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THE RECTANGULAR ELASTIC MEMBRANE FOR APERY'S AND CATALAN'S CONSTANTS CALCULATION

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Abstract: In this paper, we use the elastic membrane model to calculate Apéry's constant and others associated series. The de Saint-Venant's solution of Poisson's partial differential equation with rectangular boundary is used for the two limit cases of thin and square membranes. Gradient calculations at the boundaries give a relationship involving ratio $\pi\, G/\zeta(3)$ where G is the Catalan's constant and a rapidly convergent series for $\zeta(3)$ analogous to famous Ramanujan's series.

Keywords: zeta function; beta function; lambda function; odd integers, Apéry; Catalan; elastic; membrane.

2010 AMS Subject Classification: 11M41.

1. INTRODUCTION

The Riemann zeta function or Euler-Riemann zeta function $\zeta(s)$ is a function of complex variable s given by:

$$\zeta(s) = \sum_{k=1}^{+\infty} \frac{1}{k^s}$$

With $\mathcal{R}e(s) > 1$.

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In 1735, Euler solved the famous Basel problem [1] and found $\zeta(2)=\frac{\pi^2}{6}$ using the polynomial development of $\frac{sinx}{x}$. He also brilliantly showed that $\zeta(4)=\frac{\pi^4}{90}$ and $\zeta(6)=\frac{\pi^6}{945}$. Finally, for even integers greater than 1, he established the general formula:

$$\forall k>0 \qquad \zeta(2k)=\mid B_{2k}\mid \frac{(2\pi)^{2k}}{2(2k)!}$$

where B_{2k} are the Bernoulli numbers ($B_2=\frac{1}{6}$; $B_4=-\frac{1}{30}$; ...) [2,3].

But for odd integers, the problem appeared much more complicated [4-7] and Euler himself was not able to find a closed form for $\zeta(3)$. The Indian mathematician Ramanujan [8] worked a lot on this problem without success. We had to wait for Apery's work in 1978, who demonstrated that $\zeta(3)$ is an irrational number. This number was called Apery's constant in honour of this important result. But even now, there exists no analytical or closed form expression for $\zeta(3)$ and for all others values of zeta function for odd integers i.e. $\zeta(2k+1)\;\; k\geq 1$.

This problem remains of major interest in mathematics for numbers theory [9] particularly for prime numbers determination. From Euler work, the zeta function is related to prime numbers through Euler product formula:

$$\zeta(s)=\sum_{k=1}^{+\infty}\frac{1}{k^s}=\prod_{p\;prime}\frac{1}{1-p^{-s}}$$

And, as reported in [10-11], the reciprocal of $\zeta(3)$ is the probability that any three positive integers chosen at random will be coprime.

Moreover, odd integers zeta function is of interest in physics particularly for quantum electrodynamics [12].

The objective of this paper is to show, that famous elastic membrane deformation problem, introduced by L. Prandtl [13], and widely used in elasticity theory [14] and in fluid mechanics [15], can be of major interest for Apéry's and Catalan's constant calculation and other values of $\zeta(2k+1)\;\; k\geq 1$.

We will first recall some useful technics and results, giving important relationships, which are good candidates on the way to solve this secular problem described by Van der Poorten [16] as "a mystery wrapped in an enigma".

Then, the elastic membrane approach will be developed using old de Saint-Venant [17] solution available for rectangular shapes boundaries. Two limit cases will be particularly explored: the often called "thin membrane" and the highly symmetric square boundary.

Finally, some consequences of the results will be given for unknown series calculations.

2. PRELIMINARIES

2.1 A relationship between $\zeta(s)$ values.

As recently showed in [18-19], the sum of $\zeta(s)$ values can be related to the following series:

$$\phi(s) = \sum_{n=2}^{+\infty} \frac{1}{n^s(n-1)} \quad (1)$$

Giving,

$$\phi(s) = s - \sum_{n=2}^{s} \zeta(n) \quad (2)$$

$\phi(1)$ and $\phi(2)$ can easily be calculated as respectively a telescopic series and using partial fraction decomposition:

$\phi(1) = 1$

And

$\phi(2) = 2 - \zeta(2)$

But for larger powers of the first factor of the denominator, it is, to our knowledge, impossible to obtain a closed form. For example, we can write Apéry's constant as followed:

$\zeta(3) = 3 - \zeta(2) - \phi(3)$

But there exists no closed form for $\phi(3) = 0.15300902995 \ldots$

Using that method, Delplace [19] found:

$$\zeta(3) = 4 - \frac{9\pi}{400} - \frac{\pi^2}{6} - \frac{\pi^4}{90} = 1.202056864734 \ldots$$

As an approximative value of Apéry's constant. But until now, nobody was able to find a closed form of $\zeta(3)$ as a sum of terms involving powers of π.

2.2 A relationship with Mersenne's numbers.

From Euler's solution of Basel problem, we have:

$\zeta(2) = \pi^2/6$

For odd terms, we have the following well-known result:

$$\sum_{n=0}^{+\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8} \qquad (3)$$

Giving for even terms of this series:

$$\sum_{n=1}^{+\infty} \frac{1}{(2n)^2} = \sum_{n=1}^{+\infty} \frac{1}{n^2} - \sum_{n=0}^{+\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{24}$$

We can then form the following ratio:

$$\sum_{n=0}^{+\infty} \frac{1}{(2n+1)^2} \Bigg/ \sum_{n=1}^{+\infty} \frac{1}{(2n)^2} = \frac{\pi^2}{8} \cdot \frac{24}{\pi^2} = 3 = M_2$$

With $M_2 = 2^2 - 1$ being a Mersenne's number.

This result can easily be extended to all values of the power s whatever is its parity giving the following simple theorem:

$$\forall s \in \mathbb{N}\; s > 1 \; \sum_{n=0}^{+\infty} \frac{1}{(2n+1)^s} \Bigg/ \sum_{n=1}^{+\infty} \frac{1}{(2n)^s} = 2^s - 1 = M_s \qquad (4)$$

The proof is straightforward using:

$$\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{(2n)^s} + \sum_{n=0}^{+\infty} \frac{1}{(2n+1)^s}$$

And

$$\zeta(s) = 2^s \, . \sum_{n=1}^{+\infty} \frac{1}{(2n)^s}$$

This result can be related to Dirichlet lambda function defined as:

$$\lambda(s) = \sum_{n=0}^{+\infty} \frac{1}{(2n+1)^s} = \left(1 - \frac{1}{2^s}\right).\zeta(s) \quad (5)$$

Using above relationships for $s = 3$ gives:

$$\zeta(3) = 8.\sum_{n=1}^{+\infty} \frac{1}{(2n)^3} \; and \; \zeta(3) = \frac{8}{7}\sum_{n=0}^{+\infty} \frac{1}{(2n+1)^3} = \frac{8}{7}.\lambda(3)$$

And using another famous Euler's result:

$$\sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)^3} = \frac{\pi^3}{32} \quad (6)$$

We obtain:

$$\zeta(3) = \frac{\pi^3}{28} + \frac{16}{7}.\sum_{n=1}^{+\infty} \frac{1}{(4n-1)^3} \quad (7)$$

This last series converges to the value 0.041426822… but to our knowledge, there exists no closed form for it.

Above equation has a well-known similar one as reported in [11]:

$$\zeta(3) = -\frac{\pi^3}{28} + \frac{16}{7}.\sum_{n=0}^{+\infty} \frac{1}{(4n+1)^3} \quad (8)$$

The sum of both relationships gives:

$$\sum_{n=0}^{+\infty} \frac{1}{(2n+1)^3} = \sum_{n=0}^{+\infty} \frac{1}{(4n+1)^3} + \sum_{n=1}^{+\infty} \frac{1}{(4n-1)^3} \quad (9)$$

As reported in [11], above relationships might be regarded as leading candidates for determination of a closed form for $\zeta(2k+1)$ as found for $\zeta(2k)$. They will be useful in the following chapter 3 for interpretation of elastic membrane results and calculation of Apéry's constant.

2.3 The use of Fourier series.

The use of Fourier series is a powerful tool, used in Riemann's original work, to obtain important results about zeta function and associated series. For example, function defined as: $\forall x \in [0, \pi]\ f(x) = 1 - 2x/\pi$ gives:

$$\forall x \in \mathbb{R}\ f(x) = \frac{8}{\pi^2} \cdot \sum_{n=0}^{+\infty} \frac{cos\big((2n+1)x\big)}{(2n+1)^2}$$

Giving for $x = 0$:

$$\sum_{n=0}^{+\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$$

And also $\zeta(2)$ after simple algebraic manipulations.

And using Perseval formula:

$$\sum_{n=0}^{+\infty} \frac{1}{(2n+1)^4} = \frac{\pi^4}{96}$$

And also $\zeta(4)$ after simple manipulations.

It is also interesting to consider periodic functions of type: $\forall x \in [0, \pi];\ \forall k \in \mathbb{N}; k \geq 1\ f(x) = x^k/\pi$. For example, Fourier series expansion of $f(x) = x^3/\pi$ gives:

$$\sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)} = \frac{\pi}{4}$$

And for $f(x) = x^5/\pi$, we obtain:

$$\sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)^5} = \frac{5\,\pi^5}{1536}$$

These two above series correspond to well-known β Dirichlet function [11] defined as:

$$\beta(s) = \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)^s}\ with\ \beta(2k+1) = \frac{(-1)^k\, E_{2k}}{2(2k)!} \cdot \left(\frac{\pi}{2}\right)^{2k+1} \qquad (10)$$

Where E_{2k} are the Euler's numbers.

But despite of this successful uses of these Fourier series expansions, curious results are sometimes observed. The case of $f(x) = x^6/\pi$ reveals what we interpreted as a Gibbs phenomenon illustrated below. After 6 integration by part we obtain:

$$f(x) = \frac{\pi^5}{7} + \sum_{n=1}^{+\infty}\left(\left((-1)^n\frac{12\pi^3}{n} - (-1)^n\frac{240\pi}{n^4} + (-1)^n\frac{1440}{n^6\pi}\right)cos(nx)\right)$$

Which gives for $x = 0$: $\pi^2 = 12\,Ln2$ which is of course false. The quantity $\pi^2/12Ln2$ is a well-known quantity in numbers theory called the Lévy's constant. It is quite surprising to obtain this constant for $k = 6$. Moreover, calculation of $f(\pi)$ gives also a curious result which is $\zeta(1) = 0$ being also impossible as this series diverges. Doing the same calculations with $f(x) = x^2/\pi$ and $f(x) = x^4/\pi$ gives respectively:

$$f(x) = \frac{\pi}{3} + \sum_{n=1}^{+\infty}\frac{4}{n^2\pi}\,(-1)^n cos(nx)$$

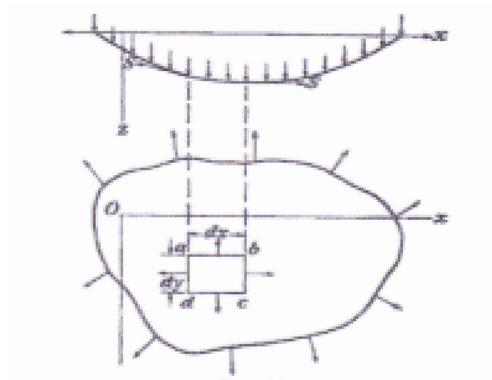
And

$$f(x) = \frac{\pi^3}{5} + \sum_{n=1}^{+\infty}\left(\left(\frac{8\pi}{n^2}(-1)^n - \frac{48}{n^4\pi}(-1)^n\right)cos(nx)\right)$$

These two equations perfectly verify $f(0) = 0$ and $f(\pi)$ being respectively equal to π and π^3. But whatever are the sometimes-surprising convergence problems encountered using Fourier series they allow important results for zeta function calculations to be obtained. Moreover, because we know, in Physics, that Fourier series describe the behaviour of waves in elastic media, we decided to study how mathematical description of an elastic membrane can be related to odd values of Euler-Riemann zeta function.

3. MAIN RESULTS

As reported in the introduction of this paper, the membrane analogy introduced by L. Prandtl [13] has proved very valuable. Imagine a homogeneous membrane supported at the edges subjected to a uniform tension at the edges and a uniform lateral pressure. If p is the pressure per unit area of the membrane and q is the uniform tension per unit length of its boundary, the tensile forces acting on the sides of a rectangular infinitesimal element of surface $dxdy$, in the case of small deflection of the membrane, a resultant in the upward direction. The following figure illustrates this elastic membrane.



<u>Figure 1</u>: The elastic membrane system from [14].

The equation of equilibrium of the surface element $dxdy$ gives:

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=-\frac{p}{q}$$

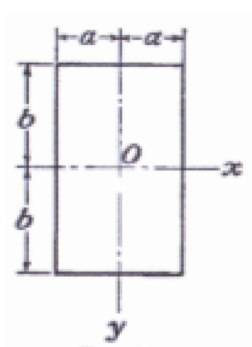
Which is a Poisson partial differential equation (PDE). Of course, membrane deflection z is zero on the boundary. We will now consider the case of a rectangular boundary and use the following notations for a general case:

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=-K$$

De Saint-Venant solution [17] of this PDE takes the following form:

$$f(x,y)=\frac{16Ka^2}{\pi^3}\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n+1)^3}\left(1-\frac{ch\left(\frac{(2n+1)\pi}{2a}y\right)}{ch\left(\frac{(2n+1)\pi}{2a}b\right)}\right)cos\left(\frac{(2n+1)\pi}{2a}x\right)$$

With $2a$ and $2b$ being lengths of rectangular membrane sides as indicated in the following figure:



<u>Figure2</u>: The rectangular membrane from [14].

We can easily calculate both the maximum and mean values of function $f(x,y)$ corresponding to the maximum and mean deflection of the elastic membrane.

$$f_{max} = f(0,0) = \frac{16Ka^2}{\pi^3}\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n+1)^3}\left(1-\frac{1}{ch\left(\frac{(2n+1)\pi}{2a}b\right)}\right)$$

And,

$$< f(x,y) >= \frac{1}{ab}\int_0^a\int_0^b f(x,y)\,dxdy$$

$$= \frac{32Ka^2}{\pi^4}\sum_{n=0}^{+\infty}\frac{1}{(2n+1)^4}\left(1-\frac{2a}{(2n+1)\pi b}th\left(\frac{(2n+1)\pi}{2a}b\right)\right)$$

For the case of the "thin membrane" corresponding to $b \gg a$, we can calculate the ratio of above quantities to obtain:

$$\frac{f_{max}}{< f >} = \frac{3}{2}$$

Which is a well-known result in fluid mechanics for the laminar flow in ducts of rectangular cross-section [15].

For the case of a square membrane i.e. $b = a$, we obtained:

$$\frac{f_{max}}{< f >} = \frac{\pi}{2}\cdot\frac{\frac{\pi^3}{32} - \sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n+1)^3}\cdot\frac{1}{ch\left(\frac{(2n+1)\pi}{2}\right)}}{\frac{\pi^4}{96} - \frac{2}{\pi}\sum_{n=0}^{+\infty}\frac{1}{(2n+1)^5}\cdot th\left(\frac{(2n+1)\pi}{2}\right)} = 2.09625538\ldots$$

From above equation, we can calculate the components of the gradient $\vec{\nabla}f$:

$$\frac{\partial f}{\partial x} = \frac{8Ka}{\pi^2}\sum_{n=0}^{+\infty}\frac{(-1)^{n+1}}{(2n+1)^2}\left(1-\frac{ch\left(\frac{(2n+1)\pi}{2a}y\right)}{ch\left(\frac{(2n+1)\pi}{2a}b\right)}\right)sin\left(\frac{(2n+1)\pi}{2a}x\right)$$

$$\frac{\partial f}{\partial y} = \frac{8Ka}{\pi^2}\sum_{n=0}^{+\infty}\frac{(-1)^{n+1}}{(2n+1)^2}\frac{sh\left(\frac{(2n+1)\pi}{2a}y\right)}{ch\left(\frac{(2n+1)\pi}{2a}b\right)}cos\left(\frac{(2n+1)\pi}{2a}x\right)$$

We can then calculate alongside of length $2a$ i.e. taking $y = b$; both the maximum and mean values of the gradient:

$$\left(\frac{\partial f}{\partial y}\right)_{max} = \left(\frac{\partial f}{\partial y}\right)_{\substack{y=b \\ x=0}} = \frac{8Ka}{\pi^2}\sum_{n=0}^{+\infty}\frac{(-1)^{n+1}}{(2n+1)^2}\, th\left(\frac{(2n+1)\pi}{2a}\, b\right)$$

$$< \left(\frac{\partial f}{\partial y}\right)_{y=b} > = \frac{1}{a}\int_0^a \left(\frac{\partial f}{\partial y}\right)_{y=b} dx = \frac{16Ka}{\pi^3}\sum_{n=0}^{+\infty}\frac{(-1)^{2n+1}}{(2n+1)^3}\, th\left(\frac{(2n+1)\pi}{2a}\, b\right)$$

We can now consider the two limit cases i.e. the case of a thin membrane meaning $b \gg a$ and the case of a square membrane meaning $b = a$.

For $b \gg a$, we obtain:

$$\left(\frac{\partial f}{\partial y}\right)_{max} = -\frac{8Ka}{\pi^2}\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n+1)^2}$$

$$< \left(\frac{\partial f}{\partial y}\right)_{y=b} > = -\frac{16Ka}{\pi^3}\sum_{n=0}^{+\infty}\frac{1}{(2n+1)^3}$$

We can now form the ratio of these two quantities and using the theorem given in chapter 2.2, we obtain:

$$\left(\frac{\partial f}{\partial y}\right)_{max} \Big/ < \left(\frac{\partial f}{\partial y}\right)_{y=b} > = \frac{4\pi G}{7\zeta(3)} = 1.367936563\ldots \quad (11)$$

Where G is well-known Catalan's constant:

$$G = \beta(2) = \sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n+1)^2} = 0.915965594\ldots$$

We know from Apéry's proof that $\zeta(3)$ is an irrational number; but until now, we ignore if Catalan's constant is irrational or not even if it is conjectured. But it is amazing that our elastic thin membrane gave us a quantity involving ratio $\pi G/\zeta(3) = 2.393888986\ldots$ which is, considering 8 first decimals, not very far from being a rational number. The thin membrane model gives a physical signification to this ratio as the ratio of gradient values along the small side of length $2a$. Let us now consider the case $b = a$.

Due to the perfect symmetry of square geometry, we have:

$$\left(\frac{\partial f}{\partial x}\right)_{x=a}=\frac{8Ka}{\pi^2}\sum_{n=0}^{+\infty}\frac{(-1)^{2n+1}}{(2n+1)^2}\left(1-\frac{ch\left(\frac{(2n+1)\pi}{2a}y\right)}{ch\left(\frac{(2n+1)\pi}{2}\right)}\right)$$

And,

$$\left(\frac{\partial f}{\partial y}\right)_{y=a}=\frac{8Ka}{\pi^2}\sum_{n=0}^{+\infty}\frac{(-1)^{n+1}}{(2n+1)^2}\,th\left(\frac{(2n+1)\pi}{2}\right)\,cos\left(\frac{(2n+1)\pi}{2a}x\right)$$

For others components of the gradient, we have:

$$\left(\frac{\partial f}{\partial x}\right)_{y=a}=\left(\frac{\partial f}{\partial y}\right)_{x=a}=0$$

We can then easily calculate the mean values of gradient components along each sides of the square giving:

$$<\left(\frac{\partial f}{\partial x}\right)_{x=a}>=\frac{8Ka}{\pi^2}\sum_{n=0}^{+\infty}\frac{(-1)^{2n+1}}{(2n+1)^2}-\frac{16Ka}{\pi^3}\sum_{n=0}^{+\infty}\frac{(-1)^{2n+1}}{(2n+1)^3}\,th\left(\frac{(2n+1)\pi}{2}\right)$$

$$<\left(\frac{\partial f}{\partial y}\right)_{y=a}>=\frac{16Ka}{\pi^3}\sum_{n=0}^{+\infty}\frac{(-1)^{2n+1}}{(2n+1)^3}\,th\left(\frac{(2n+1)\pi}{2}\right)$$

These two values of the mean gradient along each side must be the same due to square geometry symmetry giving:

$$\frac{4}{\pi}\sum_{n=0}^{+\infty}\frac{1}{(2n+1)^3}\,th\left(\frac{(2n+1)\pi}{2}\right)=\sum_{n=0}^{+\infty}\frac{1}{(2n+1)^2}=\lambda(2)$$

Because we know from chapter 2.2 that the right-side series converges to $\pi^2/8$ (equation 3), we can write:

$$\sum_{n=0}^{+\infty}\frac{1}{(2n+1)^3}\,th\left(\frac{(2n+1)\pi}{2}\right)=\frac{\pi^3}{32}=\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n+1)^3}=\beta(3)$$

Hyperbolic tangent function can be written in its exponential form giving:

$$\sum_{n=0}^{+\infty}\frac{1}{(2n+1)^3}-2\sum_{n=0}^{+\infty}\frac{1}{(2n+1)^3(e^{(2n+1)\pi}+1)}=\frac{\pi^3}{32}$$

From chapter 2.2 (equation 5), we know that:

$$\lambda(3) = \sum_{n=0}^{+\infty} \frac{1}{(2n+1)^3} = \frac{7}{8} . \zeta(3)$$

Giving,

$$\zeta(3) = \frac{\pi^3}{28} + \frac{16}{7} \sum_{n=0}^{+\infty} \frac{1}{(2n+1)^3 (e^{(2n+1)\pi} + 1)} \quad (12)$$

By identifying this equation with its equivalent given in chapter 2.2 (equation 7), we obtain:

$$\sum_{n=1}^{+\infty} \frac{1}{(4n-1)^3} = \sum_{n=0}^{+\infty} \frac{1}{(2n+1)^3 (e^{(2n+1)\pi} + 1)} \quad (13)$$

Convergence of the right-side series is extremely rapid and only five terms give 20 decimals in perfect agreement with left side series value.

And, from equation (8) in chapter 2.2, we obtain:

$$\sum_{n=0}^{+\infty} \frac{1}{(4n+1)^3} = \frac{\pi^3}{32} + \sum_{n=0}^{+\infty} \frac{1}{(2n+1)^3 (e^{(2n+1)\pi} + 1)} \quad (14)$$

It is also interesting that the above series for $\zeta(3)$ (equation 12) has a very similar form to well-known Ramanujan's expression [8]:

$$\zeta(3) = \frac{7}{180} \pi^3 - 2 \sum_{n=1}^{+\infty} \frac{1}{n^3 (e^{2\pi n} - 1)} \quad (15)$$

Which gives the following inequality:

$$\frac{\pi^3}{28} + \frac{16}{7} \sum_{n=0}^{N} \frac{1}{(2n+1)^3 (e^{(2n+1)\pi} + 1)} < \zeta(3) < \frac{7}{180} \pi^3 - 2 \sum_{n=1}^{N+1} \frac{1}{n^3 (e^{2\pi n} - 1)}$$

Because the series converges very rapidly, a value of $N = 2$ already gives a great accuracy for $\zeta(3)$.

Using above results, we also obtained the following expression for $\zeta(3)$:

$$\zeta(3) = \frac{47\pi^3}{1260} - \sum_{n=1}^{+\infty} \frac{1}{n^3 (e^{2\pi n} - 1)} + \frac{8}{7} \sum_{n=0}^{N} \frac{1}{(2n+1)^3 (e^{(2n+1)\pi} + 1)} \quad (16)$$

It is remarkable that others expressions found for $\zeta(5)$ and $\zeta(7)$ given by Plouffe [20], Borwein & al. [21], Grosswald [22] and reported in [11] also take a very similar form:

$$\zeta(5)=\frac{\pi^5}{294}-\frac{72}{35}\sum_{n=1}^{+\infty}\frac{1}{n^5(e^{2\pi n}-1)}-\frac{2}{35}\sum_{n=1}^{+\infty}\frac{1}{n^5(e^{2\pi n}+1)}$$

$$\zeta(7)=\frac{19\pi^7}{56700}-2\sum_{n=1}^{+\infty}\frac{1}{n^7(e^{2\pi n}-1)}$$

To conclude, in this paper, we used the elastic membrane model and the de Saint-Venant solution of the second order partial differential with rectangular boundary conditions.

Calculations of gradient components values along the boundaries gave interesting results in numbers theory. First of all, for the limit case of the "thin membrane", we showed that the ratio $(4\pi G/7\zeta(3))$ where G is the Catalan's constant represents the ratio of maximum and average boundary gradient values. Ratio $(\pi G/\zeta(3)) = 2.393888986\ldots$ being a quasi-rational number appeared particularly important and could be considered as a new mathematical constant.

In the case of the square membrane, average wall gradient calculations gave for $\zeta(3)$, a new rapidly convergent series, very close to famous Ramanujan's expression. By identification, we deduced, new series for both:

$$\sum_{n=1}^{+\infty}\frac{1}{(4n-1)^3}\ and\ \sum_{n=0}^{+\infty}\frac{1}{(4n+1)^3}$$

The sum of these two series being the Dirichlet lambda function $\lambda(3)$

Finally, we gave an inequality which allows a very accurate determination of Apéry's constant with few terms calculation.

Conflict of Interests

The authors declare that there is no conflict of interests.

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