Available online at http://scik.org
J. Math. Comput. Sci. 10 (2020), No. 1, 78-94
https://doi.org/10.28919/jmcs/4147
ISSN: 1927-5307

VARIABLE ORDER NESTED HYBRID MULTISTEP METHODS FOR STIFF ODES

P. O. OLATUNJI[1,*], M. N. O. IKHILE[2]

[1]Department of Mathematical Sciences, Adekunle Ajasin University, Akungba Akoko, Ondo State, Nigeria

[2]Advanced Research Laboratory, University of Benin, Benin City, Nigeria

Abstract. A family of variable order nested hybrid multistep method for the numerical integration of stiff initial value problems of an ordinary differential equation is presented in this paper. These methods possess high order with A-stability properties making it suitable for solving stiff problems. The method has been implemented on some non-linear problems.

Keywords: multistep method; nested hybrid; stiff problem; order; error; stability; variable step size.

2010 AMS Subject Classification: 65L05, 65L06.

1. Introduction

Multistep methods with off-step points, variously known as hybrid methods for the system of initial value problems (IVPs) for stiff ordinary differential equations (ODEs) of the form

$$\begin{cases} y'(x) = f(x,y), \quad x \in [x_0, X] \\ y(x_0) = y_0 \end{cases} \tag{1}$$

*Corresponding author

E-mail address: peter.olatunji@aaua.edu.ng

Received May 30, 2019

where $f : \mathbb{R}^n \to \mathbb{R}^n$, were introduced by independent authors, some of which are [1, 2, 3]. These hybrid methods were proposed in a bid to search for good numerical schemes for the numerical solution of initial value problems (IVPs) in ODEs. Most ODEs arising from scientific modelling exhibit a characteristics of high stability, which when solved by standard numerical methods, instability sets in [4]. This kind of ODEs are called stiff ODEs. In solving stiff systems of ODEs, the numerical method must have good accuracy and some reasonable wide region of absolute stability [5], in which A-stable methods are very good choice. Hence, the need to develop numerical methods with a wide region of absolute stability. However, this requirement of A-stability puts a severe limitation on the choice of suitable multistep methods, which is fondly known as the Dahlquist order barrier [6]. Hybrid methods are modified from the linear multistep methods by incorporating off-step points, this is done in an attempt to bypass the Dahlquist order barrier that exist in the linear multistep methods.

In the quest of solving stiff ODEs, several authors have developed numerical methods that circumvent the so called Dahlquist order barrier, som of which includes; the second derivative multistep methods of [7], hybrid linear multistep method of [8, 9], high order numerical method of [10], extended backward differentiation formulas of [11].

The paper presented is organized thus; in section 2 we discuss the derivation of the proposed method, as well as give some examples of the method, section 3 considers the local truncation error and order of the method proposed, we analyze the A-stability property of the proposed method in section 4, in section 5, we implemented the schemes on some test problems and compare with existing methods. The paper is concluded in section 6 with giving a brief summary.

Kulikov and Shindin [12, 13] considered nested implicit Runge-kutta formulas based on Gauss quadrature formula. Here, we consider nested hybrid multistep methods.

2. Derivation of the methods

The variable order nested hybrid multistep method (VONHM) considered in this paper for the numerical integration of (1) is the second derivative hybrid multistep method defined as

$$y_{n+k} = \sum_{j=0}^{k-1} \alpha_j^{(m)} y_{n+j} + h\left(\gamma_k^{(m)} f_{n+k} + \beta_{v_m}^{(m)} f_{n+v_m}\right) + h^2 \Omega_k^{(m)} f'_{n+k} \tag{2}$$

where $x_{n+j} = x_n + jh$ (h being the step size), k the step number ($k \in \mathbb{Z}^+$), $m = k-1$ and the hybrid value v_m is chosen as

$$v_m = k - \frac{1}{2}$$

In addition to (2), the hybrid point y_{n+v_m} is computed from the nested hybrid

$$y_{n+v_{l+1}} = y_{n+k} + h\left(\sum_{j=0}^{k} \beta_j^{(l)} f_{n+j} + \beta_{v_l}^{(l)} f_{n+v_l}\right) \qquad l = 0(1)m-1 \tag{3}$$

The hybrid method (3) is nested in that we have k number of hybrid point, that is, from (3) we have the set of hybrid points

$$y_{n+v_{l+1}}, y_{n+v_l}, y_{n+v_{l-1}}, y_{n+v_{l-2}}, \cdots, y_{n+v_2}, y_{n+v_1}, y_{n+v_0}$$

where y_{n+v_0} is the nested hybrid predictor. The hybrid value v_l; $l = 0(1)m-1$ is computed using

$$v_l = \frac{v_{l+1}+k}{2}, \qquad l = 0(1)m-1 \tag{4}$$

Two cases of hybrid predictors are considered, given as

$$y_{n+v_0} = \begin{cases} y_{n+k} + h\sum_{j=0}^{k} \beta_j^{(-1)} f_{n+j} & V_1 \\ \\ y_{n+k} + h\sum_{j=0}^{k} \beta_j^{(-1)} f_{n+j} + h^2 \lambda_k^{-1} f'_{n+k} & V_2 \end{cases} \tag{5}$$

y_{n+k} in (2) is the numerical approximation (output solution) to the exact solution $y(x_{n+k})$. Two cases of VONHM are considered above in which the first being the combination of (2), (3) and V_1, while the second is the combination of (2), (3) and V_2. The implicitness of the VONHM is on the output method (2)

We derive the schemes based on the assumption of the polynomial interpolant

$$y(x) = \sum_{j=0}^{N} a_j x^j \tag{6}$$

where $\{a_j\}_{j=0}^{N}$ are real parameter constants to be determined, differentiating (6), we have

$$f(x,y) = y'(x) = \sum_{j=0}^{N} j a_j x^{j-1} \qquad \textit{and} \qquad f'(x,y) = y''(x) = \sum_{j=0}^{N} j(j-1) a_j x^{j-2} \tag{7}$$

we then collocate at $x = x_{n+j},\ j = 0(1)k$ and x_{n+v_m}, and interpolate at x_{n+k}, as discussed in [14, 15, 16]. The hybrid values v_m and v_l; $l = 0(1)m-1$ are computed for the step number $k = 1(1)9$ in table 1.

TABLE 1. Computed values of v for each k

k	m	l	$v = (v_0, v_1, v_2, \ldots, v_{m-1})$	v_m
1	0	-	-	$\frac{1}{2}$
2	1	0	$\frac{7}{4}$	$\frac{3}{2}$
3	2	0,1	$\left(\frac{23}{8}, \frac{11}{4}\right)$	$\frac{5}{2}$
4	3	0(1)2	$\left(\frac{63}{16}, \frac{31}{8}, \frac{15}{4}\right)$	$\frac{7}{2}$
5	4	0(1)3	$\left(\frac{159}{32}, \frac{79}{16}, \frac{39}{8}, \frac{19}{4}\right)$	$\frac{9}{2}$
6	5	0(1)4	$\left(\frac{383}{64}, \frac{191}{32}, \frac{95}{16}, \frac{47}{8}, \frac{23}{4}\right)$	$\frac{11}{2}$
7	6	0(1)5	$\left(\frac{895}{128}, \frac{447}{64}, \frac{223}{32}, \frac{111}{16}, \frac{55}{8}, \frac{27}{4}\right)$	$\frac{13}{2}$
8	7	0(1)6	$\left(\frac{2047}{256}, \frac{1023}{128}, \frac{511}{64}, \frac{255}{32}, \frac{127}{16}, \frac{63}{8}, \frac{31}{4}\right)$	$\frac{15}{2}$
9	8	0(1)7	$\left(\frac{4607}{512}, \frac{2303}{256} \frac{1151}{128}, \frac{575}{64}, \frac{287}{32}, \frac{143}{16}, \frac{71}{8}, \frac{35}{4}\right)$	$\frac{17}{2}$

2.1. Example Methods. For the step number $k = 1$, $m = 0$. The method has a single hybrid predictor with hybrid value $v_0 = \frac{1}{2}$ given as

$$y_{n+1} = y_n + h\left(\frac{4}{3} f_{n+\frac{1}{2}} - \frac{1}{3} f_{n+1}\right) + \frac{1}{6} h^2 f'_{n+1} \tag{8}$$

with the hybrid predictor $y_{n+\frac{1}{2}}$ given as

(9) $$y_{n+\frac{1}{2}} = \begin{cases} y_{n+1} - \frac{1}{8}hf_n - \frac{3}{8}hf_{n+1} & V_1 \\ \\ y_{n+1} + h\left(-\frac{1}{24}f_n - \frac{11}{24}f_{n+1}\right) + \frac{1}{12}h^2 f'_{n+1} & V_2 \end{cases}$$

For the step number $k = 2$, $m = 1$, $l = 0$. The method has hybrid predictors with hybrid values $v_1 = \frac{3}{2}$, $v_0 = \frac{7}{4}$, the method is thus;

(10) $$y_{n+2} = -\frac{1}{31}y_n + \frac{32}{31}y_{n+1} + h\left(\frac{32}{31}f_{n+\frac{3}{2}} - \frac{2}{31}f_{n+2}\right) + \frac{2}{31}h^2 f'_{n+2}$$

with the hybrid

(11) $$y_{n+\frac{3}{2}} = y_{n+2} + h\left(\frac{1}{672}f_n - \frac{1}{48}f_{n+1} - \frac{3}{7}f_{n+\frac{7}{4}} - \frac{5}{96}f_{n+2}\right)$$

and the hybrid predictor $y_{n+\frac{7}{4}}$ given as

(12) $$y_{n+\frac{7}{4}} = y_{n+2} + h\left(\frac{5}{384}f_n - \frac{11}{192}f_{n+1} - \frac{79}{384}f_{n+2}\right) \qquad V_1$$

(13) $$y_{n+\frac{7}{4}} = y_{n+2} + h\left(\frac{13}{12288}f_n - \frac{29}{3072}f_{n+1} - \frac{2969}{12288}f_{n+2}\right) + \frac{49}{2048}h^2 f'_{n+2} \qquad V_2$$

For the step number $k = 3$, $m = 2$, $l = 0, 1$. The method has hybrid values are $v_2 = \frac{5}{2}$, $v_1 = \frac{11}{4}$, $v_0 = \frac{23}{8}$. The method is thus given as

(14) $$y_{n+3} = \frac{20}{3773}y_n - \frac{243}{3773}y_{n+1} + \frac{3996}{3773}y_{n+2} + h\left(\frac{3456}{3773}f_{n+\frac{5}{2}} + \frac{114}{3773}f_{n+3}\right) + \frac{18}{539}h^2 f'_{n+3}$$

with the nested hybrid

(15) $$y_{n+\frac{5}{2}} = y_{n+3} + h\left(-\frac{29}{63360}f_n + \frac{7}{1920}f_{n+1} - \frac{149}{5760}f_{n+2} - \frac{208}{495}f_{n+\frac{11}{4}} - \frac{329}{5760}f_{n+3}\right)$$

(16) $$y_{n+\frac{11}{4}} = y_{n+3} + h\left(-\frac{209}{2119680}f_n + \frac{329}{460800}f_{n+1} - \frac{769}{215040}f_{n+2} - \frac{8348}{36225}f_{n+\frac{23}{8}} - \frac{1529}{92160}f_{n+3}\right)$$

and the hybrid predictor $y_{n+\frac{23}{8}}$ given as

(17) $$y_{n+\frac{23}{8}} = y_{n+3} + h\left(-\frac{75}{32768}f_n + \frac{1027}{98304}f_{n+1} - \frac{2147}{98304}f_{n+2} - \frac{10943}{98304}f_{n+3}\right) \qquad V_1$$

$$(18)\quad y_{n+\frac{23}{8}} = y_{n+3} + h\left(-\frac{553}{8847360}f_n + \frac{281}{655360}f_{n+1} - \frac{591}{327680}f_{n+2} - \frac{2186407}{17694720}f_{n+3}\right) + \frac{19697}{2949120}h^2 f'_{n+3} \qquad V_2$$

Methods for step number $k \geq 4$ can be constructed as discussed above.

3. Error and Order

The general form of the local truncation error of the scheme (2) is

$$(19)\quad LTE_a = y(x_{n+k}) - \left(\sum_{j=0}^{k-1} \alpha_j^{(m)} y(x_{n+j}) + h\left(\gamma_k^{(m)} y'(x_{n+k}) + \beta_{v_m}^{(m)} y'(x_{n+v_m})\right) + h^2\Omega_k^{(m)} y''(x_{n+k})\right)$$

The local truncation error of the nested hybrid scheme (3) is

$$(20)\quad LTE_b = y(x_{n+v_{l+1}}) - \left(y(x_{n+k}) + h\left(\sum_{j=0}^{k} \beta_j^{(l)} y'(x_{n+j}) + \beta_{v_l}^{(l)} y''(x_{n+v_l})\right)\right) \quad l = 0(1)m-1$$

The local truncation error of the first case of hybrid predictor V_1 is

$$(21)\quad LTE_c = y(x_{n+v_0}) - \left(y(x_{n+k}) + h\sum_{j=0}^{k} \beta_j^{(-1)} y'(x_{n+j})\right)$$

The local truncation error of the first case of hybrid predictor V_2 is

$$(22)\quad LTE_d = y(x_{n+v_0}) - \left(y(x_{n+k}) + h\sum_{j=0}^{k} \beta_j^{(-1)} y'(x_{n+j}) + h^2\lambda_k^{(-1)} y''(x_{n+k})\right)$$

Proposition 1. The method (2) has order $p = k+2$ with the local truncation error given as

$$L[y(x_n),h] \quad \leq C_{k+3}^{(a)} h^{k+3} y^{(k+3)} + O(h^{k+4})$$

$$= C_{p+1}^{(a)} h^{p+1} y^{(p+1)} + O(h^{p+2})$$

Proof. Replacing $y(x_{n+k}) = y(x_n+kh)$, $y(x_{n+j}) = y(x_n+jh)$, $y'(x_{n+k}) = y'(x_n+kh)$, $y''(x_{n+k}) = y''(x_n+kh)$ and $y(x_{n+v_m}) = y(x_n+v_m h)$ in (19) and obtain the Taylor's series expansion about x_n gives

$$(23)\quad L[y(x_n),h] = C_0^{(a)} y(x_n) + C_1^{(a)} h y'(x_n) + C_2^{(a)} h^2 y''(x_n) + \cdots + C_{k+2}^{(a)} h^{k+2} y^{(k+2)}(x_n) + C_{k+3}^{(a)} h^{k+3} y^{(k+3)}(x_n) + O(h^{k+4})$$

where

$$C_0^{(a)} = 1 - \sum_{j=0}^{k-1} \alpha_j^{(m)} = 0$$

$$C_1^{(a)} = k - \beta_{v_m}^{(m)} - \gamma_k^{(m)} - \sum_{j=0}^{k-1} j\alpha_j^{(m)} = 0$$

$$C_2^{(a)} = \frac{1}{2!}\left(k^2 - 2v_m\beta_{v_m}^{(m)} - 2k\gamma_k^{(m)} - 2\Omega_k^{(m)} - \sum_{j=0}^{k-1} j^2\alpha_j^{(m)}\right) = 0$$

$$C_3^{(a)} = \frac{1}{3!}\left(k^3 - 3v_m^2\beta_{v_m}^{(m)} - 3k^2\gamma_k^{(m)} - 6k\Omega_k^{(m)} - \sum_{j=0}^{k-1} j^3\alpha_j^{(m)}\right) = 0$$

$$\vdots$$

$$C_{k+2}^{(a)} = \frac{1}{(k+2)!}\Bigg(k^{k+2} - (k+2)v_m^{k+1}\beta_{v_m}^{(m)} - (k+2)k^{k+1}\gamma_k^{(m)} - (k+1)(k+2)k^k\Omega_k^{(m)} - \sum_{j=0}^{k-1} j^{k+2}\alpha_j^{(m)}\Bigg) = 0 \tag{24}$$

$$C_{k+3}^{(a)} = \frac{1}{(k+3)!}\Bigg(k^{k+3} - (k+3)v_m^{k+2}\beta_{v_m}^{(m)} - (k+3)k^{k+2}\gamma_k^{(m)} - (k+2)(k+3)k^{k+1}\Omega_k^{(m)} - \sum_{j=0}^{k-1} j^{k+3}\alpha_j^{(m)}\Bigg) \neq 0 \tag{25}$$

therefore (23) becomes

$$\begin{aligned} L[y(x_n),h] &\leq C_{k+3}^{(a)} h^{k+3} y^{(k+3)} + O(h^{k+4}) \\ &= C_{p+1}^{(a)} h^{p+1} y^{(p+1)} + O(h^{p+2}) \end{aligned} \tag{26}$$

□

Hence, from (26), the method (2) has order $p = k+2$, where $C_{p+1}^{(a)}$ is the error constant of method (2).

Corollary 1. The method (3) has order $q = k+2$ with the local truncation error given as

$$L[y(x_n),h] \quad \leq C^{(a)}_{k+3} h^{k+3} y^{(k+3)} + O(h^{k+4})$$

$$= C^{(a)}_{q+1} h^{q+1} y^{(q+1)} + O(h^{q+2})$$

Corollary 2. The method (5), case V_1 has order $r = k+1$ with the local truncation error given as

$$L[y(x_n),h] \quad \leq C^{(a)}_{k+2} h^{k+2} y^{(k+2)} + O(h^{k+3})$$

$$= C^{(a)}_{r+1} h^{r+1} y^{(r+1)} + O(h^{r+2})$$

Corollary 3. The method (5), case V_2 has order $s = k+2$ with the local truncation error given as

$$L[y(x_n),h] \quad \leq C^{(a)}_{k+3} h^{k+3} y^{(k+3)} + O(h^{k+4})$$

$$= C^{(a)}_{s+1} h^{s+1} y^{(s+1)} + O(h^{s+2})$$

The computed error constant and order of the example methods of VONHM given in section 2.1 is given in table 2 for step number $k = 1(1)3$.

TABLE 2. Error constant and order of the VONHM in section 2.1

k	$C^{(a)}_{p+1}$	$C^{(b)}_{q+1}$	$C^{(c)}_{r+1}$	$C^{(d)}_{s+1}$	p	q	r	s
1	$-\frac{1}{72}$	-	$\frac{1}{24}$	$-\frac{5}{1152}$	3	-	2	3
2	$-\frac{1}{372}$	$-\frac{29}{92160}$	$\frac{49}{6144}$	$-\frac{59}{184320}$	4	4	3	4
3	$-\frac{3}{3430}$	$-\frac{7}{46080}, -\frac{143}{3686400}$	$\frac{19697}{11796480}$	$-\frac{25723}{943718400}$	5	5, 5	4	5

4. Stability Analysis

The first characteristics polynomial of VONHM (2) is expressed as

$$\rho(w) = w^k - \sum_{j=0}^{k-1} \alpha_j^{(m)} w^j \tag{27}$$

where w represents the shift operator. The VONHM is zero stable if the roots $w_j, j = 1(1)k$ of (27), has modulus $w_j < 1$, where the $|w_j| = 1$ is simple. The stability of the method VONHM for the cases V_1 and V_2 is investigated through the application to the scalar test problem defined as

$$y'(x) = \lambda y(x) \qquad x \geq 0, \qquad Re(\lambda) < 0 \tag{28}$$

which yields

$$\pi(w,z)_i y_n = 0, \qquad i = 1,2 \tag{29}$$

Theorem 1. The resultant stability polynomial of the VONHM for the case V_1 is

$$\pi(w,z)_1 = w^k - \sum_{j=0}^{k-1} \alpha_j^{(m)} w^j - z\left[\gamma_k^{(m)} w^k + \beta_{v_m}^{(m)}(R(w,z)_1)\right] - z^2 \Omega_k^{(m)} w^k \tag{30}$$

where

$$R(w,z)_1 = w^k + z\sum_{j=0}^{k} \beta_j^{(m-1)} w^j + \beta_{v_{m-1}}^{(m-1)}\left(\cdots\left(w^k + z\left(\sum_{j=0}^{k}\beta_j^{(1)} + \beta_{v_0}^{(0)}\left(w^k + z\sum_{j=0}^{k}\beta_j^{(-1)} w^j\right)\right)\right)\right) \tag{31}$$

Proof. Taking the right hand side (R.H.S) of (2) to the left hand side (L.H.S) and equating to zero yields

$$y_{n+k} - \sum_{j=0}^{k-1} \alpha_j^{(m)} y_{n+j} - h\left(\gamma_k^{(m)} f_{n+k} + \beta_{v_m}^{(m)} f_{n+v_m}\right) - h^2 \Omega_k^{(m)} f'_{n+k} = 0 \tag{32}$$

applying the scalar test problem (28) on (32) and taking $z = \lambda h$, yields

$$w^k - \sum_{j=0}^{k-1} \alpha_j^{(m)} w^j - z\left(\gamma_k^{(m)} w^k + \beta_{v_m}^{(m)} y_{n+v_m}\right) - z^2 \Omega_k^{(m)} w^k = 0 \tag{33}$$

where w is used as the shift operator, the stability polynomial is given as defined in (30). However, we need to prove that $y_{n+v_m} = R(w,z)_1$ as given in (31), this is obtained by applying the scalar test problem on (3) and (5). The scheme (3) is expanded as follows;

when $l = 0$

$$y_{n+v_1} = y_{n+k} + h\left(\sum_{j=0}^{k} \beta_j^{(0)} f_{n+j} + \beta_{v_0}^{(0)} f_{n+v_0}\right) \tag{34}$$

when $l = 1$

$$y_{n+v_2} = y_{n+k} + h\left(\sum_{j=0}^{k} \beta_j^{(1)} f_{n+j} + \beta_{v_1}^{(1)} f_{n+v_1}\right) \tag{35}$$

the process continues till $l = m-1$

$$y_{n+v_m} = y_{n+k} + h\left(\sum_{j=0}^{k}\beta_j^{(m-1)} f_{n+j} + \beta_{v_{m-1}}^{(m-1)} f_{n+v_{m-1}}\right) \tag{36}$$

Recall from (5) case V_1 that

$$y_{n+v_0} = y_{n+k} + h\sum_{j=0}^{k}\beta_j^{(-1)} f_{n+j} \tag{37}$$

applying the scalar test problem (28) on the R.H.S of (34) to (37), using the shift operator and taking $z = \lambda h$, we have

$$y_{n+v_1} = w^k y_n + z\left(\sum_{j=0}^{k}\beta_j^{(0)} w^j y_n + \beta_{v_0}^{(0)} y_{n+v_0}\right) \tag{38}$$

when $l = 1$

$$y_{n+v_2} = w^k y_n + z\left(\sum_{j=0}^{k}\beta_j^{(1)} w^j y_n + \beta_{v_1}^{(1)} y_{n+v_1}\right) \tag{39}$$

this process continues till

$$y_{n+v_m} = w^k y_n + z\left(\sum_{j=0}^{k}\beta_j^{(m-1)} w^j y_n + \beta_{v_{m-1}}^{(m-1)} y_{n+v_{m-1}}\right) \tag{40}$$

and

$$y_{n+v_0} = w^k y_n + z\sum_{j=0}^{k}\beta_j^{(-1)} w^j y_n \tag{41}$$

from above, the R.H.S of (41) is replaced with y_{n+v_0} in (38) and the result of this R.H.S of (38) is replaced with y_{n+v_1} in (39), this process continues recursively. Hence, we obtain

(42)

$$y_{n+v_m} = R(w,z)_1 = w^k + z\sum_{j=0}^{k}\beta_j^{(m-1)} w^j + \beta_{v_{m-1}}^{(m-1)}\left(\cdots\left(w^k + z\left(\sum_{j=0}^{k}\beta_j^{(1)} + \beta_{v_0}^{(0)}\left(w^k + z\sum_{j=0}^{k}\beta_j^{(-1)} w^j\right)\right)\right)\right)$$

the $\cdots$ denotes some recursive terms obtained between $2 \le l \le m-2$ after the application of the scalar test problem. Hence, the result of the stability polynomial given in (30) and (31). □

Theorem 2. The resultant stability polynomial of the case V_2 is

$$\pi(w,z)_2 = w^k - \sum_{j=0}^{k-1}\alpha_j^{(m)} w^j - z\left[\gamma_k^{(m)} w^k + \beta_{v_m}^{(m)}(R(w,z)_2)\right] - z^2\Omega_k^{(m)} w^k \tag{43}$$

where

$$R(w,z)_2 = w^k + z\sum_{j=0}^{k}\beta_j^{(m-1)}w^j + \beta_{v_{m-1}}^{(m-1)}\left(\cdots\left(w^k + z\left(\sum_{j=0}^{k}\beta_j^{(1)}w^j + \beta_{v_0}^{(0)}\left(w^k + z\sum_{j=0}^{k}\beta_j^{(-1)}w^j + z^2\lambda_k^{(-1)}w^k\right)\right)\right)\right) \tag{44}$$

Definition 1. The region of absolute stability of the VONHM is the set

$$\Psi = \left\{z \in C : |w_j| \leq 1, j = 1(1)k\right\}$$

i.e. if the root of $w_j, j = 0(1)k$ of (30) and (43) are less or equal to one in absolute value, such that those of magnitude one are not repeated.

Definition 2. The VONHM is Astable if the region of absolute stability lies in the entire left half of the zplane $(i.e. z \in C^-)$.

Definition 3. The VONHM is A(α)-stable for some $\alpha \in \left[0, \frac{\pi}{2}\right)$ if the wedge

$$S_\alpha = \{z : |Arg(-z)| < \alpha, z \neq 0\}$$

is contained in its region of absolute stability.

The VONHM is Astable for the step number $k = 1(1)5$ and $k = 2(1)5$ for the cases V_1 and V_2 respectively. However as at the time this research was carried out, the point at which instability sets in, was not investigated. The boundary locus plot of the VONHM, cases V_1 and V_2 for the step number $k = 1(1)9$ are shown figures 1 ans 2. Table 3 shows the angle of absolute stability of the VONHM, where those of 90^0 indicate A stable methods.

5. Numerical Experiments

We implement the third order VONHM (2) on some non-linear stiff problems using fixed step size and variable step size. Implementing the method leads to solving a system of non-linear equations in y_{n+k}. Hence, the need to resolve its implicitness by the Newton-Raphson scheme

$$y_{n+k}^{[s+1]} = y_{n+k}^{[s]} - J(y_{n+k}^{[s]})^{-1}F(y_{n+k}^{[s]}) \tag{45}$$

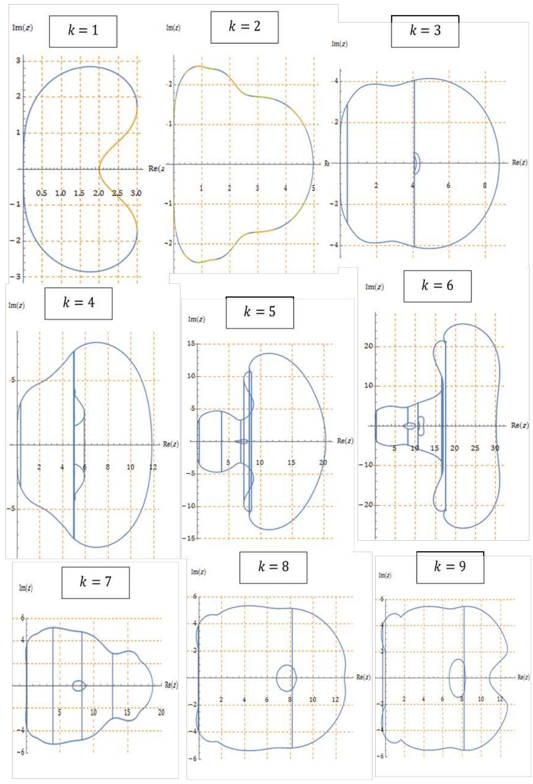

FIGURE 1. Boundary locus plot of the VONHM case V_1

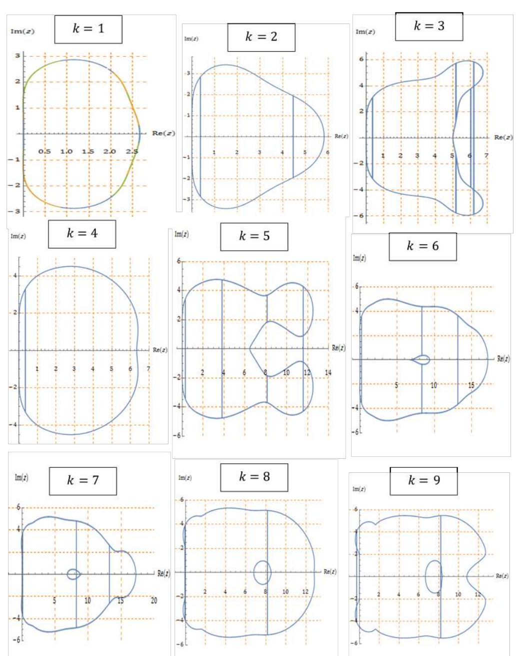

FIGURE 2. Boundary locus plot of the VONHM case V_2

TABLE 3. Angle (α) of absolute stability of the VONHM

k	V_1	V_2
1	90^0	89.2^0
2	90^0	90^0
3	90^0	90^0
4	90^0	90^0
5	90^0	90^0
6	89^0	89^0
7	87^0	87^0
8	85.5^0	85^0
9	82^0	82.5^0

where

$$F(y_{n+k}^{[s]}) = y_{n+k}^{[s]} - \left(\sum_{j=0}^{k-1} \alpha_j^{(m)} y_{n+j} + h\left(\gamma_k^{(m)} f_{n+k}^{[s]} + \beta_{v_m}^{(m)} f_{n+v_m}^{[s]} \right) + h^2 \Omega_k^{(m)} f_{n+k}^{\prime[s]} \right) \tag{46}$$

and $J(y_{n+k}^{[s]})$ is the Jacobian matrix defined by

$$J(y_{n+k}^{[s]}) = \frac{\partial}{\partial y} F(y_{n+k}^{[s]}) \tag{47}$$

therefore, the Jacobian matrix is given as

$$J(y_{n+k}^{[s]}) = \frac{\partial}{\partial y} \left(I - h\gamma_k^{(m)} f(y_{n+k}^{[s]}) + h\beta_{v_m}^{(m)} f(y_{n+v_m}^{[s]}) + h^2 \Omega_k^{(m)} f'(y_{n+k}^{[s]}) \right) \tag{48}$$

where I is the unit matrix.

We use the second derivative explicit Euler method as the starting values for (45) defined as

$$y_{n+1} = y_n + hf_n + \frac{h^2}{2} f_n' \tag{49}$$

The non-linear problems in consideration is given below.

Problem 1 Non-linear problem [17].

$$\left\{\begin{array}{lll} y_1'(x) = -0.1y_1(x) - 199.9y_2(x); & y_1(0) = 2; & y_1(x) = e^{-0.1} + e^{-200x} \\ y_2'(x) = -200y_2(x); & y_2(0) = 0; & y_2(x) = e^{-200x} \\ x \in [0, 10]; & h = 10^{-3} & \end{array}\right. \tag{50}$$

Problem 2 The Brusselator [5].

$$\left\{\begin{array}{ll} y_1'(x) = 1 + y_1^2(x)y_2(x) - 4y_1(x); & y_1(0) = 1.5 \\ y_2'(x) = 3y_1(x) - y_1^2(x)y_2(x); & y_2(0) = 3 \\ x \in [0, 20]; & h = 10^{-3} \end{array}\right. \tag{51}$$

Implementing the third order VONHM on problem 1. We verify that the method does not suffer order reduction using fixed step size. The global error is computed as

$$\varpi_h = \| \left\{ y_1(x) - y_{1,n}(x) \right\}, \left\{ y_2(x) - y_{2,n}(x) \right\} \| \tag{52}$$

where $y_1(x)$ and $y_2(x)$ are the numerical result of the exact solution, $y_{1,n}(x)$ and $y_{2,n}(x)$ are the numerical approximation obtained by the VONHM and $\| \cdot \|$ is taken as the maximum norm. The relative error is computed as

$$\bar{Re} = \frac{\varpi_h}{\varpi_{\frac{h}{2}}} \tag{53}$$

then the order p is given by

$$p + O(h) = \log_2 |\bar{Re}| \tag{54}$$

The last column of Table 4 shows that the third order method does not suffer order reduction. The VONHM is implemented using variable step size, alongside the second derivative linear multistep method (SDLMM) [5]

$$y_{n+1} = y_n + \frac{h}{3}(f_n + 2f_{n+1}) - \frac{h^2}{6}f'_{n+1}; \qquad C_4 = \frac{1}{72} \tag{55}$$

and the backward differentiation formula (BDF) [4]

$$y_{n+3} = \frac{2}{11}y_n - \frac{9}{11}y_{n+1} + \frac{18}{11}y_{n+2} + \frac{6}{11}hf_{n+3} \qquad C_4 = -\frac{3}{22} \tag{56}$$

TABLE 4. Numerical results of Problem 1 (using fixed step size) for $x \in [0,2]$

h	ϖ	$\bar{Re}$	p (approx.)
0.001	1.110481203949743e-004	-	-
0.0005	1.455972370728587e-005	7.627076078332752	2.93113 ≈ (3)
0.00025	1.866506438574778e-006	7.800521555341296	2.96357 ≈ (3)
0.000125	2.363607967126313e-007	7.896852881419614	2.98128 ≈ (3)
0.0000625	2.974006951816932e-008	7.947553605018632	2.99051 ≈ (3)
0.00003125	3.729839104238408e-009	7.973552930037692	2.99522 ≈ (3)

for the numerical integration of the non linear problems (50) and (51). We adopt the idea in [4] to compute a new step size

(57) $$h_{new} = 0.9\left(\frac{TOL}{\varpi}\right)^{\frac{1}{p}} h_{old}$$

where, h_{old} is the last successful or failed step size, h_{new} is the new step size to be used if the last step size is a failed attempt, TOL is the specified users tolerance (i.e. allowable error estimate), p is the order of the method and ϖ is the maximum error, nfe represents the number of function evaluation, ns represents the number of steps, and nrs represents the number of rejected steps. We choose $TOL = 10^{-6}$. In Tables 5 and 6, our method has a good maximum error than the

TABLE 5. Numerical result of Problem 1

	VONHM	SDLMM [5]	BDF [4]
nfe	220000	114080	159970
ns	1000	1000	1000
nrs	0	2	5
ϖ	1.287858708565182e-014	9.9990050970576e-002	3.67879441171442e-001

SDLMM and BDF with no rejected steps. Hence, this shows that our method outperforms the SDLMM and BDF for problems 1 and 2.

TABLE 6. Numerical result of Problem 2

	VONHM	SDLMM [5]	BDF [4]
nfe	440000	239978	288320
ns	20000	20000	2000
nrs	0	11	198
ϖ	3.159554022663061e-010	1.005387230111084e-006	8.20707935075017e-001

6. CONCLUSION

Second derivative variable order nested hybrid multistep method for the numerical integration of stiff initial value problems of ordinary differential equation is presented. The method presented possess high order with A-stability properties for the step number $k = 1(1)5$. The method is also implemented using fixed and variable step size. The fixed step size implementation on the non-linear problem 1 suggest that the method did not suffer from order reduction, while the variable step size implementation of our method shows that the scheme presented outperforms the SDLMM and BDF.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

REFERENCES

[1] W. B. Gragg and H.J. Stetter, Generalized multistep predictor-corrector methods, J. Assoc. Comput. Mach. 11 (2) (1964), 188-209.

[2] C. W. Gear, Hybrid methods for initial value problems in ordinary differential equations, SIAM. J. Numer. Anal. 2 (1965), 69-86.

[3] J. C. Butcher and A. E. OSullivan, Nordsieck methods with an off-step point, Numer. Algorithms, 31 (2002), 87 - 101.

[4] J.C. Butcher, Numerical methods for ordinary differential equivalent, John Willey & Sons, Ltd Chichester, 2016.

[5] W. H. Enright, Second derivative multistep method for stiff ordinary differential equations, SIAM J. Numer. Anal. 11 (2) (1974), 321-331.

[6] G. Dahlquist, A special stability problem for Linear Multistep Methods, Academic Press, New York, 1963.

[7] E. Hairer, and G. Wanner, Solving ordinary differential equations II. Stiff and Differential - Algebraic problems Vol. II, Springer-Verlag, 2010.

[8] R. I. Okuonghae and M. N. O. Ikhile, On the construction of high order A(α)-stable hybrid linear multistep methods for stiff IVPs and ODEs, J. Numer. Anal. Appl. 15(3) (2012), 231-241.

[9] R. I. Okuonghae and M. N. O. Ikhile, A class of hybrid linear multistep methods with A(α)-stability properties for stiff IVPs in ODEs, J. Numer. Math. 21 (2) (2013), 157-172.

[10] J. C. Butcher, High order A-stable numerical methods for stiff problems, J. Sci. Comput. 25 (2005), 51-66.

[11] J.R. Cash, Second derivative extended backward differential formulas for the numerical integration of stiff systems, SIAM J. Numer. Anal. 18 (1981), 21-36.

[12] G. Yu Kulikov and S. K. Shindin, Numerical Tests with Gauss-Type Nested Implicit Runge-Kutta Formulas, International Conference on Computational Science. Springer, Berlin, Heidelberg, 2007, 136-143.

[13] G. Yu Kulikov and S. K. Shindin, Adaptive nested IRK formulas of Gauss Type, J. Appl. Numer. Math. 59 (2009), 707-722.

[14] P. O. Olatunji, Second Derivative Multistep methods with Nested Hybrid Evaluation, M.Sc. Thesis, 2017.

[15] P. O. Olatunji, and M. N. O. Ikhile, Modified Backward Differentiation Formulas with Recursively Nested Hybrid Evaluation; J. Nigerian Assoc. Math. Phys. 40 (2017), 83-90.

[16] P. O. Olatunji, and M. N. O. Ikhile, Second Derivative Multistep Method with Nested Hybrid Evaluation; Asian Res. J. Math. 11 (4) (2018), 1-11.

[17] S. O. Fatunla, Numerical Methods for Initial Value Problems in Ordinary Differential equations; Academic Press Inc. New York, 1988.