{1/2}$	$2^{1/2}$
2	8	2	8	$2^{1/2}$	$2^{7/8}$	$2^{11/8}$	$2^{3/4}$	$2^{3/4}$
4	16	4	16	2	$2^{11/8}$	$2^{3/2}$	2	2
4	16	4	16	$2^{5/4}$	$2^{13/8}$	$2^{31/16}$	$2^{11/8}$	$2^{11/8}$
8	32	8	32	$2^{3/2}$	$2^{31/16}$	4	$2^{3/2}$	$2^{3/2}$

$d^2_{\min}(\Lambda), \gamma(\Lambda)$ *are squared minimum distance and fundamental coding gain of lattice* Λ.

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